VERY ROUGH OUTLINE OF DUSTIN'S LECTURES

Be warned that the numbering of lectures may not correspond to reality. In particular I may use the overflow lecture at any point. But the material described is the material I intend to cover.

Some sources:

- Serre's "A Course in Arithmetic"
- Lam's "Introduction to quadratic forms over fields"
- Milnor-Husemoller "Symmetric bilinear forms"
- Milnor "An introduction to algebraic K-theory"
- Cox, "Primes of the form $x^2 + ny^2$"

1. Hilbert's product formula

(1) Statement of Hilbert's product formula.
(2) Deduction of quadratic reciprocity and the supplementary laws for 2 and -1.
(3) Reminder about Milnor K-theory, definition of tame symbol.
(4) Statement of Tate's calculation of $K_2(\mathbb{Q})$, deduction of Hilbert's product formula from it.
(5) Proof of Tate's calculation.

Problem set ideas:

- Continuous Steinberg symbols on $\mathbb{Q}_p$ and $\mathbb{R}$.
- Hilbert's product formula for $k(T)$.
- Give your favorite proof of QR.

2. Proof of Hasse-Minkowski theorem

(1) Three forms of Hasse-Minkowski: local zero $\Rightarrow$ global zero, number locally represented $\Rightarrow$ globally represented, and two forms locally equivalent $\Rightarrow$ globally equivalent.
(2) Proof of Hasse-Minkowski, at least for $n \leq 4$.
(3) Corollary: classification of isomorphism classes of quadratic forms over $\mathbb{Q}$ by numerical invariants.

Problem set ideas:

- Weak approximation theorem, converse to the classification by numerical invariants.
- Geometric perspective on zeroes of quadratic forms in 3 variables: rational parametrization of conics. The example of the circle, with applications to trig formulas and Pythagorean triples. See Silverman-Tate.

3. Quadratic forms over $\mathbb{Z}$, in two variables.

(1) Equivalent definitions of "being in the same genus": i) equivalent over $\mathbb{Q}$, ii) equivalent over $\mathbb{Z}_p$ for all p, iii) equivalent over $\mathbb{Z}/N\mathbb{Z}$ for all N, iv) have same discriminant.
(2) Definition of class number: number of equivalence classes of forms in the same genus.
(3) Restrictions on the discriminant; notion of fundamental discriminant.

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(4) Reduction theory: finiteness of the class number.
(5) Statement of Dirichlet's formula for negative fundamental discriminants.

Problem set ideas:

- Calculate some class numbers by hand using reduction theory, compare with Dirichlet's formula.
- Geometric picture for reduction theory using $SL_2(\mathbb{Z})$ action on upper half plane, as in Serre's book
- Geometry of numbers: asymptotics for size of $\{n \leq N : \exists x, y \in \mathbb{Z}, x^2 + y^2 = n\}$ by drawing pictures of circles and lattice points and calculating area.

4. Rings of integers in quadratic number fields

(1) Definition and calculation of ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{d})} \subset \mathbb{Q}(\sqrt{d})$ for d a negative fundamental discriminant
(2) Proof that the ring of integers is additively a free abelian group of rank two. Definition of norm map as an example of a quadratic form of discriminant d.
(3) Generalization to nonzero ideals. Definition of ideal class group; identification of ideal class group with the set of equivalence classes of quadratic forms of discriminant d. Identification of automorphism group of any such quadratic form in terms of the group of roots of unity in $\mathbb{Q}(\sqrt{d})$.

Problem set ideas:

- What are the roots of unity in $\mathbb{Q}(\sqrt{d})$?
- Finiteness of class number from this perspective, following your favorite algebraic number theory text.

5. Multiplicative classification of ideals

(1) Statement of unique prime factorization of ideals for $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.
(2) Classification of maximal ideals in terms of the primes they lie over.
(3) Back to quadratic forms: deduce simple criterion for a prime number to be represented by *some* quadratic form of discriminant d. Consequence in case of class number one: criterion for prime number to be represented by a given quadratic form.

Problem set ideas:

- Abstract theory of Dedekind domains, proof of unique prime factorization of ideals
- Classification of all numbers, not necessarily prime, of the form $x^2 + y^2$, by combining (1) and (2)

6. Dedekind zeta function

(1) Dedekind zeta function as a formal Dirichlet series. Using Euler product, interpretation parts (1) and (2) of the previous lecture as the equality

$$\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta_{\mathbb{Q}}(s) \cdot L(s, \chi_d).$$

(2) Corollary: formula for number of ways of representing n by a quadratic form of discriminant d in case of class number one.

(3) Statement of Dedekind's class number formula in the case of $\mathbb{Q}(\sqrt{d})$, statement of calculation of $L(1, \chi_d)$. Deduction of Dirichlet's class number formula from these two claims, yet unproven.

Problem set ideas:

- Dirichlet's theorem on primes in arithmetic progressions, part 1

7. Dirichlet L-series

(1) Proof of calculation of $L(1, \chi_d)$.
(2) Proof of Dedekind's class number formula.

Problem set ideas:

- Dirichlet's theorem on primes in arithmetic progressions, part 2

8. Overflow lecture

To give extra space in case something takes more time...