Open Dynamics @IAS

Summer Collab Research Summary

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We are grateful for the excellent working conditions provided by the IAS for our group to work (amidst periodic confectionary distractions) on our multi-volume monograph in-progress, which is tentatively titled: *Open Dynamical Systems Avoiding Arbitrary Balls: Statistics, Geometry, and Thermodynamic Formalism.* Our team is normally spread between Wisconsin, Ontario, Warsaw, and Texas. Thus having this dedicated time in Princeton to focus our energies on this large project provided the perfect momentum to complete approximately 5 chapters of the first volume, which we hope to have submitted by the end of this year. Beyond our work on open systems, we also started fruitful discussions on a few other projects that we hope to revisit in the future.

FIGURE 1. Selfie-breaks while working in our office, and on the last day of our visit.

The theory of *open dynamical systems*, also known as *dynamical systems with holes*, naturally models physical systems where some escape of mass, or "leakage" occurs. One of the first mathematical treatments of such systems goes back to the 1980's ([**45**], [**46**], see also [**47**]) for the doubling map of the circle. One considers the doubling map $T(x) := 2x \mod 1$ acting on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. For $\epsilon > 0$, one takes as a "hole" the open interval $U_\epsilon := (0, \epsilon)$ of the circle, and defines the set

$$K(\varepsilon) := \{x \in \mathbb{R}/\mathbb{Z} \ : \ T^n(x) \notin (0, \varepsilon) \ \ \forall n \geq 0\} \tag{0.1}$$

that is, the set of points that never "fall" into the hole under forward iteration.

The set $K(\epsilon)$ is usually a Cantor set with measure zero, but positive Hausdorff dimension. The dependence of $K(\epsilon)$ on ϵ is rather subtle: for instance, it is natural to consider the dimension function

$$\eta(\epsilon) := \mathrm{HD}(K(\epsilon)). \tag{0.2}$$

Such a function is continuous [**45**], but not smooth: in fact, it is locally constant on a countable union of intervals moreover, the local Hölder exponent of η in a neighborhood of some ϵ is equal to the value of $\eta(\epsilon)$ [**5**].

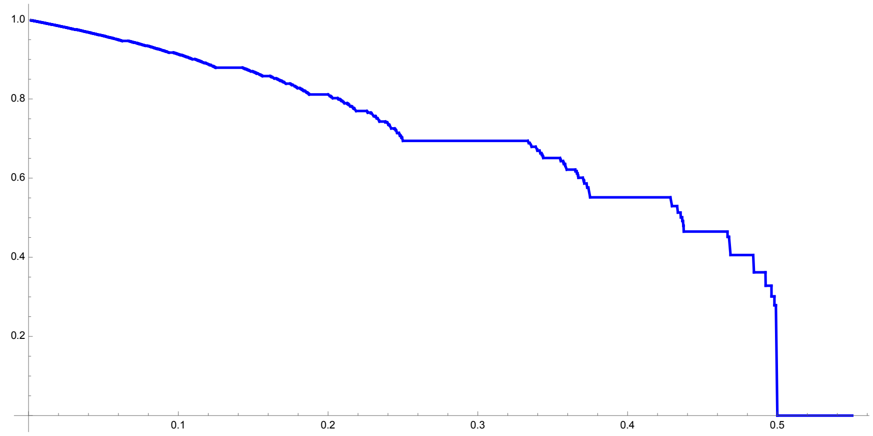

FIGURE 2. The Hausdorff dimension function $\eta(\epsilon) = \mathrm{HD}(K(\epsilon))$. One can see the intervals where the function is locally constant.

The above example can be generalized to a wide class of dynamical systems, and this has sparked, in the following three decades, a growing body of literature: see e.g. [**1**], [**4**], [**11**], [**10**], [**13**], [**14**], [**15**], [**16**], [**17**], [**18**], [**22**], [**26**], [**28**], [**29**], [**36**] [**37**].

