EXERCISES AND SUPPLEMENTAL TOPICS FOR LECTURE 3

1. Group completions of abelian monoids

Let M be a commutative monoid (i.e., M is equipped with a commutative, associative, and unital addition law – for instance $M = \mathbb{Z}_{\geq 0}$ is an example). The *group completion* of M is the abelian group M^{gp} generated by symbols $x_m \in M$ and relations $x_{m+m'} = x_m + x_{m'}$. By construction, there is a homomorphism of abelian monoids $M \to M^{\mathrm{gp}}$ sending $m \mapsto x_m$. Conversely, any homomorphism $M \to N$ for N an abelian group extends over M^{gp} (the "universal property" of M^{gp}).

- When $M = \mathbb{Z}_{\geq 0}$, then $M^{\mathrm{gp}} = \mathbb{Z}$. In general, we can think of M^{gp} as sort of analogous to the construction of the fraction field of an integral domain.
- Alternatively, we can think of elements of M^{gp} as represented by pairs $(m_1, m_2) \in M^2$ (corresponding to the "formal difference" $m_1 - m_2$). Two pairs (m_1, m_2) and (m_1', m_2') are equivalent if there exists a third element n such that $m_1 + m_2' + n = m_1' + m_2 + n$.
- We say that a monoid M is *cancellative* if whenever $m + n = m' + n$, then $m = m'$. If M is cancellative, check that $M \to M^{\mathrm{gp}}$ is injective. What is an example of a non-cancellative monoid when this fails?

The construction of the Grothendieck–Witt ring is obtained as follows. Let M be the collection of isomorphism classes of quadratic forms over a field F. Then M is an abelian monoid with respect to direct sum. The group completion of M is the Grothendieck–Witt ring. (The fact that M has a multiplication coming from tensor product – it is a *commutative semiring* – gives M^{gp} the additional ring structure.) Note that the Witt cancellation theorem gives that M is cancellative.

2. General exercises on Witt rings

(1) What is the Witt ring of a finite field? How many isomorphism classes of anisotropic forms are there?

(2) Let $F \subset E$ be a finite extension, and let $\ell : E \to F$ be any nonzero F-linear map. As shown in the exercises on the first day, given a quadratic form (V, q) over E, we obtain a quadratic form $(V, \ell \circ q)$ over F. Show that this induces a map of Grothendieck–Witt groups (not rings) $GW(E) \to GW(F)$ and Witt groups $W(E) \to W(F)$.

(3) If $F \subset E$ is any field extension, then check that there is a homomorphism of Witt rings $W(F) \to W(E)$ given by extension of scalars of quadratic forms. If $E = F(\sqrt{d})$, then show that the kernel of $W(F) \to W(E)$ is the ideal generated by $\langle 1, -d \rangle$.

(4) Let $F \subset E$ be a cubic extension. We will show that a quadratic form over F, $(V, q) = \langle a_1, \ldots, a_n \rangle$, whose base change to E is isotropic must be isotropic itself. In fact, let $\alpha \in E$ be a primitive element with minimal polynomial $p(x) \in F[x]$. Our assumption is that we can find polynomials $f_i(x) \in F[x]$ of degree ≤ 2 such that $m(x) \mid \sum_i a_i f_i(x)^2$. Show that the right-hand-side has necessarily a root in F and that (V, q) is therefore isotropic over F. (A slight extension of this argument works for any odd degree field extension, by

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using induction on the odd degree; this is a theorem of Springer.) Deduce that the map $W(F) \to W(E)$ is injective in this case.

3. Exercises on the Witt ring and orderings

For more on the following topics, see the book by Scharlau, *Quadratic and Hermitian Forms.*

(1) (*The following is also repeated earlier.*) Let $d \in F^\times$. Then the kernel of $W(F) \to W(F(\sqrt{d}))$ is the ideal of $W(F)$ generated by $\langle 1, -d\rangle$ or equivalently $\langle 1\rangle - \langle d\rangle$. (If (V, q) is anisotropic over F and $(V \otimes_F F(\sqrt{d}), q_{F(\sqrt{d})})$ is hyperbolic, then show that (V, q) is a direct sum of copies of forms $\langle a, -da\rangle_{a \in F^\times}$.)

