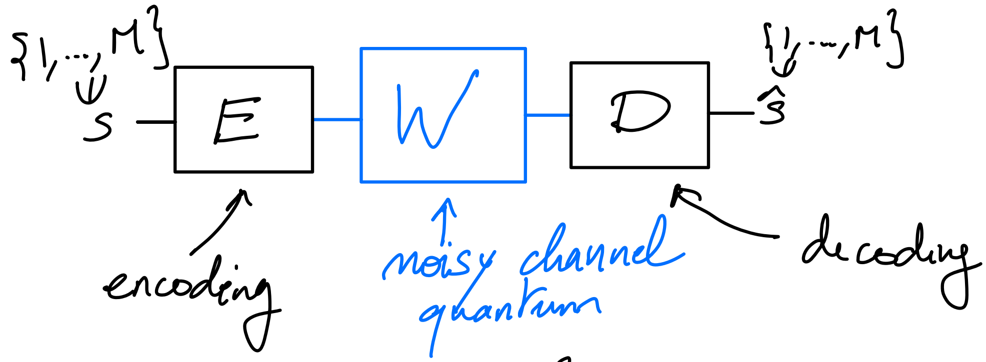
Note: answer questions from Whova: contractive, operational intr Choi, Kraus.



Objective: $*$ $\mathbb{P}\{s \neq \hat{s}\}$ small

$*$ M large.

1. Classical - quantum channels.

Input of W is classical

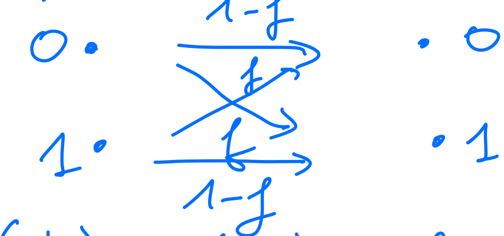
Def: A classical-quantum channel W with input space $\mathcal{X}$ (finite set) and output space B (Hilbert space) is a collection $\{W_x\}_{x \in \mathcal{X}}$ of density operators W_x acting on B.

Ex: • Classical channel: $\{W(y|x)\}_{x\in\mathcal{X}, y\in\mathcal{Y}}$

$W(y|x)$ = probability output y for input x.

For example: Binary symmetric channel flip probability f ($\in [0,1]$)

$\mathcal{X} = \{0,1\}$ $\qquad$ $\mathcal{Y} = \{0,1\}$



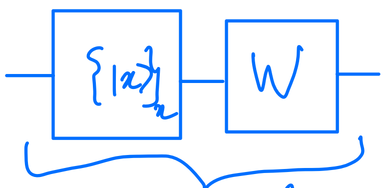
$W(0|0) = W(1|1) = 1-f, \quad W(0|1) = W(1|0) = f.$

Can see it as a classical-quantum channel with output B a Hilbert space of dimension $|\mathcal{Y}|$

$$W_x = \sum_{y \in \mathcal{Y}} W(y|x)\, |y\rangle\langle y|$$

where $\{|y\rangle : y \in \mathcal{Y}\}$ is a fixed orthonormal basis.

- $W_0 = |0\rangle\langle 0|$ and $W_1 = |+\rangle\langle +|$ $\quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Can see W as a quantum channel that starts by measuring in a basis $\{|x\rangle\}_{x \in \mathcal{X}}$ followed by preparation



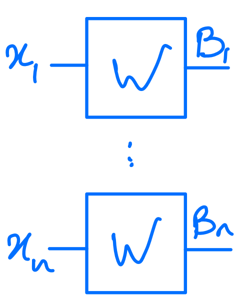
quantum channel $\mathcal{W}$ satisfying $\mathcal{W}(|x\rangle\langle x|) = W_x$ for $x \in \mathcal{X}$.

and $\mathcal{W}(|x\rangle\langle x'|) = 0$ for $x \neq x'$

- Given W and $n \geq 1$ integer, can define

$W^{\otimes n}$: input $\mathcal{X}^n$ and output $B^{\otimes n}$

$$(W^{\otimes n})_{x_1 \cdots x_n} = W_{x_1} \otimes W_{x_2} \otimes \cdots \otimes W_{x_n}.$$



$[M] := \{1, \dots, M\}$.