The goal of our book is to present a general theory of open dynamical systems in arbitrary metric spaces, which is both as self-contained as possible, but also introduces new results, in particular by considering arbitrary metric balls as holes.

Basic definitions. In general, one considers a transformation $T : X \to X$ of a metric space (X, ρ). Let us fix a reference point $\xi \in X$ and, for every $\varepsilon > 0$, consider the ball $B(\xi, \varepsilon)$, of center ξ and radius ε, which we will treat as a "hole". Then one has the set

$$K_{T,\xi}(\varepsilon) := \big\{x \in X : T^n(x) \notin B(\xi, \varepsilon)\ \forall n \geq 0\big\} = \bigcap_{n=0}^{\infty} T^{-n}(X \setminus B(\xi, \varepsilon)), \tag{0.3}$$

of points that never fall into the hole under forward iteration, which is usually called the survival set. The set $K_{T,\xi}$ is a closed, forward invariant set. Usually, the survival set has measure zero, but its Hausdorff dimension is positive.

Another natural way of measuring the "size" of the hole is to consider its escape rate, which is the exponential rate of decay of the measure of the set of points that are not absorbed by the hole up to time n. More precisely, given a Borel probability T-invariant measure μ on X, one defines the set

$$A_n := \big\{x \in X : T^k(x) \notin B(\xi, \varepsilon)\ \forall\ 0 \leq k \leq n-1\big\}$$

of points that do not fall into the hole before n steps, and the lower escape rate

$$\underline{R}_{T,\mu}(B(\xi, \varepsilon)) := -\limsup_{n \to \infty} \frac{1}{n} \log \mu\,(A_n) \tag{0.4}$$

and, analogously, the upper escape rate $\overline{R}_{T,\mu}(B(\xi,\varepsilon))$. If these are equal, then the common value

$$R_{T,\mu}(B(\xi,\varepsilon)) := \underline{R}_{T,\mu}(B(\xi,\varepsilon)) = \overline{R}_{T,\mu}(B(\xi,\varepsilon))$$

is called the escape rate generated by the map T and the hole $B(\xi,\varepsilon)$.

In a slightly more general context, one can replace the hole $B(\xi,\varepsilon)$ by an arbitrary measurable subset Γ of X in (0.3) and (0.4), and form the corresponding set $K_T(\Gamma)$. Following [**11**] we call a Borel probability measure ν on $X \setminus \Gamma$ conditionally invariant if there exists $\alpha \in (0,1]$ such that

$$\nu\big((X \setminus \Gamma) \cap T^{-1}(A)\big) = \alpha\nu(A) \tag{0.5}$$

for every Borel set $A \subset X \setminus \Gamma$. In slightly different terms, a Borel probability measure ν on X is conditionally invariant with respect to $X \setminus \Gamma$ if $\nu(X \setminus \Gamma) = 1$ and

$$\nu \circ T^{-1} = \alpha\nu. \tag{0.6}$$

If μ is an equilibrium state of some potential $\psi : X \to \mathbb{R}$, then one can ask about the existence, uniqueness, and geometric and stochastic properties of equilibrium states (commonly called survival equilibria) for the potentials $\psi_\Gamma := \psi|_{K_T(\Gamma)} : K_T(\Gamma) \to \mathbb{R}$ and the dynamical system $T|_\Gamma : \Gamma \to \Gamma$.

Motivating questions. Using these notions, the fundamental questions in the theory of open systems can be formulated as follows:

(1) Do the escape rates exist? If, so how are these related to the measures $\mu(B(\xi,\varepsilon))$? And, more subtly: how do escape rates change as the hole changes? In particular, what is the relation between the quantities $|R_{T,\mu}(B(\xi,\varepsilon)) - R_{T,\mu}(B(\xi,r))|$ and $|\mu(B(\xi,\varepsilon)) - \mu(B(\xi,r))|$?