(2) All torsion in the Witt ring $W(F)$ is 2-power torsion (due to Pfister, Scharlau). Prove this as follows:
  (a) Let $\alpha, \beta \in F^\times$ such that $\alpha^2 + \beta^2 \neq 0$. Then $\langle 1, -(\alpha^2 + \beta^2)\rangle \in W(F)$ is 2-torsion. (Check this directly; in fact twice this form is hyperbolic.)
  (b) Let $\alpha, \beta \in F^\times$ such that $\alpha^2 + \beta^2 \neq 0$. Then the kernel of $W(F) \to W(F(\sqrt{\alpha^2 + \beta^2}))$ consists of 2-torsion. (Combine with Exercise 1.)
  (c) Let F be a field such that any sum of squares is a square (such fields are called *pythagorean*). Then either F is quadratically closed (every element is a square) and $W(F) = \mathbb{Z}/2$ or $W(F)$ is torsion-free (if F is pythagorean but not quadratically closed, i.e., -1 is not a sum of squares, show that if (V, q) is anisotropic, then $(V^{\oplus n}, q^{\oplus n})$ is anisotropic for any $n > 0$).
  (d) Combine a, b, and c to deduce the claim.

(3) The Witt ring $W(F)$ is torsion if and only if -1 is a sum of squares in F. (Use the previous exercise and its various steps.)

(4) The signature map gives an isomorphism $W(\mathbb{R}) \simeq \mathbb{Z}$. More generally, given any ordered field F,[1] show that we obtain a ring homomorphism $\sigma : W(F) \to \mathbb{Z}$ given by taking the "generalized signature" of a quadratic form.
  (a) Suppose given any ring map $f : W(F) \to \mathbb{Z}$. Define a subset $P \subset F^\times$ (of "positive elements") as follows: say that $\alpha \in F^\times$ belongs to P if $f(\langle\alpha\rangle) = 1$. Note in fact that for any $\alpha \in F^\times$, we have $f(\langle\alpha\rangle) = \pm 1$.
  (b) Suppose given the situation as in part (a). Show that P is closed under sums and products. Use that for $\alpha, \beta \in F^\times$, one has $\langle\alpha, \beta\rangle \simeq \left\langle \alpha + \beta, \frac{\alpha\beta}{\alpha+\beta}\right\rangle$. Moreover, $F^\times = P \cup (-P)$ and the union is disjoint. Therefore, show that conversely, the homomorphism f defines an ordering.
  Thus, the following are in correspondence: orderings of F, and ring homomorphisms $W(F) \to \mathbb{Z}$.

(5) (Pfister's local-to-global principle). Given an element $x \in W(F)$, x is torsion (necessarily 2-power torsion) if and only if its signature relative to each ordering to F vanishes, or equivalently (by the previous exercise) if x is annihilated by all homomorphisms $W(F) \to \mathbb{Z}$. Prove this as follows:
  (a) Let $x \in W(F)$ be annihilated by all homomorphisms $W(F) \to \mathbb{Z}$ (which we have seen come from orderings). We want to show that x is torsion. Without loss of generality, we can assume by Zorn's lemma that for all proper extensions $F \subsetneq F'$, x maps to a

[1] An *ordering* on a field F can be specified by a collection of *positive elements* $P \subset F$ such that P is closed under sums and products, and $F = (-P) \cup \{0\} \cup P$ with the union disjoint. This defines the order relation $<$ on F.

torsion element in $W(F')$. (That is, we assume our counterexample to the statement is maximal.)

(b) Show that for each $d \in F^\times \setminus F^{\times 2}$, $2^n x$ is divisible by $\langle 1, -d \rangle$ in $W(F)$ for $n \gg 0$. (Use exercise 1.) Conclude that $2^n x = 2^n x \langle -d \rangle$ in $W(F)$.

(c) Suppose that an element $d \in F^\times$ has the property that neither d nor $-d$ is a square. Then show that $2^n x \langle d \rangle = 2^n x \langle -d \rangle$ in $W(F)$. Since for any $a \in F^\times$, we have $\langle a \rangle = -\langle -a \rangle$ in $W(F)$, conclude that $2^{n+1} x = 0$ in $W(F)$. Since x was assumed non-torsion in $W(F)$, we obtain a contradiction. Conclude that in fact $F^\times$ is the union of the squares and the negatives of the squares. This union is necessarily disjoint, or -1 would be a square and $W(F)$ would be torsion.

(d) Conclude that the squares give an ordering on F and in fact that $W(F) = \mathbb{Z}$, completing the proof of Pfister's local-to-global theorem.

(e) As an example, we recover the following classical fact: F admits an ordering if and only if -1 is not a sum of squares in F.

(6) Let $\gamma \in F^\times$. Show that γ is a sum of squares if and only if $\langle 1, -\gamma \rangle$ is a torsion element of the Witt ring if and only if γ is positive with respect to any ordering of F.