SHEET 2

1 State discrimination

1. We start with the average error probability setting.

   (a) Let ρ, σ be density operators. Show that $\Delta(\rho, \sigma) = \max\{\mathrm{tr}(P(\rho - \sigma))\}$, where the maximum is over all orthogonal projections P. Show also that the maximization can be taken over all operators P satisfying $0 \leq P \leq I$.

   (b) Conclude that the minimum average error probability for distinguishing ρ_0 and ρ_1 is given by $\frac{1}{2} - \frac{1}{2}\Delta(\rho_0, \rho_1)$.

2. Now consider the asymmetric setting. Assume $\mathrm{supp}(\rho)$ is not included in $\mathrm{supp}(\sigma)$. We show Stein's lemma in this case.

   (a) Show that for some $\epsilon < 1$, we have $D_H^\epsilon(\rho\|\sigma) = +\infty$.

   (b) Conclude that for any $\epsilon > 0$, there is an n_ϵ such that for $n \geq n_\epsilon$, we have $D_H^\epsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = +\infty$ and that Stein's lemma holds in this case.

3. Assume that ρ and σ commute and let $\{P(x)\}_{x\in\mathcal{X}}$ and $\{Q(x)\}_{x\in\mathcal{X}}$ be their vector of eigenvalues. Show that $D_H^\epsilon(\rho\|\sigma) = D_H^\epsilon(P\|Q)$, where $D_H^\epsilon(P\|Q) = \max\{-\log\sum_{x\in\mathcal{X}} E(x)Q(x) : \sum_{x\in\mathcal{X}} E(x)P(x) \geq 1 - \epsilon\}$.

2 Properties of quantum entropies

1. Recall that the von Neumann entropy $H(A)_\rho = -D(\rho_A\|I_A)$. Show that $0 \leq H(A)_\rho \leq \log\dim A$. You might want to use Jensen's inequality.

2. Show that $H(A)_\rho = 0$ if and only if ρ is pure and $H(A)_\rho = \log\dim A$ if and only if ρ is maximally mixed.

3. Show that if $\rho_{AB} = \rho_A \otimes \rho_B$, $H(AB)_\rho = H(A)_\rho + H(B)_\rho$.

4. Recall we defined $H(A|B)_\rho = -D(\rho_{AB}\|I_A \otimes \rho_B)$. Show that $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$.

5. Show that if ρ_{AB} is classical, i.e., $\rho_{AB} = \sum_{a,b} P(a,b)|a\rangle\langle a|_A \otimes |b\rangle\langle b|_B$ for some orthonormal bases $\{|a\rangle\}_a$ and $\{|b\rangle\}_b$, then $H(A|B)_\rho \geq 0$. Is this still the case for general ρ?

3 Pinching

Recall that for a Hermitian operator σ, the pinching map $\mathcal{P}_\sigma$ is defined by $\mathcal{P}_\sigma(S) = \sum_{\lambda\in\mathrm{spec}(\sigma)} \Pi_\lambda S \Pi_\lambda$, where Π_λ is the projector onto the eigenspace of λ for the operator σ.

1. If $\sigma = I$, what is $\mathcal{P}_\sigma$?

2. Show that $\mathcal{P}_\sigma(S)$ commutes with σ.

3. Show that $\mathrm{tr}(\mathcal{P}_\sigma(S)\sigma) = \mathrm{tr}(S\sigma)$.

4. Let $m = |\mathrm{spec}(\sigma)|$ and label the eigenvalues by λ_x with $x \in \{0, 1, \ldots, m-1\}$. Show that for any $y \in \{0, 1, \ldots, m-1\}$, the operator $U_y := \sum_{x\in\{0,1,\ldots,m-1\}} e^{\frac{2\pi i x y}{m}} \Pi_{\lambda_x}$ is unitary. Show that $\mathcal{P}_\sigma$ can be written as choosing $y \in \{0, 1, \ldots, m-1\}$ at random and then applying U_y.

5. Show that for a positive operator ρ, we have $\mathcal{P}_\sigma(\rho) \geq \frac{1}{|\mathrm{spec}(\sigma)|}\rho$.