(2) What is the asymptotic behavior of $\mathrm{HD}(X) - \mathrm{HD}(K_{T,\xi}(\varepsilon))$ when $\varepsilon \searrow 0$? And, more generally: given $r \geq 0$, what is the asymptotic behavior of $\mathrm{HD}(K_{T,\xi}(r)) - \mathrm{HD}(K_{T,\xi}(\varepsilon))$ when $\varepsilon \to r$?

(3) What do the conditionally invariant measures generated by the holes $B(\xi,\varepsilon)$ look like?

(4) What about the dynamics of $T|_{K_{T,\xi}(\varepsilon)} : K_{T,\xi}(\varepsilon) \to K_{T,\xi}(\varepsilon)$? Are there any natural significant invariant measures, with respect to which the dynamics of $T|_{K_{T,\xi}(\varepsilon)}$ does not differ too much from the dynamics of $T : X \to X$? More specifically, what about survival equilibria?

We develop several techniques to answer the above questions for a large class of systems. Our book intends to provide an expository account of these concepts and will simultaneously contains many original theorems.

In particular, we extend the work of [**22**], where the authors study thermodynamic formalism for open systems with holes, under the constraint that holes are a finite union of cylinders of the same depth. This was improved in [**37**], where the hole is allowed to be a countable union of cylinders, still of the same depth. In our current project we no longer require this assumption, thus the “hole” generating the open system may be an arbitrary metric ball, provided that it satisfies a certain “thin boundary” condition.

Symbolic dynamics, Perron Frobenius operators, and Banach spaces. The general approach to these questions is based on the following framework. First, one starts working in the symbolic dynamic context, more precisely with countable alphabet subshifts of finite type, whose thermodynamic formalism was laid down in [**32**] and [**33**].

We consider a countable (finite or infinite) set E, which we treat as our "alphabet", a finitely primitive incidence matrix $A : E \times E \to \{0, 1\}$, and a summable Hölder continuous potential $\varphi : E_A^\infty \to \mathbb{R}$, and we work with the shift map $\sigma : E_A^\infty \to E_A^\infty$. The holes are then quite general open sets $U \subseteq E_A^\infty$, while the σ–invariant measure is μ_φ, the Gibbs/equilibrium state of the potential φ.

The first key idea is to define, for each open set U, the singularly perturbed Perron–Frobenius operator

$$g \longmapsto \mathcal{L}_U(g) := \mathcal{L}_\varphi(g\mathbb{1}_{U^c}).$$

Usually, as in [**32**] and [**33**], Perron-Frobenius operators act on the Banach space $\mathrm{H}_\theta^b(E_A^\infty)$ of bounded Hölder continuous functions. However, if the hole U is quite general, and we want it to be such, the characteristic function $\mathbb{1}_{U^c}$ is not Hölder, and, as a consequence, the operator $\mathcal{L}_U$ does *not* preserve $\mathrm{H}_\theta^b(E_A^\infty)$. We tackle this issue by defining a different, bigger Banach space, denoted in [**22**] and in [**37**] by $\mathcal{B}_\theta(m_\varphi)$, endowed with two new norms $\|\cdot\|_{\mathcal{B}_\theta(m_\varphi)}$ and $\|\cdot\|_{m_\varphi,\theta,\mathcal{U}}$. The first crucial point is that the characteristic function $\mathbb{1}_{U^c}$ belongs to the Banach space $\mathcal{B}_\theta(m_\varphi)$ for a large collection of sets U. Then,

$$\mathcal{L}_U\big(\mathcal{B}_\theta(m_\varphi)\big) \subseteq \mathcal{B}_\theta(m_\varphi)$$

and the operator $\mathcal{L}_U : \mathcal{B}_\theta(m_\varphi) \to \mathcal{B}_\theta(m_\varphi)$ is bounded. In order for this operator to result in a rich thermodynamic formalism, one would like to show that the operator $\mathcal{L}_U$ is quasi–compact and $r(\mathcal{L}_U)$, the spectral radius of $\mathcal{L}_U$, is a unique eigenvalue of $\mathcal{L}_U$ of modulus $r(\mathcal{L}_U)$; moreover that this eigenvalue is simple.

