IAS 2023 Summer Collaborators Final Report

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1 Acknowledgements

We are immensely grateful for support provided to us by the IAS Summer Collaborators Program. The time spent together in a common location allowed us to make significant progress on our projects, solidify our working relationship that we intend to maintain throughout our careers, and network with local mathematicians in ways that would not have been feasible through remote collaboration.

2 Summary of Progress

We accomplished our two primary goals during our time at the IAS. First, we fine-tuned a result within the scheme of the Maximum Distance Problem that we began during the 2022-2023 academic year; this project is now in preprint form. Second, we chose a problem and established foundational results in a new direction that will we work on together during the 2023-2024 academic year. Below, we summarize our Maximum Distance Problem results.

2.1 Maximum Distance Problems for Hölder Curves

Roughly speaking, the Maximum Distance Problem is the following:

Given some compact subset $E \subset \mathbb{R}^n$ and some neighborhood radius $r > 0$, find a curve of least length whose closed r-neighborhood covers E.

To formulate this problem precisely, we must define what we mean by *curve* and fix a notion of *length*. Traditionally, in the study of the Maximum Distance Problem "curves" has meant *finite continuum*, i.e., closed connected subset $\Gamma \subseteq \mathbb{R}^n$ with $\mathcal{H}^1(\Gamma) < \infty$, and length has been measured by 1-dimensional Hausdorff measure, $\mathcal{H}^1(\Gamma)$. With these choices, the *Maximum Distance Problem* may be stated as:

$$(\mathcal{H}^1, E, r) \begin{cases} \text{minimize } \mathcal{H}^1(\Gamma) \\ \text{among finite continua } \Gamma \subseteq \mathbb{R}^n \text{ such that } B(\Gamma, r) \supseteq E \end{cases} \tag{2.1}$$

where $B(A, r) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leq r\}$ denotes the *closed r-neighborhood* of any subset $A \subset \mathbb{R}^n$, and $\operatorname{dist}(x, A) := \inf_{y \in A} |x - y|$.

More than simply identifying solutions to the Maximum Distance Problem, there is interest in finding length minimizers of the Maximum Distance Problem and determining the regularity of such minimizers. To this end, we define an infimum:

$$\Lambda(\mathcal{H}^1, E, r) := \inf\{\mathcal{H}^1(\Gamma) : \Gamma \subseteq \mathbb{R}^n \text{ is a finite continuum and } B(\Gamma, r) \supseteq E\}.$$

Compactness arguments show that $(\mathcal{H}^1, E, r)$-minimizers exist [AKV21, Theorem 2.15], and we denote minimizers of (2.1) by Γ^* so that Γ^* is a finite continuum such that $B(\Gamma^*, r) \supseteq E$ and $\mathcal{H}^1(\Gamma^*) = \Lambda(\mathcal{H}^1, E, r)$. We call Γ^* an *$(\mathcal{H}^1, E, r)$-minimizer*, or an *r-maximum distance minimizer of E*.

For our project, we investigated the asymptotic behavior of r-maximum distance minimizers for various types of subsets $E \subset \mathbb{R}^n$. In [AKV21], the authors' motivation to study the maximum distance problem came from the *Analyst's Traveling Salesman Problem*, which asks for necessary and sufficient conditions for a set $E \subseteq \mathbb{R}^n$ to be covered by a finite continuum. We showed that whenever E can be covered by a finite continuum, that solutions to the Analyst's Traveling Salesman Problem can be obtained as the limit of r-maximum distance minimizers as $r \to 0+$. Formally, we state this result as follows:

Theorem 2.1. *Suppose E can be covered by a finite continuum. For any positive numbers $r_i \to 0$, there exists a sequence $\{\Gamma_i^*\}$ of $(\mathcal{H}^1, B(E, r_i), r_i)$-minimizers and a finite continuum Γ^* such that*

1. $\lim\limits_{i\to\infty} d_H(\Gamma_i^*, \Gamma^*) = 0$;
2. $E \subseteq \Gamma^*$;
3. $\lim\limits_{i\to\infty} \mathcal{H}^1(\Gamma_i^*) = \mathcal{H}^1(\Gamma^*) = \inf\{\mathcal{H}^1(\Gamma) : \Gamma$ *is a rectifiable curve and* $\Gamma \supseteq E\}$.

Note that any finite continuum has a Lipschitz parameterization whose 1-variation is comparable to its 1-dimensional Hausdorff measure. In particular, there is a constant $C = C(n)$ such that whenever $K \subseteq \mathbb{R}^n$ is a finite continuum, there exists an $L > 0$ and a Lipschitz map $\gamma : [0, L] \to \mathbb{R}^n$ such that $\widehat{\gamma} = K$, $\mathcal{H}^1(K) \leq L \leq C\mathcal{H}^1(K)$, and $|\gamma'(t)| = 1$ for $\mathcal{L}^1$-almost every $t \in [0, L]$ [DS93, Theorem 1.8]. Thus, we may state an equivalent version of problem (2.1) where we are minimizing over such Lipschitz maps. A common way of measuring the length of a Lipschitz map $\gamma : [a, b] \to \mathbb{R}^n$ is with its *1-variation*, defined by

$$\ell(\gamma) := \sup \sum_{k=1}^{N} |\gamma(t_k) - \gamma(t_{k-1})|,$$

where the supremum is taken over all *partitions* $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of $[a, b]$. We used the 1-variation as a way to measure the length of a Lipschitz curve for what we call the *maximum distance problem with Lipschitz parameterizations*:

$$(\ell, E, r) \begin{cases} \text{minimize } \ell(\gamma) \\ \text{among } \gamma \in \mathcal{C}^{1,1}([0,1]) \text{ such that } B(\widehat{\gamma}, r) \supseteq E. \end{cases} \tag{2.2}$$

