SHEET 1

1 Basic questions about density operators

1. Let $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C + |1\rangle_A \otimes |1\rangle_B \otimes |1\rangle_C)$. Compute $\operatorname{tr}_C |\psi\rangle\langle\psi|_{ABC}$.
2. Let $\rho_{AB} \in \mathrm{S}(A \otimes B)$ be a pure state, i.e., ρ_{AB} has rank one. Show that the non-zero eigenvalues of ρ_A and ρ_B are the same. Recall that we use the notation ρ_A to denote the marginal of ρ on the system A, i.e., $\rho_A = \operatorname{tr}_B \rho_{AB}$.
3. Let $\rho \in \mathrm{S}(A)$. Show that there exists a Hilbert space $\bar{A}$ and $|\psi\rangle_{A\bar{A}} \in A \otimes \bar{A}$ such that $\rho = \operatorname{tr}_{\bar{A}} |\psi\rangle\langle\psi|_{A\bar{A}}$.
4. Given a tripartite pure quantum state ρ_{ABC}, define the vector of ranks $(r_A, r_B, r_C, r_{AB}, r_{BC}, r_{AC})$ of the marginals (e.g., $r_{AC} = \operatorname{rank}(\rho_{AC})$). Derive as many equalities and inequalities satisfied by all such vectors. Can you find all such inequalities? i.e., a set of inequalities such that for any vector of ranks satisfying the inequalities, one can construct a state having these ranks.
5. Assume $\rho_{AB} \in \mathrm{S}(A \otimes B)$ is such that ρ_A is pure. Show that $\rho_{AB} = \rho_A \otimes \rho_B$.

2 Basic questions about quantum channels

1. Define $\top : \mathrm{L}(A) \to \mathrm{L}(A)$ be the transpose map in some fixed orthonormal basis $\{|a\rangle\}_a$ of A, i.e., $\top(|a\rangle\langle a'|) = |a'\rangle\langle a|$. Show that $\top$ is a positive map, but that it is not completely positive.
2. Show that if $\mathcal{E}$ and $\mathcal{F}$ are completely positive, then $\mathcal{E} \otimes \mathcal{F}$ is completely positive.
3. For $p \in \mathbb{R}$, let $\mathcal{D}_p : \mathrm{L}(\mathbb{C}^2) \to \mathrm{L}(\mathbb{C}^2)$ with parameter p by $\mathcal{D}_p(S) = (1-p)S + p \operatorname{tr}(S)\frac{\mathrm{id}}{2}$. Compute the Choi operator for $\mathcal{D}_p$, a Kraus representation and a Stinespring dilation of it. For which values of p is $\mathcal{D}_p$ a quantum channel?
4. Note that $\mathrm{L}(A)$ can be seen as a Hilbert space with the Hilbert Schmidt inner product defined by $\langle S, T\rangle = \operatorname{tr}(S^*T)$. Thus, for a map $\mathcal{E} : \mathrm{L}(A) \to \mathrm{L}(B)$, we define the adjoint map $\mathcal{E}^* : \mathrm{L}(B) \to \mathrm{L}(A)$ such that for any $S \in \mathrm{L}(A), T \in \mathrm{L}(B)$, we have $\langle T, \mathcal{E}(S)\rangle = \langle \mathcal{E}^*(T), S\rangle$. Show that a map $\mathcal{E} : \mathrm{L}(A) \to \mathrm{L}(B)$ is trace-preserving if and only if $\mathcal{E}^*(\mathrm{id}_B) = \mathrm{id}_A$ (we say that $\mathcal{E}^*$ is unital). Show that $\mathcal{E}$ is completely positive if and only if $\mathcal{E}^*$ is completely positive.

3 Separable and block positive operators and interpretation in terms of maps

Let A, B be fixed Hilbert spaces. Define the set of separable operators as $\mathrm{Sep}(A : B) = \{\sum_i p_i \rho_A \otimes \sigma_B : \rho_A \in \mathrm{Pos}(A), \sigma_B \in \mathrm{Pos}(B), \sum_i p_i = 1\}$. A separable state is a separable operator of trace 1.

1. A Hermitian matrix $M \in \mathrm{L}(A \otimes B)$ is called block positive if for any $\rho \in \mathrm{Sep}(A : B)$, $\operatorname{tr}(M\rho) \geq 0$. Construct a block positive matrix M that is not positive semidefinite. *Remark:* Note that in the language of cones, the cone of block positive operators is dual to the cone of separable operators.
2. Let $\mathcal{E} : \mathrm{L}(A) \to \mathrm{L}(B)$ be a linear map. Show that $\mathcal{E}$ is positive if and only if the Choi operator $J^{\mathcal{E}}$ is block positive.
3. A quantum channel $\mathcal{E} : \mathrm{L}(A) \to \mathrm{L}(B)$ is called entanglement breaking if there exists a POVM $\{M_x\}$ and density operators σ_x such that $\mathcal{E}(S) = \sum_x \operatorname{tr}(M_x S)\sigma_x$. Show that a quantum channel $\mathcal{E}$ is entanglement breaking if and only if the Choi operator of $\mathcal{E}$ is separable.

4 Operator convexity and monotonicity

1. Show that $x \mapsto 1/x$ is operator convex. You can start writing down the definition of convexity for $A = \mathrm{id}$ and B an arbitrary positive operator, and then get to the general setting by considering $B \to A^{-1/2}BA^{-1/2}$.

2. Show that $x \mapsto -\ln x$ is operator convex. You might want to show and use the following integral representation: $-\ln x = \int_0^\infty \left(\frac{1}{x+t} - \frac{1}{1+t}\right) dt$ combined with the previous question.

3. Use the same technique to show that $x \mapsto \ln x$ is operator monotone.