EXTREMAL GRAPH THEORY 4 - COUNTING RESULTS

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The celebrated conjectures of Erdős and Simonovits and Sidorenko suggest that for any bipartite H there is a $\gamma(H) > 0$ such that the number of copies of H in any graph G on N vertices with edge density $p > N^{-\gamma(H)}$ is asymptotically at least the same as in the N-vertex random graph with edge density p.

The original formulation of the conjecture by Sidorenko is in terms of graph homomorphisms. A homomorphism from a graph H to a graph G is a mapping $f : V(H) \to V(G)$ such that for each edge (u, v) of H, $(f(u), f(v))$ is an edge of G. Let $\mathrm{Hom}(H, G)$ denote the number of homomorphisms from H to G. We also consider the normalized function $t_H(G) = \mathrm{Hom}(H, G)/|G|^{|H|}$, which is the fraction of mappings $f : V(H) \to V(G)$ which are homomorphisms. Sidorenko's conjecture states that for every bipartite graph H with m edges and every graph G, $t_H(G) \geq t_{K_2}(G)^m$. In this lecture, we give two different proofs for one of the basic results on Sidorenko's conjecture.

Theorem 1 (Conlon–Fox–Sudakov)**.** *Sidorenko's conjecture holds for every bipartite graph H which has a vertex complete to the other part.*

Our first lemma is a counting variant of dependent random choice. For a vertex v in a graph G, the *neighborhood* $N(v)$ is the set of vertices adjacent to v. For a sequence S of vertices of a graph G, the *common neigbhorhood* $N(S)$ is the set of vertices adjacent to every vertex in S.

Lemma 1. *Let G be a graph with N vertices and $pN^2/2$ edges. Call a vertex v bad with respect to k if the number of sequences of k vertices in $N(v)$ with at most $(2n)^{-n-1}p^kN$ common neighbors is at least $\frac{1}{2n}|N(v)|^k$. Call v good if it is not bad with respect to k for all $1 \leq k \leq n$. Then the sum of the degrees of the good vertices is at least $pN^2/2$.*

Proof. We write $v \sim k$ to denote that v is bad with respect to k. Let X_k denote the number of pairs v, S with S a sequence of k vertices, v a vertex adjacent to every vertex in S, and $|N(S)| \leq (2n)^{-n-1}p^kN$. We have

$$(2n)^{-n-1}p^kN \cdot N^k \geq X_k \geq \sum_{v, v\sim k} \frac{1}{2n}|N(v)|^k \geq \frac{1}{2n}N(\sum_{v,v\sim k}|N(v)|/N)^k = \frac{1}{2n}N^{1-k}(\sum_{v,v\sim k}|N(v)|)^k.$$

The first inequality is by summing over S, the second inequality is by summing over vertices v which are bad with respect to k, and the third inequality is by convexity of the function $f(x) = x^k$. We therefore get

$$\sum_{v,v\sim k}|N(v)| \leq (2n)^{-n/k}pN^2 \leq \frac{1}{2n}pN^2.$$

Hence,

$$\sum_{v,v \text{ good}}|N(v)| \geq \sum_v |N(v)| - \sum_{k=1}^{n}\sum_{v,v\sim k}|N(v)| \geq pN^2 - n \cdot \frac{1}{2n}pN^2 = pN^2/2,$$

as required. □

Lemma 2. *Suppose $\mathcal{H}$ is a hypergraph with v vertices and at most e edges, and $\mathcal{G}$ is a hypergraph on N vertices with the property that for each k, $1 \le k \le v$, the number of sequences of k vertices of $\mathcal{G}$ that do not form an edge of $\mathcal{G}$ is at most $\frac{1}{2e}N^k$. Then the number of homomorphisms of $\mathcal{H}$ in $\mathcal{G}$ is at least $\frac{1}{2}N^v$.*

Proof. Consider a random mapping of the vertices of $\mathcal{H}$ to the vertices of $\mathcal{G}$. The probability that a given edge of $\mathcal{H}$ does not map to an edge of $\mathcal{G}$ is at most $\frac{1}{2e}$. By the union bound, the probability that there is an edge of $\mathcal{H}$ that does not map to an edge of $\mathcal{G}$ is at most $e \cdot \frac{1}{2e} = 1/2$. Hence, with probability at least $1/2$, a random mapping gives a homomorphism, so there are at least $\frac{1}{2}N^v$ homomorphisms from $\mathcal{H}$ to $\mathcal{G}$. □

