EXERCISES AND SUPPLEMENTAL TOPICS FOR LECTURE 7

Let F be a field and let A be an abelian group. A *symbol* on F with values in A is a function

$$\phi : F^\times \times F^\times \to A$$

which is bilinear and such that $\phi(a, 1-a) = 0$. Using the identity

$$-a = \frac{1-a}{1-1/a},$$

one deduces as well that $\phi(a, -a) = 0$. Consequently, with a little work one checks that ϕ is *antisymmetric*, i.e., $\phi(a,b) = -\phi(b,a)$.

For the purpose of quadratic forms, one is most interested in symbols that take values in $\mathbb{F}_2$-vector spaces, in which case (i.e., $2A = 0$) such a ϕ defines a *symmetric* bilinear form $F^\times/F^{\times 2} \times F^\times/F^{\times 2} \to A$.

(1) For $p > 2$, the formula for the p-adic Hilbert symbol is as follows. If $x, y \in \mathbb{Q}_p^\times$ and $x = p^a u, y = p^b v$ for $a, b \in \mathbb{Z}$ and $u, v \in \mathbb{Z}_p^\times$, then

$$(x,y)_{\mathbb{Q}_p} = (-1)^{ab\epsilon(p)} \left(\frac{u}{p}\right)^b \left(\frac{v}{p}\right)^a$$

where $\left(\frac{\cdot}{p}\right)$ denotes the quadratic (Legendre symbol), i.e., ± 1 according to whether the input is a square in $\mathbb{F}_p^\times$, and $\epsilon(p) = \frac{p-1}{2}$. Check this by reducing to a handful of basic cases and using the results on quadratic forms over $\mathbb{Q}_p$ proved earlier.

(2) Let $p > 2$. We define a function

$$\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \to \mathbb{F}_p^\times$$

by sending

$$(a,b) \mapsto (-1)^{v_p(a)v_p(b)} \overline{\frac{b^{v_p(a)}}{a^{v_p(b)}}}$$

where v_p denotes the p-adic valuation and $\bar{\cdot}$ refers to reduction modulo p (for a p-adic unit). Show that this function is a symbol (called the *tame symbol*). In fact, when one composes with the quadratic symbol $\mathbb{F}_p^\times \to \{\pm 1\}$, the composite is just the p-adic Hilbert symbol (this follows from the formula given in the lecture and above).

(3) The Hilbert symbol for $\mathbb{Q}_2$ is as follows. Given $x = 2^a u, y = 2^b v$ with $u, v \in \mathbb{Z}_2^\times$, then

$$(x,y)_{\mathbb{Q}_2} = (-1)^{\epsilon(u)\epsilon(v) + a\omega(v) + b\omega(u)}.$$

Here $\epsilon, \omega : \mathbb{Z}_2^\times \to \mathbb{Z}/2$ are defined by $\epsilon(z) = \frac{z-1}{2}, \omega(z) = \frac{z^2-1}{8}$ (and reducing mod 2). Check that these are both homomorphisms.

In particular:

- We have $(u,v)_{\mathbb{Q}_2} = (-1)^{\epsilon(u)\epsilon(v)}$.
- We have $(2,v)_{\mathbb{Q}_2} = (-1)^{\omega(v)}$.

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(4) (Non-dyadic fields). Let $(E, |\cdot|)$ be any nonarchimedean local field. The *residue characteristic* of E is the unique prime number p such that $|p| < 1$. If E actually has characteristic p (so that E is isomorphic to $\mathbb{F}_q((t))$), then E has residue characteristic p. But $\mathbb{Q}_p$ or any finite extension has characteristic zero but residue characteristic p. Given such an E, we let $\mathcal{O}_E = \{x \in E : |x| \leq 1\}$ be the unit disk (which is a subring of E),

A nonarchimedean local field E is said to be *non-dyadic* if the residue characteristic is $\neq 2$ (so $|p| = 1$). Much of the treatment of $\mathbb{Q}_p$ for $p > 2$ can be generalized to any non-dyadic local field; for instance, for such E we have $E^\times/E^{\times 2}$ is isomorphic to $\mathbb{F}_2^2$, and the Hilbert symbol can be computed explicitly.