Biregularity in Sidorenko's Conjecture

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Abstract

Sidorenko's Conjecture says that the minimum density of a bigraph G in a bigraphon W of a given edge density is attained when W is a constant function. A consequence of a result by B. Szegedy is that it is enough to show Sidorenko's Conjecture under the further assumption that W is biregular. In this paper, we retrieve this result with a more elementary proof. With this biregularity result and some ideas of its proof, we also obtain simple proofs of several other results related to Sidorenko's Conjecture. Furthermore, we also show that bigraphs that have a special type of tree decomposition, called reflective tree decomposition, satisfy Sidorenko's conjecture. This both unifies and generalizes the notions of strong tree decompositions and N-decompositions from the literature.

1 Introduction

In [Sid91] (see also [Sid93]), Sidorenko conjectured that if $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ are probability spaces, $W\colon X \times Y \to \mathbb{R}_+$ is a bounded measurable function (a *bigraphon*), and $G = (V_1, V_2, E)$ is a bipartite graph with a given bipartition (V_1, V_2) (a *bigraph*), then

$$t(G, W) \geq t(\rho, W)^{e(G)}, \tag{1}$$

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where ρ denotes the bigraph consisting of a single edge, $e(G) \stackrel{\text{def}}{=} |E(G)|$ is the number of edges in G and the density of G in W is naturally defined as

$$t(G, W) \stackrel{\text{def}}{=} \int_{X^{V_1} \times Y^{V_2}} \prod_{(v,w) \in E(G)} W(x_v, y_w) \, d(\mu^{V_1} \otimes \nu^{V_2})(x, y). \tag{2}$$

Bigraphs G that satisfy (1) for every W are called *Sidorenko bigraphs*.

A tensor power trick [Sid91, Remark 2] implies that to show that G is a Sidorenko bigraph, it is sufficient to prove that there exists $c_G > 0$ such that for every W, we have

$$t(G, W) \geq c_G \cdot t(\rho, W)^{e(G)}. \tag{3}$$

With this tensor power trick, Sidorenko showed [Sid91, Theorem 10] that his conjecture is equivalent to a conjecture by Simonovits [Sim84, Conjecture 8]. Since Simonovits's Conjecture is a weak version of another joint conjecture with Erdős [ES84, Conjecture 2] on supersaturation (see also [Sim84, Conjecture 7]), Sidorenko's Conjecture is sometimes referred to as Erdős–Sidorenko–Simonovits Conjecture, possibly with some permutation of these names. While quite easy to prove (see Lemma 4.1 below for a general version), this tensor power trick has been essential to several results in the literature, see [CFS10, LS11, KLL16, CKLL18b, CKLL18a, Sid21] for some examples.

From an easy adaptation of the graphon theory (see [Lov12] for an introduction to the topic) to the case of bigraphs, (3) is in turn equivalent to

$$t(G, H) \geq c_G \cdot t(\rho, H)^{e(G)} \tag{4}$$

for every bigraph $H = (U_1, U_2, F)$, where

$$t(G, H) \stackrel{\text{def}}{=} \frac{|\text{Hom}(G, H)|}{|U_1|^{|V_1|} \cdot |U_2|^{|V_2|}}$$

and $\text{Hom}(G, H)$ is the set of all bigraph homomorphisms from G to H, i.e., functions $f \colon V_1 \cup V_2 \to U_1 \cup U_2$ such that $f(V_i) \subseteq U_i$ ($i = 1, 2$) and $(v, w) \in E \implies (f(v), f(w)) \in F$. In fact, Sidorenko's Conjecture is often studied under the further assumption that W is symmetric (i.e., $\Omega = \Lambda$ and $W(x, y) = W(y, x)$ for every $x, y \in X$), in which case G and H become ordinary graphs (but G is still bipartite).

It was proved in [Sze15b, Theorem 4] that it is sufficient to show (4) under the further assumption that H is edge-vertex transitive (i.e., the natural actions of the automorphism group of H on the sets $V_1(H)$, $V_2(H)$ and $E(H)$ are transitive). In this paper we recover an important consequence of this result through a different, more elementary, method. More specifically, we prove (Theorem 3.1) that in order to show that a bigraph is Sidorenko, it is sufficient to show (3) under the additional assumption that W is biregular in the sense

$$\int_Y W(x_0, y)\ d\nu(y) = \int_X W(x, y_0)\ d\mu(x) = t(\rho, W) \tag{5}$$

for almost every $x_0 \in X$ and almost every $y_0 \in Y$.

Our techniques and the biregularity assumption allow us to both retrieve several results from the literature in the non-symmetric setting with a much simpler proof and provide some generalizations.

For example, not only can we obtain an easy proof of the non-symmetric analogue of a result of [LS11, Lemmas 3.2 and 3.4 and Theorem 5] that amalgamations of Sidorenko bigraphs along a vertex are Sidorenko bigraphs (Theorem 3.2), but we can prove a weak converse: if G' is the amalgamation of k copies of G along the same vertex, then G is Sidorenko if and only if G' is Sidorenko (Theorem 3.3). Of course, these two results also follow from [Sze15b].