Def: An M-code (E, D) for W is given by

- $E : [M] \to \mathcal{X}$ encoding function
- Decoding is a POVM $\{D_s\}_{s \in [M]}$ on B.

Ex: For a classical channel, we may assume $\{D_s\}$ are diagonal $D_s = \mathrm{diag}(D_s(y) : y \in \mathcal{Y})$

$D_s(y)$ = Probability of decoding to s when seeing y.

POVM condition: $\sum_s' D_s(y) = 1$.

Def: The error probability $p_{err}(E, D)$ of an M-code for W is defined by

$$p_{err}(E, D) = 1 - \underbrace{\frac{1}{M} \sum_{s \in [M]} \mathrm{Tr}\left(D_s W_{E(s)}\right)}_{\text{probability of successful decoding}}$$

If $p_{err}(E, D) \leq \varepsilon$, we say that (E, D) is an (M, ε)-code

Remark: Used a uniform prior on $[M]$, another natural choice is

$$p_{err,max}(E, D) = \max_{s \in [M]} 1 - \mathrm{Tr}\left(D_s W_{E(s)}\right)$$

p_{err} and $p_{err,max}$ are related

Ex: • $\dim B = |\mathcal{X}|$. $W_x = |x\rangle\langle x|$.

$(|\mathcal{X}|, 0)$ code given by

$E(s) = |s\rangle\langle s|$ (identifying $\mathcal{X}$ with $[|\mathcal{X}|]$)

$D_s = |s\rangle\langle s|$.

$$\frac{1}{|\mathcal{X}|} \sum_{s} \mathrm{Tr}\big(|s\rangle\langle s| \cdot |s\rangle\langle s|\big) = 1.$$

• Let $\rho \in S(B)$ and $W_x = \rho$ for all $x \in \mathcal{X}$.

(useless channel, output does not depend on input)

For any choice of (E, D), we have

$$\sum_{s \in [M]} \mathrm{Tr}(D_s \rho) = 1. \Rightarrow p_{err}(E, D) = 1 - \frac{1}{M}$$

Question: Fixed ε, largest M for which there exists an (M, ε)-code for W?

$$M^{opt}(W, \varepsilon) = \max\{M : \exists\ (M, \varepsilon)\text{-code for } W\}.$$

Objective: Characterize $M^{opt}(W, \varepsilon)$ in terms of "simple" properties of W.

Important special case: • $W^{\otimes n}$ with large n.

• ε small

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \underbrace{\frac{\log_2 M^{opt}(W^{\otimes n}, \varepsilon)}{n}}_{\text{number of bits transmitted per channel use}} = ?$$

Intuition: $M^{opt}(W,\varepsilon)$ should be given by a correlation measure between input and output of W.

Given a probability measure P_X on $\mathcal{X}$ let

$$\rho_{XB} = \sum_{x\in\mathcal{X}} P_X(x)\, |x\rangle\langle x| \otimes W_x \quad \text{cq-state}$$

Recall we write $\rho_X = \mathrm{Tr}_B\, \rho_{XB}$ and $\rho_B = \mathrm{Tr}_X\, \rho_{XB}$.

To characterize $M^{opt}(W,\varepsilon)$ need:

* upper bound (called converse)
* lower bound (called achievability)

Converse

Th: If there exists an (M,ε) code for W

then $$\log M \leq \sup_{P_X} D_H^{\varepsilon}\underbrace{(\rho_{XB} \| \rho_X \otimes \rho_B)}_{\text{correlation measure}}$$

Rk: Channel W is arbitrary, "one-shot" entropy measure expected. For W iid, D_H^{ε} will become a relative entropy D.

Proof: Consider an (M,ε) code (E,D).