Perturbative schemes. Moreover, in order to answer the previous questions (1)-(4), we need to be able to control the statistical and geometric properties of the system under "deformations" of the hole. This is done by considering perturbations of the Perron-Frobenius operators as follows.

To set this up, we introduce the notion of a perturbative scheme, namely a family $\mathcal{U} = (U_\varepsilon)_{\varepsilon\in\Gamma}$ of open holes in E_A^∞, where the space of parameters Γ is a topological space with a marked element denoted by 0, satisfying certain natural conditions. For any $\epsilon \in \Gamma$, we define the perturbed operator $\mathcal{L}_\varepsilon$ as

$$\mathcal{L}_\varepsilon(g) := \mathcal{L}_\varphi(g\mathbb{1}_{U_\varepsilon^c})$$

The problem is, however, that the standard perturbation theory of linear operators (see [**24**]) does not apply as the operators $\mathcal{L}_\varepsilon$, acting on $\big(\mathcal{B}_\theta(m_\varphi), \|\cdot\|_{\mathcal{B}_\theta(m_\varphi)}\big)$ (not to mention $\mathrm{H}_\theta(E_A^\infty)$), are not in general small perturbations of $\mathcal{L}_\varphi$.

The remedy comes from introducing yet another norm, which we denote by $\|\cdot\|_{m_\varphi,\theta,\mathcal{U}}$. The key point is that although the differences between the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varphi$ are actually always large in the $\mathcal{B}_\theta$-norm, these are, however, small if the input Banach space is $\mathcal{B}_\theta(m_\varphi)$

endowed with the "strong" norm $\|\cdot\|_{\mathcal{B}_\theta(m_\varphi)}$ while the output is the same vector space but endowed with the appropriately defined "weak" norm $\|\cdot\|_{m_\varphi,\theta,\mathcal{U}}$.

We assume all the holes in our perturbation scheme to satisfy a uniform thin boundary condition. The norm $\|\cdot\|_{m_\varphi,\theta,\mathcal{U}}$ is crafted so that, if the hole U_ε shrinks to a point, then the perturbed operators

$$\mathcal{L}_\varepsilon : (\mathcal{B}_\theta(m_\varphi), \|\cdot\|_{\mathcal{B}_\theta(m_\varphi)}) \longrightarrow (\mathcal{B}_\theta(m_\varphi), \|\cdot\|_{m_\varphi,\theta,\mathcal{U}})$$

are close to $\mathcal{L}_\varphi$, and converge to $\mathcal{L}_\varphi$ sufficiently fast, so that the perturbation theorem of Keller and Liverani (see [**28**], compare also [**27**]) applies. This yields our main perturbative result, viz. the perturbed operators $\mathcal{L}_\varepsilon$, with all $\varepsilon \in \Gamma$ sufficiently close to 0, have the same spectral structure as the operator $\mathcal{L}_\varphi : \mathcal{B}_\theta(m_\varphi) \longrightarrow \mathcal{B}_\theta(m_\varphi)$, in particular these are quasi–compact with spectral gap.