Lemma 3. *Let $H = (V_1, V_2, E)$ be a bipartite graph with n vertices and m edges such that there is a vertex $u \in V_1$ which is adjacent to all vertices in V_2. Let G be a graph with N vertices and $pN^2/2$ edges, so $t_{K_2}(G) = p$. Then the number of homomorphisms from H to G is at least $(2n)^{-n^2}p^m N^n$.*

Proof. Let $n_i = |V_i|$ for $i \in \{1, 2\}$. We will give a lower bound on the number of homomorphisms $f : V(H) \to V(G)$ that map u to a good vertex v of G. Suppose we have already picked $f(u) = v$. Let $\mathcal{H}$ be the hypergraph with vertex set V_2, where $S \subset V_2$ is an edge of $\mathcal{H}$ if there is a vertex $w \in V_1 \setminus \{v\}$ such that $N(w) = S$. The number of vertices of $\mathcal{H}$ is n_2, which is at most n, and the number of edges of $\mathcal{H}$ is n_1, which is at most n. Let $\mathcal{G}$ be the hypergraph on $N(v)$, where a sequence R of k vertices of $N(v)$ is an edge of $\mathcal{G}$ if $|N(R)| \ge (2n)^{-n-1}p^k N$. Since v is good, for each k, $1 \le k \le v$, the number of sequences of k vertices of $\mathcal{G}$ that are not the vertices of an edge of $\mathcal{G}$ is at most $\frac{1}{2n}N^k$. Hence there are at least $\frac{1}{2}|N(V)|^{n_2}$ homomorphisms g from $\mathcal{H}$ to $\mathcal{G}$. Pick one such homomorphism g, and let $f(x) = g(x)$ for $x \in V_2$. By construction, once we have picked $f(u)$ and $f(V_2)$, there are at least $(2n)^{-n-1}p^{|N(w)|}N$ possible choices for $f(w)$ for each vertex $w \in V_1$. Hence, the number of homomorphisms from H to G is at least

$$\begin{aligned}
\sum_{v \text{ good}} \frac{1}{2}|N(v)|^{n_2} \prod_{w \in V_1 \setminus \{v\}} (2n)^{-n-1}p^{|N(w)|}N &= \frac{1}{2}(2n)^{-(n-1)(n_1-1)}p^{m-n_2}N^{n_1-1} \sum_{v \text{ good}} |N(v)|^{n_2} \\
&\ge \frac{1}{2}(2n)^{-(n-1)(n_1-1)}p^{m-n_2}N^{n_1-1}N \left(\sum_{v \text{ good}} |N(v)|/N \right)^{n_2} \\
&\ge \frac{1}{2}(2n)^{-(n-1)(n_1-1)}p^{m-n_2}N^{n_1}(pN/2)^{n_2} \\
&\ge (2n)^{-n^2}p^m N^n.
\end{aligned}$$

The first inequality is by convexity of the function $q(x) = x^k$. □

We next complete the proof of Theorem 1 by improving the bound in the previous lemma on the number of homomorphisms from H to G using the tensor power trick. The tensor product $G \times F$ of two graphs G and F has vertex set $V(G) \times V(F)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \times F$ if and only if u_i is adjacent with v_i for $i \in \{1, 2\}$. Let $G^1 = G$ and $G^r = G^{r-1} \times G$. Note that $t_H(G \times F) = t_H(G) \times t_H(F)$ for all H, G, F.

Proof of Theorem 1: Suppose for contradiction that there is a graph G such that $t_H(G) < t_{K_2}(G)^m$. Denote the number of edges of G as $pN^2/2$, so $t_{K_2}(G) = p$. Let $c = \frac{t_H(G)}{t_{K_2}(G)^m} < 1$. Let r be such that $c^r < (2n)^{-n^2}$. Then

$$t_H(G^r) = t_H(G)^r = c^r t_{K_2}(G)^{mr} = c^r t_{K_2}(G^r)^m < (2n)^{-n^2} t_{K_2}(G^r)^m.$$

However, this contradicts Lemma 3 applied to H and G^r. This completes the proof. □

We now give a second proof of this result, essentially due to Li and Szegedy, using entropy techniques. These ideas, originating in work of Kopparty and Rossman, have been used to considerable effect in exploring the conjecture.