Let $C = \{x\in\mathcal{X} : \exists s\in[M] : E(s)=x\}$

and define $P_X(x) = 1/|C|$ for $x\in C$ and $P_X(x)=0$ $x\notin C$

Then

$$\rho_{XB} = \frac{1}{|C|}\sum_{x\in C} |x\rangle\langle x| \otimes W_x .$$

and

$$F = \frac{|C|}{M}\sum_{x\in C} |x\rangle\langle x| \otimes \Big(\sum_{s: E(s)=x} D_s\Big)$$

As $\{D_s\}$ is a POVM, $0 \leq F \leq I$.

$$\begin{aligned}
\mathrm{Tr}(F\rho_{XB}) &= \frac{1}{M}\sum_{x\in C} \mathrm{Tr}\Big(\Big(\sum_{s:E(s)=x} D_s\Big) W_x\Big) \\
&= \frac{1}{M}\sum_{s=1}^{M} \mathrm{Tr}\big(D_s W_{E(s)}\big) \\
&\geq 1-\varepsilon .
\end{aligned}$$

by the fact that (E,D) is an (M,ε)-code.

On the other hand,

$\frac{1}{|C|}\sum_{x\in C}|x\rangle\langle x| = \rho_X$

$$\begin{aligned}
\mathrm{Tr}(F\rho_X\otimes\rho_B) &= \frac{|C|}{M}\,\mathrm{Tr}\Big(\Big(\sum_{x\in C}|x\rangle\langle x|\otimes\Big(\sum_{s:E(s)=x} D_s\Big)\Big)(\rho_X\otimes\rho_B)\Big) \\
&= \frac{|C|}{M}\sum_{x\in C}\mathrm{Tr}\Big(\Big(|x\rangle\langle x|\otimes\sum_{s:E(s)=x} D_s\Big)\Big(\frac{|x\rangle\langle x|}{|C|}\otimes\rho_B\Big)\Big) \\
&= \frac{1}{M}\,\mathrm{Tr}\Big(\Big(\sum_{x\in C}\sum_{s:E(s)=x} D_s\Big)\rho_B\Big) \\
&= \frac{1}{M}
\end{aligned}$$

So $D_H^{\varepsilon}(\rho_{XB}\|\rho_X\otimes\rho_B) \geq \log M$ ∎

Achievability

Th: For any $\varepsilon \in (0,1)$ and $\delta \in (0,\varepsilon)$

Tunable parameters for the bound.

and M satisfying:

$$(*) \quad \log M \leq \sup_{P_X} \underbrace{D_H^{\varepsilon-\delta}\left(\rho_{XB} \| \rho_X \otimes \rho_B\right) - \log(1/\delta)}_{\text{Think of this as a small error term}}$$

then exists an (M,ε)-code.

Not the same ε but can choose it arbitrarily close to ε.

Rk: * Achievability statement matches converse up to error terms that are "small" in many settings of interest.

* Proof uses the probabilistic method: does not give an explicit (E,D) that is an (M,ε) code but rather we choose (E,D) at random and show that on average, it has an error probability $\leq \varepsilon$.

Proof:

Assume E is fixed (chosen later), we construct a decoder $\{D_s\}$

We use pretty good measurement:

$$D_s = \Lambda^{-1/2} W_{E(s)} \Lambda^{-1/2} \qquad \Lambda = \sum_{s' \in [M]} W_{E(s')}.$$

notation $\rightarrow$ $= \dfrac{W_{E(s)}}{\Lambda}$

Rk: Interpretation of this strategy in classical case

On sample y, output s with prob $\frac{W_{E(s)}(y)}{\sum_{s'} W_{E(s')}(y)}$

Maximum likelihood would be:

On sample y, output s maximizing $W_{E(s)}(y)$

$$p_{err}(E,D) = \frac{1}{M}\sum_{s\in[M]} \mathrm{Tr}\Big(\sum_{s'\neq s} D_{s'} W_{E(s)}\Big)$$
$$= \frac{1}{M}\sum_{s\in[M]} \mathrm{Tr}\left(\frac{\left(\sum_{s'\neq s} W_{E(s')}\right)}{W_{E(s)} + \left(\sum_{s'\neq s} W_{E(s')}\right)} W_{E(s)}\right)$$

Note that for $a,b\geq 0$ scalars: $\frac{ab}{a+b} \leq \min(a,b)$.