In fact, one has the spectral decomposition

$$\mathcal{L}_\varepsilon = \lambda_\varepsilon Q_\varepsilon + \Delta_\varepsilon,$$

where $Q_\varepsilon : \mathcal{B}_\theta(m_\varphi) \to \mathcal{B}_\theta(m_\varphi)$ is a projector ($Q_\varepsilon^2 = Q_\varepsilon$) onto the 1–dimensional eigenspace of the real eigenvalue λ_ε (of maximal modulus), and there exist $\kappa \in (0, e^{\mathrm{P}(\varphi)})$ and $C > 0$ such that

$$\|\Delta_\varepsilon^k\|_{\mathcal{B}_\theta(m_\varphi)} \leq C\kappa^k$$

for all integers $k \geq 0$. Moreover,

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = e^{\mathrm{P}(\varphi)}$$

Furthermore, $\|Q_\varepsilon\|_{\mathcal{B}_\theta(m_\varphi)} \leq C$ and

$$\lim_{\varepsilon\to 0} |\!|\!| Q_\varepsilon - Q_\varphi |\!|\!| = 0,$$

where $|\!|\!|\cdot|\!|\!|$ is the norm of bounded operators acting from $(\mathcal{B}_\theta(m_\varphi), \|\cdot\|_{\mathcal{B}_\theta(m_\varphi)})$ to $(\mathcal{B}_\theta(m_\varphi), \|\cdot\|_{m_\varphi,\theta,\mathcal{U}})$, and Q_φ is the projector associated to the original operator $\mathcal{L}_\varphi$.

We go on to prove a far reaching continuity strengthening of the above results. Namely that for all $r \in \Gamma$ sufficiently close to 0, we have that

$$\lim_{\varepsilon\to r} \lambda_\varepsilon = \lambda_r$$

and

$$\lim_{\varepsilon\to r} |\!|\!| Q_\varepsilon - Q_r |\!|\!| = 0.$$

The latter two results are starting points to calculate the asymptotic of escape rates at any point sufficiently close to $0 \in \Gamma$.

We prove that any perturbative scheme can be enlarged so that each of its holes can be sufficiently well approximated by holes consisting of carefully chosen unions of cylinders of the same length. This enlargement/approximation is key, and we will use it throughout our book. Its first consequence is that the eigenfunctions ρ_ε produced as elements of the Banach space $\mathcal{B}_\theta(m_\varphi)$ can be globally extended, i.e. as Borel bounded functions defined on all of E_A^∞ satisfying pointwise the eigenvalue equation

$$\mathcal{L}_\varepsilon(\rho_\varepsilon) = \lambda_\varepsilon \rho_\varepsilon.$$

We then study analytic families of Perron–Frobenius operators all of which act, and are bounded, on the same Banach space $\mathcal{B}_\theta(m_\psi)$ where ψ is a 1–cylinder Hölder continuous potentials. Following some technical work we obtain analytic dependence of eigenvalues, eigenfunctions, projectors, and other relevant spectral objects.

Next we consider the restriction of the potential φ to these survival sets $K(\varepsilon)$ and ask about significant invariant measures it generates. What is really remarkable is that although the maps $\sigma : K(\varepsilon) \longrightarrow K(\varepsilon)$ need not be (and usually are not!) Markov or topologically mixing, these maps and potentials $\varphi|_{K(\varepsilon)}$ nevertheless bear lots of similarities with finitely primitive countable alphabet subshifts of finite type. This is transparently reflected in the nearly classical version of the Variational Principle, the uniqueness of (survival) equilibrium states for the map $\sigma : K(\varepsilon) \longrightarrow K(\varepsilon)$ and potentials $\varphi|_{K(\varepsilon)}$, and stochastic properties of these states such as ergodicity, and most notably the Almost Sure Invariance Principle, which also implies an exponential decay of correlations, the Central Limit Theorem and the Law of Iterated Logarithm. Both proofs of the existence and uniqueness of survival equilibrium states are highly technical and subtle, the latter using in particular the globally defined eigenfunctions ρ_ε mentioned above, and adapting the method of differentiability of topological pressure on survival sets.

Going beyond continuous parameter dependence of the eigenvalues λ_ε, we determine the rate of convergence, i.e. that the limits

$$\lim_{\varepsilon \searrow r} \frac{\lambda_r - \lambda_\varepsilon}{\tilde{\mu}_r(A(r,\varepsilon))} \tag{0.7}$$

exist and we can determine their values. Here $A(r,\varepsilon) = U_\varepsilon \ominus U_r$, where $\ominus$ denotes the symmetric difference between sets, and $\tilde{\mu}_r$ is the survival equilibrium state on the set $K(U_r)$ we discussed in the previous paragraph.

