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Memories of Sir Michael Atiyah

Nigel Hitchin, Editor

Sir Michael Atiyah, a hugely influential figure in mathematics, died on January 11, 2019. A tribute appeared in the November 2019 issue of the Notices, but what follows is a collection of more personal recollections—what it was like to be his student, to work alongside him, to have avenues of exploration pointed out, or to be inspired and energized by his unique personality. The contributions cover a full fifty years in which his interests moved from mathematics to incursions into physics. They are arranged in chronological order.

This is Part 2 of a two-part series on Sir Michael Atiyah. Part 1 was included in the November issue.

Michael Atiyah and Physics

Edward Witten

Michael Atiyah played a major role, starting in the mid-1970s, in redefining the relationship between mathematics and physics.

By that time, theoretical physics had reached a major turning point with the emergence of the Standard Model of particle physics, based on nonabelian gauge theory or Yang–Mills theory. In a sense, theory had caught up with experiment, though it took a while for that to be clear. The decades-long process that led to the Standard Model had been largely driven by experiment and by considerations of quantum theory that were rather far afield from the vantage point of most mathematicians. Conversely, during this period the ideas of modern mathematics seemed largely irrelevant to physicists grappling with elementary particles. A physics graduate student of the period, for example, would most probably never hear about a homology group, let alone something more contemporary like the Atiyah–Singer index theorem.

By the mid-1970s, the gauge theory revolution had created for physicists a new situation that would call for greater mathematical sophistication. But this was understood only gradually. Michael Atiyah and other mathematicians who became interested in what physicists were doing in quantum gauge theory played an important role in the process.

An early turning point came in 1976. A puzzle about the Standard Model known as the U(1) problem, identified by Murray Gell-Mann and Steve Weinberg, among others, was abruptly solved by Gerard 't Hooft by studying the Dirac equation in the field of a gauge theory instanton. Soon, Albert Schwarz showed that the key facts were best understood in the context of the Atiyah–Singer index theorem. Few physicists at the time knew what to make of this.

I first met Atiyah when he visited MIT in the spring of 1977, invited by Roman Jackiw. At the time, he was explaining his work with Richard Ward, solving the instanton equation on $\mathbb{R}^4$ by use of the Penrose twistor transform. His lectures had a big impact in the math and physics communities in the Cambridge (Massachusetts) area. Physicists at the time were very interested in solving the instanton equations because of speculation by Alexander Polyakov about the dynamics of gauge theories. However, the ingredients in the twistor transform of the instanton equation—complex manifolds, sheaf cohomology, fiber bundles—were quite unfamiliar to me and most other physicists.

By January 1978, when Atiyah invited me to visit Oxford for a few weeks, he was lecturing at the Maths Institute about a more precise understanding of instantons—the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction. I can well remember my perplexity in this period. Clearly, Atiyah and his colleagues were saying interesting and remarkable things about the nonlinear classical equations of nonabelian gauge theory. At the same time, it was very hard to imagine how their results could be applied to the questions of quantum dynamics that most interested physicists. Physical applications of the ADHM construction seemed far away.

Toward the end of my visit, Atiyah showed me two

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physics papers that I had not seen before. By this time, it was known that nonabelian gauge theories can describe magnetic monopoles as well as ordinary elementary particles. Ordinary particles arise by quantization of fields and are in representations of the gauge group G; monopoles are classified by topology. But Peter Goddard, Jean Nuyts, and David Olive (GNO) had suggested that magnetic charge can be interpreted as a weight of a "dual gauge group" $G^\vee$, and Olive with Claus Montonen had gone on to suggest a quantum duality between theories with gauge groups G and $G^\vee$. Atiyah told me that the GNO dual gauge group was the same as the dual group introduced by Robert Langlands in the Langlands program and that he thought there was something very deep here. He urged me to go to London to discuss the matter with Olive.

By the time I got to London, I was skeptical. Like other physicists of the time, I had never heard of the Langlands program and I had no idea what to make of Atiyah's observation that the Langlands and GNO dual groups were the same. But I could see that technically the Montonen–Olive proposal was not correct if taken literally. However, by the end of the day, Olive and I understood that the technical objections to Montonen–Olive duality are absent in the supersymmetric case, and we had formulated a number of ideas that eventually (in the mid-1990s) were important in understanding it more fully.

Atiyah's deeper idea—that the Langlands program should somehow be tied up with electric-magnetic duality in four-dimensional gauge theory—remained in limbo for much longer. A concrete understanding depended on many intervening developments in both math and physics and only emerged in the mid-2000s. The full scope of this relationship is probably still far out of sight.

I will mention just a few highlights of the following decade. At the 1979 Cargése summer school, Atiyah and Raoul Bott undertook to educate physicists about Morse theory. I and most (or all?) of the physicists there had certainly never been exposed to Morse theory before. Another highlight was a conference in Texas where Atiyah and Is Singer began to elucidate the topological meaning of what physicists know as perturbative anomalies in gauge theory. This helped introduce physicists to a deeper understanding of fermion path integrals. Two papers by Atiyah and Bott in these years were ultimately influential for physicists. Their 1983 paper "The Yang–Mills equations over Riemann surfaces" introduced ideas that were important later in understanding quantum gauge theories in two dimensions. Their 1984 paper "The moment map and equivariant cohomology" helped lead to the important technique of "localization" in supersymmetric quantum field theory. Starting in the mid-1980s, the emergence of string theory greatly widened the horizons of physicists and expanded the scope of interaction between physicists and mathematicians. Among many other things, this led to unexpected applications of the ADHM construction in physics. Unfortunately, to explain all that here would take us too far afield.