Definition 1. *Let X be a discrete random variable taking values in a set S. The* entropy *of X is then*

$$H(X) = -\sum_{s \in S} p_s \log_2 p_s,$$

where $p_s = \mathbb{P}[X = s]$. One key property of entropy, that follows from the concavity of the log function, is

$$H(X) \le \log_2 |S|$$

with equality if and only if X is uniformly distributed on S.

Definition 2. *If X, Y are jointly distributed discrete random variables, then*

$$H(X, Y) = -\sum_{(x,y)} \mathbb{P}[X = x, Y = y] \log_2 \mathbb{P}[X = x, Y = y].$$

In particular, if X and Y are independent, then

$$H(X, Y) = H(X) + H(Y).$$

We also define the conditional entropy

$$H(X|Y) = \sum_{y} \mathbb{P}[Y = y] H(X|Y = y),$$

where

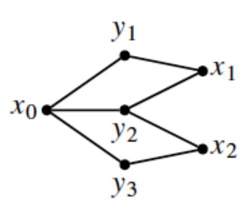
$$H(X|Y = y) = -\sum_{x} \mathbb{P}[X = x|Y = y] \log_2 \mathbb{P}[X = x|Y = y].$$

A key property of conditional entropy is the chain rule.

Lemma 4 (Chain rule)**.**

$$H(X, Y) = H(X|Y) + H(Y).$$

To use entropy concretely, we will look at the special case



though the same method applies in full generality.

Proof. We will study the entropy of a random embedding of this graph, chosen as follows:

- first take $X_0 Y_1$ to be a uniform random edge,
- choose Y_2 and Y_3 to be uniform random neighbours of X_0,
- choose X_1 to be a conditionally independent copy of X_0 given (Y_1, Y_2),
- choose X_2 to be a conditionally independent copy of X_0 given (Y_2, Y_3).

Then

$$\begin{aligned} H(X_0, X_1, X_2, Y_1, Y_2, Y_3) &= H(X_0, X_1, X_2|Y_1, Y_2, Y_3) + H(Y_1, Y_2, Y_3) \\ &= H(X_0|Y_1, Y_2, Y_3) + H(X_1|Y_1, Y_2, Y_3) + H(X_2|Y_1, Y_2, Y_3) + H(Y_1, Y_2, Y_3) \\ &= H(X_0|Y_1, Y_2, Y_3) + H(X_1|Y_1, Y_2) + H(X_2|Y_2, Y_3) + H(Y_1, Y_2, Y_3) \\ &= H(X_0, Y_1, Y_2, Y_3) + H(X_1, Y_1, Y_2) + H(X_2, Y_2, Y_3) - H(Y_1, Y_2) - H(Y_2, Y_3). \end{aligned}$$

We now use the fact that Sidorenko holds for trees (or, more correctly, that the entropy method works for trees) to conclude that

$$\begin{aligned} H(X_0, Y_1, Y_2, Y_3) &\geq 3 \log_2(2e(G)) - 2 \log_2(v(G)) \\ H(X_1, Y_1, Y_2) &\geq 2 \log_2(2e(G)) - \log_2(v(G)) \\ H(X_2, Y_2, Y_3) &\geq 2 \log_2(2e(G)) - \log_2(v(G)). \end{aligned}$$

Since also $H(Y_1, Y_2) = H(Y_2, Y_3) \leq 2 \log_2(v(G))$, we have

$$\log_2 \mathrm{Hom}(H, G) \geq H(X_0, X_1, X_2, Y_1, Y_2, Y_3) \geq 7 \log_2(2e(G)) - 8 \log_2(v(G)).$$

Rewriting, we get that

$$\mathrm{Hom}(H, G) \geq (2e(G))^7 (v(G))^{-8} = (2e(G)/v(G)^2)^7 v(G)^6,$$

which is exactly $t_H(G) \geq t_{K_2}(G)^7$, as required. □

There are several further interesting results regarding Sidorenko's conjecture. For instance, a result of Lovász says that the conjecture holds locally around the uniform graphon, while a result of Conlon and Lee says that for any bipartite graph H there exists a constant k such that gluing k copies of H along one side gives a graph satisfying the conjecture. For example, for the smallest open case, $H = K_{5,5} - C_{10}$, it suffices to take $k = 2$.