We proved the following

THEOREM 0.0.1. *Let E be a countable alphabet, let $A : E \times E \to \{0,1\}$ be a finitely primitive incidence matrix, let $\varphi : E_A^\infty \longrightarrow \mathbb{R}$ be a 1–cylinder Hölder continuous summable potential, and let $\mathcal{U} = (U_\varepsilon)_{\varepsilon\in\Gamma}$ be a totally linear perturbative scheme with respect to the measure m_φ. Then, for all positive $r < \varepsilon \in \Gamma$ small enough, the escape rates $R_{\tilde{\mu}_r}(U_\varepsilon)$ exist, and moreover*

$$R_{\tilde{\mu}_r}(U_\varepsilon) = R_{\tilde{\mu}_r}(A(r,\varepsilon)) = \log\lambda_r - \log\lambda_\varepsilon.$$

Applying this formula and Eq. (0.7), we get another remarkable expression for escape rates, namely the existence of the limit

$$\lim_{\varepsilon \searrow r} \frac{R_{\widetilde{\mu}_r}(U_\varepsilon)}{\widetilde{\mu}_r(U_\varepsilon)} = \lim_{\varepsilon \searrow r} \frac{R_{\widetilde{\mu}_r}(A(r,\varepsilon))}{\widetilde{\mu}_r(A(r,\varepsilon))}$$

and an explicit formula expressing it.

The base for all our forthcoming applications to smooth and conformal dynamical systems is via studying conformal Graph Directed Markov Systems (GDMS), whose foundations were laid down in [**33**]. In this chapter, we deal with measures on the limit set J_S of a GDMS

$\mathcal{S}$ that are projections under the map $\pi_\mathcal{S} : E_A^\infty \longrightarrow J_\mathcal{S}$ of Gibbs states for 1–cylinder Hölder continuous summable potentials. One of our main results in this vein is the following.

THEOREM 0.0.2. *Let $\mathcal{S} = \{\varphi_e\}_{e\in E}$ be a finitely irreducible conformal Graph Directed Markov System. Let $\varphi : E_A^\infty \longrightarrow \mathbb{R}$ be a Hölder continuous summable potential. As usual, denote its equilibrium/Gibbs state by μ_φ. Fix a point $z \in J_\mathcal{S}$ which is not periodic, i.e. $\varphi_\omega(z) \neq z$ for all $\omega \in E_A^*$.*

If $\Gamma \subseteq [0, +\infty)$ is an infinite set clustering at 0 and $\big(\pi_\mathcal{S}^{-1}(B(z, \varepsilon)\big)_{\varepsilon\in\Gamma}$ is a perturbative scheme with respect to the measure m_φ, then for all $0 < b \in \Gamma$ small enough the escape rates

$$R_{\mathcal{S},\varphi}(B(z, b)) := R_{\tilde{\mu}_a}\big(\pi_\mathcal{S}^{-1}(B(z, b))\big) = \mathrm{P}(\varphi) - \log \lambda_b, \tag{0.8}$$

exist and moreover,

$$\lim_{\varepsilon\searrow 0} \frac{R_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\tilde{\mu}_\varphi \circ \pi_\mathcal{S}^{-1}(B(z, \varepsilon))} = \mathrm{P}(\varphi). \tag{0.9}$$