In 1987, Atiyah twice visited the Institute for Advanced Study, and he was more excited than I could remember. What he was excited about was Floer theory (of symplectic manifolds or of flat connections on a three-manifold), which he thought should be interpreted as the Hamiltonian formulation of a quantum field theory. This quantum field theory was supposed to be, in language that was introduced later, a topological quantum field theory, which would be related to Gromov invariants of a symplectic manifold or Donaldson invariants of a four-manifold. The idea of topological quantum field theory was mostly Atiyah's conception. Atiyah set for me the task of trying to interpret what he was saying in the language of physicists. At first, this was difficult, for a variety of technical reasons. For example, the fermionic symmetry used by Floer had spin 0, as opposed to the half-integral spin of spacetime supersymmetries as studied by physicists. But eventually I realized that a simple "twisting" of supersymmetric field theories could give a theory with the properties that Atiyah wanted. This gave, at a formal level, a reformulation of the Gromov and Donaldson invariants in a language that was natural to physicists.

The other problem that Atiyah recommended for physicists in the years 1987–8 was to understand the Jones polynomial of a knot via quantum field theory. I had never heard of the Jones polynomial before Atiyah recommended this problem, and this certainly put me in the majority among physicists. The challenge about the Jones polynomial that Atiyah posed was specifically to find a description of it with manifest topological invariance. By 1987–8, a number of constructions of the Jones polynomial were known, but topological invariance was never manifest a priori; it was always proved by checking generators and relations.

The following year brought many new clues about the Jones polynomial in work by, among others, Erik Verlinde, Greg Moore and Nathan Seiberg, and Akihiro Tsuchiya and Yukihiro Kanie. Eventually, at a meeting in Swansea, where I had the benefit of further discussions with Atiyah and with Graeme Segal, I had the good fortune to put some of the pieces together and interpret the Jones polynomial in terms of a three-dimensional gauge theory with the Chern–Simons function as its action.

This answered some of the questions, but actually Atiyah's vision about the Jones polynomial had two important aspects that were vindicated only long afterwards. First, Atiyah predicted that the argument q of the Jones polynomial should ultimately be understood as a parameter that counts instantons on a four-manifold. In 1988–9, this idea looked to me like a bridge too far, but something along these lines was actually understood in the last decade. Various things were needed first, including the invention of a refinement of the Jones polynomial, known as Khovanov homology, and a greatly enriched understanding by physicists of the consequences of electric-magnetic duality. Also, Atiyah advocated that a natural explanation of the three-dimensional invariance of the Jones polynomial should have an extension to explain the spectral parameter of integrable systems and the associated Yang–Baxter equation. Something along these lines was understood only in the last few years in the work of Kevin Costello.

Actually, one facet of Atiyah's vision is still unclear as of 2019. Atiyah was always extremely interested in the spectral parameter that appears in the twistor transform of instantons—as in the ADHM construction—and in the construction of monopoles—as explored in a book that he wrote with Nigel Hitchin. He often expressed a suspicion that the monopole spectral parameter should be related to the spectral parameter of integrable spin systems and integrable models of lattice statistical mechanics. As of this writing, there is hope that something along these lines will emerge in further developments from the work of Costello.

Going back to Donaldson theory, after formulating it in terms of a twisted version of supersymmetric gauge theory, I thought that this would lead to immediate progress, but that was not the case. In the period around 1990, Atiyah probably understood better than I did the following essential point: to contribute something new to Donaldson theory, physicists would need some sort of strong coupling methods, since anything that could be said for weak coupling would involve retracing the steps that mathematicians had already taken. Eventually, Seiberg and I were able to apply strong coupling methods to this problem, leading to a relationship between Donaldson theory and an abelian theory with monopoles (Seiberg–Witten theory).

Michael Atiyah worked with physicists on many occasions. I will describe the background to one of the papers he wrote with physicists, "An M-theory flop as a large N duality," written in the year 2000 with Juan Maldacena and Cumrun Vafa. Early in his career, Atiyah had explored the "small resolutions" of certain complex threefold singularities. By the mid-1990s, it was known that these small resolutions are important in the physics of Calabi–Yau manifolds, and it was also understood that nonperturbative "dualities" can relate string theory on a Calabi–Yau threefold to M-theory on a manifold of G_2 holonomy. Putting these two lines of thought together, Atiyah, Maldacena, and Vafa explored the significance in M-theory of isolated singularities of G_2 manifolds and their resolutions. This was new at the time but is now regarded as an important direction. In the spring of 2001, Atiyah spent several months at Caltech, where I was on sabbatical. We had a memorable collaboration on this topic, leading to our paper "M-theory dynamics on a manifold of G_2 holonomy."

Atiyah, along with colleagues such as Raoul Bott and Is Singer, played an enormous role in introducing new ideas and encouraging and teaching physicists to study quantum field theory from new points of view. It took many twists and turns for these lessons to be really learned and absorbed in the physics world. Atiyah always believed that the study of quantum field theory as a tool in geometry had to be integrated with the study of more "physical" aspects of quantum field theory. His vision and clairvoyance have had a truly far-reaching influence.

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