Noncommutative minimum:

For $A,B\geq 0$,
$$A\wedge B := \frac{1}{2}\left(A+B-|A-B|\right)$$

Properties: (i) $\mathrm{Tr}(A\wedge B) = \max_{M=M^*}\{\mathrm{Tr}(M) : M\leq A, M\leq B\}$
$$= \min_{0\leq \Lambda\leq I}\{\mathrm{Tr}(A(\mathbb{1}-\Lambda)) + \mathrm{Tr}(B\Lambda)\}$$

(ii) $\mathrm{Tr}\left(A(A+B)^{-1/2}B(A+B)^{-1/2}\right) \leq \mathrm{Tr}(A\wedge B)$.

(iii) $(A,B)\mapsto \mathrm{Tr}(A\wedge B)$ jointly concave

See arXiv: 2208.02132 for proofs and more properties.

$$p_{err}(E,D) \le \frac{1}{M}\sum_{s\in[M]} \mathrm{Tr}\Big(W_{E(s)} \wedge \sum_{s'\neq s} W_{E(s')}\Big)$$

Now how to choose E?

choose $E(s)$ random with distribution P_X maximizing ✱ independently for every $s\in[M]$.

$$\underset{\substack{\uparrow\\ \text{randomness}\\ \text{in } E}}{\mathbb{E}}\big\{ p_{err}(E,D)\big\} \le \mathbb{E}_{\substack{E(1)\sim P_X\\ E(s')\sim P_X}} \mathrm{Tr}\Big(W_{E(1)} \wedge \sum_{s'\neq 1} W_{E(s')}\Big)$$

$$\le \mathbb{E}_{E(1)\sim P_X}\Big\{\mathrm{Tr}\Big(W_{E(1)} \wedge \mathbb{E}_{E(s')\sim P_X}\Big(\sum_{s'\neq 1} W_{E(s')}\Big)\Big)\Big\}$$

$$= \mathbb{E}_{x\sim P_X} \mathrm{Tr}\big(W_x \wedge (M-1)\rho_B\big)$$

$$= \mathrm{Tr}\big(\rho_{XB} \wedge (M-1)\rho_X\otimes\rho_B\big).$$

Now want to relate this to hypothesis testing.

$$= \inf_{0\le\Lambda\le I} \mathrm{Tr}\big((I-\Lambda)\rho_{XB}\big) + (M-1)\mathrm{Tr}\big(\Lambda\rho_X\otimes\rho_B\big)$$

By definition, there exists $0\le F\le I$ s.t.

$$\mathrm{Tr}(F\rho_{XB}) \ge 1-\varepsilon+\delta \;\;\&\;\; \mathrm{Tr}(F\rho_X\otimes\rho_B) \le 2^{-D_H^{\varepsilon-\delta}(\rho_{XB}\|\rho_X\otimes\rho_B)}$$

As a result

$$\mathbb{E}\{p_{err}(E,D)\} \leq \varepsilon - \delta + (M-1)\, 2^{-D_H^{\varepsilon-\delta}(\rho_{XB} \| \rho_X \otimes \rho_B)}$$

Using condition on M in (20)

$$\mathbb{E}\{p_{err}(E,D)\} \leq \varepsilon$$

$\Rightarrow$ There exists (E,D) s.t. $p_{err}(E,D) \leq \varepsilon$ ∎

This characterization of $\log M^{opt}(W, \varepsilon)$ is very general.

Important special case where we can evaluate the expression more explicitly: Memoryless channel $W^{\otimes n}$.

<u>Def</u>: Let W be a cq channel.

The classical capacity $C(W)$ of W is defined by

$$C(W) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log M^{opt}(W^{\otimes n}, \varepsilon)}{n}$$

optimal rate for transmitting information.