Geometric potentials. Next we move to geometry. In order to capture the conformal structure of the limit sets we are interested in, we introduce complex potentials of the following form, usually called geometric potentials. For every $\xi \in \mathbb{C}$ the complex-valued function ζ_ξ is defined as

$$\zeta_\xi(\omega) := \xi \log |\varphi'_{\omega_1}(\pi_\mathcal{S}(\sigma(\omega)))|,$$

and for every $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \in \Gamma_\zeta$ the operator $\mathcal{L}_\xi : C_b(E_A^\infty) \to C_b(E_A^\infty)$ is given by the formula

$$\mathcal{L}_\xi g(\omega) := \sum_{\substack{e\in E\\ A_{e\omega_1}=1}} g(e\omega) \exp(\zeta_\xi(e\omega)) = \sum_{\substack{e\in E\\ A_{e\omega_1}=1}} g(e\omega)|\varphi'_e(\pi_\zeta(\omega))|^\xi.$$

Furthermore, given $\varepsilon \in \Gamma$, the perturbed operator $\mathcal{L}_{\xi,\varepsilon}$, acting on the "common" Banach space $\mathcal{B}_\theta\big(m_{\zeta_t}\big)$ with appropriately chosen t, is given by the formula

$$\mathcal{L}_{\xi,\varepsilon} g(\omega) := \sum_{\substack{e\in E\\ A_{e\omega_1}=1}} g(e\omega) \mathbb{1}_{U_\varepsilon}(e\omega) \exp\big(\xi \log |\varphi'_e(\pi_\mathcal{S}(\omega))|\big),$$

where, we recall, $\mathcal{U} = (U_\varepsilon)_{\varepsilon\in\Gamma}$ is the collection of holes. Denoting by $\lambda_{t,\varepsilon}$ the leading eigenvalues of the operators $\mathcal{L}_{t,\varepsilon}$, we show the existence, for every ε, of a unique parameter t, denoted by b_ε, such that

$$\lambda_{t,\varepsilon} = 1.$$

We then show that

$$\mathrm{HD}(K(U_\varepsilon)) = b_\varepsilon$$

which is a generalization of Bowen's formula for the Hausdorff dimension of survival sets $K(U_\varepsilon)$. We do need here real analytic dependence of $\lambda_{t,\varepsilon}$ on t in order to get the asymptotics of

$$|\mathrm{HD}(K(U_\varepsilon)) - \mathrm{HD}(K(U_r))| = |b_\varepsilon - b_r|,$$

and we reap the fruits of our previous work on analytic dependence of perturbed eigenvalues on complexified parameters.

As an ultimate result we get the following theorem, in which $K_z(\varepsilon) := K(B(z,\varepsilon))$ and $\widetilde{\mu}_{b_{\mathcal{S}}(\varepsilon),r}$ are the appropriate survival equilibrium states.

THEOREM 0.0.3. *Let $\mathcal{S}$ be a finitely irreducible strongly regular conformal GDMS. For $\varepsilon > 0$ sufficiently small, let b_ε denote the Bowen parameters of the system $\mathcal{S}$. Fix $z \in J_{\mathcal{S}}$. Assume that $\Gamma \subseteq [0,+\infty)$ is an infinite set clustering at 0 and $\left(\pi_{\mathcal{S}}^{-1}(B(z,\varepsilon)\right)_{\varepsilon\in\Gamma}$, is a perturbative scheme. We then have that*

$$\lim_{\varepsilon\searrow r} \frac{\mathrm{HD}(K_z(r)) - \mathrm{HD}(K_z(\varepsilon))}{\widetilde{\mu}_{b_{\mathcal{S}}(r),r}\left(\pi^{-1}(A_+^-(z;r,\varepsilon))\right)} = -\frac{\lambda_r^2}{\lambda_r'(b_r)} \tag{0.10}$$

if $r \in \Gamma_+$ is sufficiently small, and

$$\lim_{\varepsilon\nearrow r} \frac{\mathrm{HD}(K_z(\varepsilon)) - \mathrm{HD}(K_z(r))}{\widetilde{\mu}_{b_{\mathcal{S}}(\varepsilon),r}\left(\pi^{-1}(A_+^-(z;\varepsilon,r))\right)} = -\frac{\lambda_r^2}{\lambda_r'(b_r)} \tag{0.11}$$

if $r \in \Gamma_-$ is sufficiently small.

