Mathematics is the art of reducing any problem to linear algebra.
– William Stein

Complex Inner Product Spaces **PCMI USS, Summer 2023**

Quantum states live in a **complex inner product space**. This handout will give the definition and introduce their properties, asking you to prove some of the basic ones.

1 The definition

While a real inner product $\langle x, y\rangle$ is bilinear (linear in each variable separately), there is a slight difference for complex inner products. A complex inner product $\langle x|y\rangle$ is linear in y and *conjugate linear* in x.

Definition 1 *A* **complex inner product space** *is a vector space V over the field $\mathbf{C}$ of complex numbers together with a function*

$$\langle\cdot|\cdot\rangle : V \times V \longrightarrow \mathbf{C},$$

satisfying for all $x, y, y_1, y_2 \in V$, and $\alpha \in \mathbf{C}$

$$\begin{aligned}
\langle x, y_1 + y_2\rangle &= \langle x, y_2\rangle + \langle x, y_2\rangle \\
\langle x, \alpha y\rangle &= \alpha\langle x, y\rangle \\
\langle y, x\rangle &= \overline{\langle x, y\rangle} \\
\langle x, x\rangle &> 0 \text{ if } x \neq 0.
\end{aligned}$$

Exercise 1 *Prove that if $\langle\cdot|\cdot\rangle$ is a complex inner prodcut on V, then $\langle\cdot|\cdot\rangle$ is conjugate linear in the first variable, i.e.,*

$$\begin{aligned}
\langle x_1 + x_2, y\rangle &= \langle x_1, y\rangle + \langle x_2, y\rangle \\
\langle \alpha x, y\rangle &= \overline{\alpha}\langle x, y\rangle
\end{aligned}$$

Example 1 *Let $V = \mathbf{C}^n$. For $z, w \in \mathbf{C}^n$, we may write $z = (z_1, \ldots, z_n)$, and $w = (w_1, \ldots, w_n)$, and define*

$$\langle z|w\rangle = \sum_{j=1}^{n} \overline{z_j} w_j.$$

Exercise 2 *Prove (or at least convince yourself) that the formula above satisfies the definition of a complex inner product.*

One then has a notion of norm:

Definition 2 *If V is a complex inner product space and $v \in V$, we define the* **norm** *of v by*

$$||v|| = \sqrt{\langle v|v\rangle}.$$

Just like real inner product spaces, complex inner product spaces enjoy these three properties

1. (Triangle Inequality)
$$||x + y|| \leq ||x|| + ||y||$$
.

2. (Cauchy-Schwarz Inequality)
$$| \langle x|y\rangle | \leq ||x|| \cdot ||y||$$

3. (Gram-Schmidt Orthonormalization) Any orthonormal set in V can be extended to an orthonormal basis of V.

2 Adjoints

If A is an matrix with entries in $\mathbf{C}$, then we use $A^\dagger$ to represent the conjugate transpose of A. Thus, the (j, k) entry of $A^\dagger$ is $\overline{a_{kj}}$, the conjugate of the (k, j) entry of A.

Just like the regular transpose, we have the following properties of the conjugate transpose, which you may check if you like:

$$\begin{aligned} (A + B)^\dagger &= A^\dagger + B^\dagger \\ (\alpha A)^\dagger &= \overline{\alpha} A^\dagger \\ (AB)^\dagger &= B^\dagger A^\dagger \\ (A^\dagger)^\dagger &= A \end{aligned}$$

If A is a complex $n \times n$ matrix, then $A^\dagger$ is characterized by the following property:

$$\langle z|Aw\rangle = \langle A^\dagger z|w\rangle \tag{1}$$

known as the **adjoint** property.

Exercise 3 *Prove that the equation above holds for all $n \times n$ complex matrices A, and all $z, w \in \mathbf{C}^n$.*

3 Unitary Maps

An $n \times n$ complex matrix U is called **unitary** if

$$U^\dagger U = I.$$

There are many equivalent ways to say that a matrix is unitary.

Theorem 3 *Let U be an $n \times n$ complex matrix. Show that the following are equivalent*

1. *U is unitary.*
2. *$UU^\dagger = I$.*
3. *U preserves the inner product, i.e. $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbf{C}^n$.*
4. *The rows (or columns) of A form an orthonormal basis of $\mathbf{C}^n$.*

Exercise 4 *Prove this theorem.*

Exercise 5 *Suppose that U is unitary and let $\lambda \in \mathbf{C}$ be an eigenvalue of U. Prove that $|\lambda| = 1$.*

It is also true that U is unitary if and only if U preserves the complex norm. (Postponed for later. Or try it now.)