<u>Corollary</u>: For any cq channel W

$$\sup_{P_X} I(X:B)_{\rho_{XB}} \leq C(W) \leq \sup_n \frac{1}{n} \sup_{P_{X_1 \dots X_n}} I(X_1 \dots X_n : B_1 \dots B_n)_{\rho_{X_1 \dots X_n B_1 \dots B_n}}$$

where $$\rho_{X_1 \dots X_n B_1 \dots B_n} = \sum_{x_1 \dots x_n} P_{X_1 \dots X_n}(x_1 \dots x_n) W_{x_1} \otimes W_{x_2} \otimes \dots \otimes W_{x_n}$$

Notation: $X^n := X_1 \dots X_n \qquad B^n = B_1 \dots B_n$

<u>Proof</u>:

Apply one-shot theorem with $W^{\otimes n}$

$$\sup_{P_{X^n}} D_H^{\varepsilon/2}(\rho_{X^n B^n} \| \rho_{X^n} \otimes \rho_{B^n}) - \log\left(\frac{2}{\varepsilon}\right) - 1 \leq \log M^{opt}(W^{\otimes n}, \varepsilon) \leq \sup_{P_{X^n}} D_H^{\varepsilon}(\rho_{X^n B^n} \| \rho_{X^n} \otimes \rho_{B^n})$$

Multiply by $\frac{1}{n}$ and take limits $n \to \infty$ then $\varepsilon \to 0$: $$C(W) = \lim_{\substack{\varepsilon \to 0 \\ n \to \infty}} \frac{1}{n} \sup_{P_{X^n}} D_H^{\varepsilon}(\rho_{X^n B^n} \| \rho_{X^n} \otimes \rho_{B^n})$$

We should evaluate $\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \sup_{P_{X^n}} D_H^{\varepsilon}(\rho_{X^nB^n} \| \rho_{X^n} \otimes \rho_{B^n}) =: \alpha$

- $\alpha \geq \sup_{P_X} I(X:B)_{\rho_{XB}}$ Let P_X achieve the sup.

Choose $P_{X^n} = P_X \otimes P_X \cdots \otimes P_X$ ($X_1 \ldots X_n$ independent distribution P_X).

$$\alpha \geq \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{D_H^{\varepsilon}(\rho_{XB}^{\otimes n} \| \rho_X^{\otimes n} \otimes \rho_B^{\otimes n})}{n}$$

Stein lemma:

$$= D(\rho_{XB} \| \rho_X \otimes \rho_B)$$
$$= I(X:B)_{\rho_{XB}}.$$

- $\alpha \leq \sup_n \sup_{P_{X^n}} \frac{1}{n} I(X^n : B^n)_{\rho_{X^nB^n}}$

In the converse part of Stein's lemma, we showed

$$D_H^{\varepsilon}(\rho \| \sigma) \leq \frac{D(\rho\|\sigma) + 1}{1-\varepsilon}$$

$$\alpha \leq \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \sup_{P_{X^n}} \frac{D(\rho_{X^nB^n} \| \rho_{X^n} \otimes \rho_{B^n}) + 1}{1-\varepsilon}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sup_{P_{X^n}} I(X^n : B^n)_{\rho_{X^nB^n}}$$

$$= \sup_n \frac{1}{n} \sup_{P_{X^n}} I(X^n : B^n)_{\rho_{X^nB^n}}$$

$f(n) := \sup_{P_{X^n}} I(X^n : B^n)$ is superadditive (ex) $f(n+m) \geq f(n) + f(m)$. + Fekete lemma.

Rk: Actually easy to see $C(W) = \sup_n \frac{1}{n} \sup_{P_{X^n}} I(X^n : B^n)_{\rho_{X^nB^n}}$.

How to compute $\underbrace{\sup_n \frac{1}{n} \sup_{P_{X^n}} I(X^n : B^n)_{\rho_{X^n B^n}}}_{= \sup_{P_X} I(X:B)}$?