As an illustration of the previous theorem, we can consider the Gauss map $G(x) := \lfloor \frac{1}{x} \rfloor$, whose orbits generate the continued fraction expansion of x. This map has countably many smooth (but not linear) branches, hence it defines a conformal GDMS over the full shift with countably many symbols. Then, for each r one considers the hole $U_r = [0,r)$, and the corresponding survival set

$$\mathcal{B}(r) := \{x \in [0,1] \ : \ G^n(x) \geq r \ \forall n \geq 0\}.$$

These sets are closely related to Diophantine approximation, and are called sets of *numbers of generalized bounded type* in [**6**]. Our results imply that for each r one has a conditionally invariant measure $\tilde{\mu}_r$ on $\mathcal{B}(r)$. Then for any $\varepsilon > r$ we have $A(0;r,\varepsilon) = (r,\varepsilon]$. Then, as a corollary of Theorem 0.0.3, we have the limit

$$\lim_{\varepsilon\to r^+} \frac{\mathrm{HD}(\mathcal{B}(r)) - \mathrm{HD}(\mathcal{B}(\varepsilon))}{\tilde{\mu}_r((r,\varepsilon])} = -\frac{\lambda_r^2}{\lambda_r'(b_r)}.$$

In fact, much more is true: we do *not* need to consider as a hole a ball centered at 0, but rather we can consider for *any* point $z \in [0,1]$ the family of holes given by balls centered at z and of radius r, and an analogous result as above holds.

Note that, if instead of G we take the doubling map $T(x) = 2x \mod 1$, and we take as holes the family $U_r = (0,r)$, the survival set is the set $K(\epsilon)$ considered at the beginning of the introduction, (Eq. (0.1)), and the dimension function is $\eta(r) := \mathrm{HD}(K(r))$ as in (0.2)). In this simple case, where the maps are linear, the alphabet has two symbols, and the space is one-dimensional, one shows that the conditionally invariant measures satisfy $\tilde{\mu}_r((r,\varepsilon]) \asymp |r-\varepsilon|^{\eta(r)}$; hence, as a corollary of Theorem 0.0.3, we obtain that the local Hölder exponent of the function η at r equals the value of $\eta(r)$; this recovers the result from [**5**] mentioned at the beginning of the introduction. The technical tour de force presented in our book provides, among other things, a vast generalization of this result to nonlinear systems on arbitrary metric spaces. Let us point out that, even for the Gauss map, such results were not known in full generality up until now.

Future applications. We end with some highlights of applications to "real" dynamical systems $T : X \to X$, where X is most often, but not always, a closed subset of a (finitely dimensional) Euclidean space. The general idea to get such applications is based on utilizing the powerful technique of *first return maps*. By building on the method of the first-return map techniques and developing the appropriate large deviations theory we will fully answer the questions (1)–(4) for conformal dynamical systems T such as Topological Collet–Eckmann maps of the unit interval $[0, 1]$ and the Riemann sphere $\widehat{\mathbb{C}}$; for dynamically semi-regular meromorphic functions from $\mathbb{C}$ to $\widehat{\mathbb{C}}$ in the sense of [**35**]; as well to coarse expanding dynamical systems, whose thermodynamic formalism was recently laid down by the four of us and Feliks Przytycki [**12**]. The point is that such systems admit nice sets (or families), in the sense of [**42**] and [**39**], and these canonically give rise to conformal countable alphabet iterated function systems via the first return maps. So, in particular, the results for such iterated function systems (more generally graph directed Markov systems) are usable. Furthermore, by utilizing Young's towers, we also answer such questions for a large class of smooth dynamical systems and billiards such as those considered for example in [**51**], [**7**], [**2**], and [**8**].

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