We let $\Lambda(\ell, E, r)$ be the infimum of that problem:

$$\Lambda(\ell, E, r) := \inf\{\ell(\gamma) : \gamma \in \mathcal{C}^{1,1}([0,1]) \text{ such that } B(\widehat{\gamma}, r) \supseteq E\}.$$

We showed the existence of *(ℓ, E, r)-minimizers*. That is, we showed that there exists a Lipschitz curve $\gamma^* : [0,1] \to \mathbb{R}^n$ such that $B(\widehat{\gamma}^*, r) \supseteq E$ and $\ell(\gamma^*) = \Lambda(\ell, E, r)$. Although $(\mathcal{H}^1, E, r)$-minimizers and (ℓ, E, r)-minimizers may be different (even for fixed E and r), their minimum values, $\Lambda(\mathcal{H}^1, E, r)$ and $\Lambda(\ell, E, r)$ are comparable in the sense that there exists a constant $C = C(n)$ such that

$$\Lambda(\mathcal{H}^1, E, r) \leq \Lambda(\ell, E, r) \leq C\Lambda(\mathcal{H}^1, E, r).$$

The situation is more complicated whenever E cannot be covered by a finite continuum as is the case, for example, with the von Koch snowflake. One immediate complication is that we cannot use compactness arguments since the function $r \mapsto \Lambda(\mathcal{H}^1, B(E, r), r)$ is unbounded. In order to understand the asymptotic behaviour of minimizers for more general subsets, we studied such asymptotics in the context of Hölder curves.

In order to introduce the general techniques in a more familiar context, we began by first looking at r-maximum distance minimizers of r-neighborhoods of the standard $\frac{1}{3}$-von Koch snowflake, which has infinite $\mathcal{H}^1$ measure but postive and finite $\mathcal{H}^s$ measure when $s = \log_3(4)$. We were able to show that the growth of the function $r \mapsto \Lambda(\mathcal{H}^1, B(E, 1/r), 1/r)$ is comparable to a logarithmic exponentiation of $4/3$.

Theorem 2.2. *Let S be the $\frac{1}{3}$-von Koch snowflake. There exists a constant $C > 1$ such that*

$$\frac{1}{C}\left(\frac{4}{3}\right)^{k(r)} \leq \Lambda(\mathcal{H}^1, B(S, 1/r), 1/r) \leq C\left(\frac{4}{3}\right)^{k(r)}$$

for all $r > 1/3$ where $k(r) := \log_3(r)$.

In a sense, this result quantifies how $\mathcal{H}^1$ can approximate higher dimensional Hausdorff measures $\mathcal{H}^{\log_3(4)}$ in the context of the von Koch snowflake at scale r. To extend our result to more general Hölder curves, we begin with a definition. For $\alpha \in (0, 1]$, we say that a map $\gamma : [0, 1] \to \mathbb{R}^n$ is an *α-Hölder curve* with constant $1 \leq C < \infty$ if

$$|\gamma(x) - \gamma(y)| \leq C|x - y|^\alpha$$

for all $x, y \in [0, 1]$. We denote the class of α-Hölder curves by $\mathcal{C}^{0,\alpha}([0, 1])$. If in addition, there exists a $\beta \geq \alpha$ such that

$$\frac{1}{C}|x - y|^\beta \leq |\gamma(x) - \gamma(y)| \leq C|x - y|^\alpha$$

for all $x, y \in [0, 1]$, then we call γ a *weak (α, β)-bi-Hölder curve with constant C*, or simply an *(α, β)-bi-Hölder curve*. We were able to show that the maximum distance function $r \mapsto \Lambda(\mathcal{H}^1, B(\widehat{\gamma}, 1/r), 1/r)$ for neighborhoods of (α, β)-bi-Hölder curves is $O(2^{(1-\alpha)k_\alpha(r)})$ and $\Omega(2^{(1-\beta)k_\beta(r)})$.

Theorem 2.3. *Let $0 < \alpha \leq \beta$ and let $\gamma : [0, 1] \to \mathbb{R}^n$ be a weak (α, β)-bi-Hölder curve with constant $1 \leq C_\gamma < \infty$. Then there exists $C = C(\alpha, \beta, C_\gamma)$ such that*

$$\frac{1}{C}2^{(1-\beta)k_\beta(r)} \leq \Lambda(\mathcal{H}^1, B(\widehat{\gamma}, 1/r), 1/r) \leq C2^{(1-\alpha)k_\alpha(r)}$$

for all small enough $r = r(\beta) > 0$ where $k_\eta(r) := \log_{2^\eta}(r)$.

2.2 Future Direction

Going forward, we plan to continue to explore ways in which results for Lipschitz curves can be extended to Hölder curves as well as ways in which standard tools break down when regularity is decreased from Lipschitz to Hölder. We intend to continue our collaboration during the upcoming academic year, and we look forward to sharing our new problem statement and results soon.

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