For $\underline{\text{cq channels}}$ $\longrightarrow$ $= \sup_{P_X} I(X:B)$, ie $f(n)$ additive

$\underline{\text{Lemma:}}$ For any n

$$\frac{1}{n} \sup_{P_{X^n}} I(X^n : B^n) = \sup_{P_X} I(X:B)$$

$\underline{\text{Proof}}$: • $\geq$ simple (always true, not only cq channels)

• $\leq$ Let P_{X^n} be arbitrary $\quad \rho_{X^n B^n} = \sum_{x^n} P_{X^n}(x^n) |x^n\rangle\langle x^n| \otimes W_{x^n}$

$$I(X^n : B^n) = H(B^n) - H(B^n | X^n)$$

* $H(B^n) \leq \sum_{i=1}^n H(B_i)$ (subadditivity).

* $H(B^n | X^n) = \sum_{x_1 \ldots x_n} P_{X^n}(x_1 \ldots x_n) H(B^n)_{W_{x_1} \otimes W_{x_2} \otimes \ldots \otimes W_{x_n}}$

Property of von Neumann entropy:

conditional entropy = average of entropy of conditional state.

$$= \sum_{x_1 \ldots x_n} P_{X^n}(x_1 \ldots x_n) \underbrace{\left(H(B)_{W_{x_1}} + H(B)_{W_{x_2}} + \cdots + H(B)_{W_{x_n}} \right)}_{\text{entropy of a product state is sum of entropies.}}$$

$$= \sum_{i=1}^n \sum_{x_i} P_{X_i}(x_i) H(B)_{W_{x_i}}$$

$$= \sum_{i=1}^n H(B_i | X_i)$$

So $I(X^n : B^n) \leq \sum_{i=1}^{n} H(B_i) - H(B_i | X_i)$

$$= \sum_{i=1}^{n} I(X_i : B_i)$$

$$\leq n \cdot \sup_{P_X} I(X:B)_{\rho_{XB}} \quad \blacksquare.$$

<u>Th</u> (Shannon theorem for cq channels)

The capacity of a cq channel is given by

$$C(W) = \sup_{P_X} I(X:B)_{\rho_{XB}}$$

$$\rho_{XB} = \sum_x P_X(x) |x\rangle\langle x| \otimes W_x$$

Rk: If $W_x = W$ for all $x \Rightarrow C(W) = 0$

Surprisingly, converse also true

$$C(W) = 0 \iff W_x = W \ \forall x$$

This is surprising, just using "repetition" will not work.

2. General quantum channels.

Now $\mathcal{W}: L(A) \to L(B)$ quantum channel.

Very similar definitions:

Def: An M-code (E,D) for $\mathcal{W}$ is given by

- $E: [M] \to S(A)$ encoding function
- Decoding is a POVM $\{D_s\}_{s\in[M]}$ on B.

Def: The error probability $p_{err}(E,D)$ of an M-code for $\mathcal{W}$ is defined by

$$p_{err}(E,D) = 1 - \underbrace{\frac{1}{M}\sum_{s\in[M]} \mathrm{Tr}\left(D_s\, \mathcal{W}(E(s))\right)}_{\text{probability of successful decoding.}}$$

If $p_{err}(E,D) \leq \varepsilon$, we say that (E,D) is an (M,ε)-code

Looking back at the proofs for cq channels, we see that it suffices to optimize over choice of $\{\sigma_A^x \in S(A)\}_{x\in\mathcal{X}}$ and consider the corresponding cq channel $W_x = \mathcal{W}(\sigma_A^x)$

We then define (as before) for $\{P_X(x), \sigma_A^x\}_{x\in\mathcal{X}}$

This is called an ensemble

$$\rho_{XB} = \sum_{x\in\mathcal{X}} P_X(x)\, |x\rangle\langle x| \otimes \mathcal{W}(\sigma_A^x)$$

Th: Any (M,ε)-code for $\mathcal{W}$ satisfies

$$\log M \leq \sup_{\sigma_A^x} \sup_{P_X} D_H^{\varepsilon}(\rho_{XB} \| \rho_X \otimes \rho_B)$$

and there exists and (M,ε)-code for $\mathcal{W}$

$$\log M \geq \sup_{\sigma_A^x} \sup_{P_X} D_H^{\varepsilon-\delta}(\rho_{XB} \| \rho_X \otimes \rho_B) - \log(1/\delta)$$

Basically the same proof. (good exercice to redo it yourself)

Rk: we take supremum over arbitrarily large X but in many cases can bound it.

