EXERCISES AND SUPPLEMENTAL TOPICS FOR LECTURE 4

1. Exercises on absolute values

(1) Let $|\,|\cdot|$ be a nonarchimedean absolute value on the field K. Prove that if $x, y \in K$ and $|x| < |y|$, then $|x+y| = |y|$.
(2) In a complete nonarchimedean field, an infinite sum $\sum_{i=0}^{\infty} x_i$ converges if and only if $x_i \to 0$ as $i \to \infty$.
(3) We say that two absolute values on a field K, say $|\cdot|_1$ and $|\cdot|_2$, are *equivalent* if they define the same topology on K. Prove that if $|\cdot|_1$ and $|\cdot|_2$ are equivalent, then there exists $\alpha > 0$ such that $|\cdot|_1 = |\cdot|_2^{\alpha}$. Start by showing that $|x|_1 < 1$ if and only if $|x|_2 < 1$, for any element $x \in K$.
(4) Let $(K, |\cdot|)$ be a field with an absolute value. Suppose that for each $n \in \mathbb{Z}$, $|n| \leq 1$. Then $|\cdot|$ is nonarchimedean. Expand out $|(x+y)^r|$ using the binomial theorem and let $r \to \infty$.
(5) Let K be a field of positive characteristic. Prove that any (non-trivial) absolute value on K is nonarchimedean (use the previous exercise).
(6) Prove Ostrowski's theorem: any absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent either to the archimedean absolute value $|\cdot|_{\infty}$ or to the p-adic absolute value for some p.
    - First consider the case where $|\cdot|$ is ≤ 1 on $\mathbb{Z}$. In this case, we have seen that the absolute value is necessarily nonarchimedean. Argue directly in this case (how do we find the right prime p?).
    - Suppose there exists a positive integer n with $|n| > 1$. Fix one such n. For each $m \in \mathbb{Z}$, show that $|m| \leq C\,|n|^{\log_n(m)}$ for some constant C independent of m. Deduce that $|m| \leq |n|^{\log_n(m)}$ by taking powers appropriately, and finally conclude.

(7) Let K be any field and consider the field $K(t)$ of rational functions in one variable t. Define the "t-adic valuation" on $K(t)$ and show that the completion is the field of Laurent series $K((t))$.
(8) There is an analog of Ostrowski's classification for absolute values on $\mathbb{F}_p(t)$. The polynomial ring $\mathbb{F}_p[t]$ is a euclidean domain. For each prime element $f(t) \in \mathbb{F}_p[t]$, we have the f-adic valuation on $\mathbb{F}_p(t)$ (defined analogously) and hence the f-adic absolute value. In addition, we have the ("∞-adic") valuation (and induced absolute value) which carries a quotient $f(t)/g(t)$ to $\deg(g) - \deg(f)$. Every absolute value on $\mathbb{F}_p(t)$ is of the above form (up to equivalence). Similarly, one can form the completions of the field $\mathbb{F}_p(t)$ with respect to these absolute values. (In fact, they are all of the form $\mathbb{F}_q((u))$, i.e., Laurent series fields.) Many of the results discussed in this minicourse (such as the Hasse–Minkowski theorem) have analog for *global function fields* – that is, finite extensions of $\mathbb{F}_p(t)$.

2. Exercises on $\mathbb{Q}_p$

(1) As a topological space, $\mathbb{Z}_p$ is compact. In fact, it is homeomorphic to the Cantor set in $\mathbb{R}$.

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(2) Let $m \in \mathbb{Q}_p$ be such that $|m-1|_p < 1$. Then $|m^p - 1|_p < |m-1|_p$. (Informally, raising to the pth power is contracting towards 1 on the open unit disk around 1.) In particular, the sequence $m, m^p, m^{p^2}, \dots$ converges to 1.

(3) Let $x \in \mathbb{Q}_p$ be such that $|x|_p = 1$. Then the sequence $x^{p-1}, x^{p(p-1)}, x^{p^2(p-1)}, \dots$ converges to 1 in $\mathbb{Q}_p$.

(4) Show that $\mathbb{Q}_p$ contains a primitive $(p-1)$st root of unity. (One way to think about this: consider the pth power map $x \mapsto x^p$ on the unit disk of radius 1 at zero. The map preserves each subdisk of radius $1/p$ (note that there are p of them) and on each subdisk the map is contracting and therefore has a unique fixed point by the contraction mapping principle.

(5) Suppose $p > 2$. Consider again the open unit disk $\{m \in \mathbb{Q}_p : |m-1|_p < 1\}$. Show that this is a group under multiplication. Show that the there is a continuous isomorphism between this group and $p\mathbb{Z}_p$ (under addition), given by the exponential and logarithm maps. In particular, show that the exponential function $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for $x \in \mathbb{Q}_p$ with $|x-1|_p < 1$. Note that this convergence needs to be checked because the denominators $\frac{1}{n!}$ do not converge to zero p-adically! It will be helpful to use the general fact (why is this true?)

$$\mathrm{ord}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots.$$

(6) Let S be a collection of primes, with ∞ allowed as well. Suppose given $x_i \in \mathbb{Q}$ for each $i \in S$. Given $\epsilon > 0$, show that there exists $x \in \mathbb{Q}$ with $|x - x_i| \le \epsilon$ for each $i \in S$. In other words, $\mathbb{Q}$ is dense in any finite product of its completions. (This is in fact a general statement about absolute values.)

(7) Consider the field $\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$. Let p be a prime such that $p \equiv 3 \bmod 4$. Show that the function

$$v : \mathbb{Q}(i)^\times \to \mathbb{Z}, \quad v(a+bi) = \mathrm{ord}_p(a^2+b^2)$$

defines a valuation on $\mathbb{Q}(i)$. Why is the condition $p \equiv 3 \bmod 4$ necessary? What is the completion of $\mathbb{Q}(i)$ with respect to the induced absolute value?