Important special case: $\mathcal{W}^{\otimes n}$.

Def: The classical capacity $C(\mathcal{W})$ of a quantum channel $\mathcal{W}$ is defined as:

same def as for cq channels

$$C(\mathcal{W}) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log M^{opt}(\mathcal{W}^{\otimes n}, \varepsilon)}{n}$$

Same as before: using Stein lemma $\frac{1}{n} D_H^{\varepsilon} \to D$.

Notation: $\chi(\mathcal{W}) := \sup_{\{\sigma_A^x, P_X(x)\}} \underbrace{D(\rho_{XB} \| \rho_X \otimes \rho_B)}_{I(X:B)_{\rho_{XB}}}$

where $\rho_{XB} = \sum_x P_X(x) |x\rangle\langle x| \otimes \mathcal{W}(\sigma_A^x)$

Rk: * $\chi(\mathcal{W})$ is called the Holevo information of $\mathcal{W}$. See [Wilde, Ch 13] or [Watrous, Ch 8] for properties.

* The Holevo information of an ensemble $\{P_X(x), \sigma_A^x\}$ also commonly denoted $\underbrace{I(X:A)_{\rho_{XA}}}_{\overset{\shortparallel}{\chi(\{P_X(x),\sigma_A^x\})}}$ for $\rho_{XA} = \sum_x P_X(x) |x\rangle\langle x| \otimes \sigma_A^x$

With this notation, for a cq channel W: $C(W) = \sup_{P_X} \chi(\{P_X(x), W_x\})$

Th (Holevo-Schumacher-Westmorland, HSW)

Let $\mathcal{W}$ be a quantum channel

$$C(\mathcal{W}) = \lim_{n\to\infty} \frac{1}{n} \chi(\mathcal{W}^{\otimes n}) = \sup_n \frac{1}{n} \chi(\mathcal{W}^{\otimes n})$$

Proof is the same as what we did in cq case.

Question: Is χ additive under tensor product?

i.e. $\chi(\mathcal{W}^{\otimes n}) \overset{?}{=} n\,\chi(\mathcal{W})$

Note that $\chi(\mathcal{W}^{\otimes n}) \geq n\,\chi(\mathcal{W})$ is simple, follows from the fact that $D(\rho\otimes\rho \| \sigma\otimes\sigma) = 2D(\rho\|\sigma)$.

Answer: NO in general, i.e., there exists channels $\mathcal{W}$ s.t. $\chi(\mathcal{W}^{\otimes 2}) > 2\,\chi(\mathcal{W})$.

This means that optimal choices of state $\sigma_{A_1A_2}^x$ will be entangled

Construction in [Hastings, 2009] by choosing $\mathcal{W}$ "random" and a very involved analysis. [See book Alice & Bob meet Banach]

- But there are families of channel for which additivity can be proved.

<u>Research question</u>:

Consider the amplitude damping channel $\mathcal{A}_\gamma$ (parameter $\gamma \in [0,1)$)

$$\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \xrightarrow{\mathcal{A}_\gamma} \begin{pmatrix} \rho_{00}+\gamma\rho_{11} & \sqrt{1-\gamma}\,\rho_{01} \\ \sqrt{1-\gamma}\,\rho_{10} & (1-\gamma)\rho_{11} \end{pmatrix}$$

$C(\mathcal{A}_\gamma)$ unknown.

[Simplistic model for decay of 2-level atom due to spontaneous emission of photon]