When studying (1), Sidorenko in fact introduced a stronger (a priori) conjecture [Sid91, Equation (2)] that in particular implies

$$t(G, W) \geq \left(\int_X \left(\int_Y W(x, y)\ d\nu(y)\right)^{e(G)/|V_1(G)|} d\mu(x)\right)^{|V_1(G)|}.$$

One way of interpreting the right-hand side is as $t(K_{1,d}, W)^{e(G)/d}$, where $K_{1,d}$ is the (left) d-star bigraph, except that in the above $d \stackrel{\text{def}}{=} e(G)/|V_1(G)|$, which is not necessarily an integer. Our methods allow us to retrieve a weaker version of this implication in the "ordinary" setting; namely, we show that every Sidorenko bigraph $G = (V_1, V_2, E)$ in which all vertices of V_1 have degree at least d satisfies $t(G, W) \geq t(K_{1,d}, W)^{e(G)/d}$ (Theorem 3.4).

Finally, by using the biregularity assumption we are able to unify and generalize (Theorem 3.5) the results of [CKLL18b, Theorem 1.2] on strong tree decompositions and of [CL17, Theorem 5.12] on N-decompositions as particular cases of what we call reflective tree decompositions (see Definition 2.2); this result holds in both the non-symmetric and symmetric settings.

This paper is organized as follows. In Section 2 we establish necessary notation. In Section 3 we state our results. In Section 4 we present the main lemma used in the proofs. In Section 5 we prove our biregularity result. In Section 6, we prove the aforementioned applications of our biregularity result and our main lemma to amalgamations and the strengthened $K_{1,d}$ version of Sidorenko's Conjecture. In Section 7 we prove the result on reflective tree decompositions. In Section 8 we show how to adapt the material from Sections 4, 5 and 7 to the symmetric setting. We finish the paper with a brief discussion and some open problems in Section 9.

2 Preliminaries

Throughout the text, we will use the notation $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, \ldots\}$ for non-negative integers and $\mathbb{N}_+ \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$ for positive integers. For $n \in \mathbb{N}$, we let $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. We also let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ the set of non-negative real numbers. Given a set V, we denote its power set by $2^V \stackrel{\text{def}}{=} \{W \mid W \subseteq V\}$.

2.1 Bigraphs

A *bigraph* is a triple $G = (V_1, V_2, E)$, where V_1 and V_2 are disjoint finite sets and $E \subseteq V_1 \times V_2$. We will also use the following notation ($i = 1, 2$):

$$V_i(G) \stackrel{\text{def}}{=} V_i, \qquad v_i(G) \stackrel{\text{def}}{=} |V_i|, \qquad V(G) \stackrel{\text{def}}{=} V_1 \cup V_2,$$
$$E(G) \stackrel{\text{def}}{=} E, \qquad e(G) \stackrel{\text{def}}{=} |E|, \qquad v(G) \stackrel{\text{def}}{=} |V_1| + |V_2|.$$

For $v \in V(G)$, we denote by $d_G(v)$ its degree. We also let

$$\delta_i(G) \stackrel{\text{def}}{=} \min_{v \in V_i(G)} d_G(v), \qquad \Delta_i(G) \stackrel{\text{def}}{=} \max_{v \in V_i(G)} d_G(v).$$

We say that G is *left d-regular* (*right d-regular*, respectively) if $d_G(v) = d$ for every $v \in V_1(G)$ ($v \in V_2(G)$, resp.). We say that G is *biregular* if it is both left d_1-regular and right d_2-regular for some $d_1, d_2 \in \mathbb{N}$. An *isomorphism* between bigraphs G_1 and G_2 is a bijection $f\colon V(G_1) \rightarrowtail V(G_2)$ such that $f(V_i(G_1)) = V_i(G_2)$ ($i = 1, 2$) and $(v, w) \in E(G_1) \iff (f(v), f(w)) \in E(G_2)$ ($(v, w) \in V_1(G_1) \times V_2(G_1)$); when such an isomorphism exists, we say that G_1 and G_2 are *isomorphic* and denote this as $G_1 \cong G_2$.

For $U \subseteq V(G)$, we let $G|_U$ be the *subgraph induced by* U *in* G, that is, we let

$$V_i(G|_U) \stackrel{\text{def}}{=} V_i(G) \cap U, \quad E(G|_U) \stackrel{\text{def}}{=} E(G) \cap ((U \cap V_1(G)) \times (U \cap V_2(G))).$$

For $v \in V(G)$, we let $G - v \stackrel{\text{def}}{=} G|_{V(G)\setminus\{v\}}$ be the bigraph obtained from G by removing v. For $E \subseteq E(G)$, we also let $G - E \stackrel{\text{def}}{=} (V_1(G), V_2(G), E(G) \setminus E)$ be the spanning subgraph obtained from G by removing the edges in E. The *dual bigraph* of G is the bigraph $G^* \stackrel{\text{def}}{=} (V_2, V_1, E^*)$, where $E^* \stackrel{\text{def}}{=} \{(w, v) \mid (v, w) \in E(G)\}$. We denote the *edge bigraph* $(\{1\}, \{2\}, \{(1, 2)\})$ by ρ and the d-*star bigraph* $(\{0\}, [d], \{(0, i) \mid i \in [d]\})$ by $K_{1,d}$ (thus, $\rho \cong K_{1,1}$).

2.2 Flags

It will be convenient to also work with partially labeled bigraphs and for this purpose we will borrow some terminology from the theory of flag algebras [Raz07].

More specifically, we work in the theory T^2_{Graph} of graphs augmented with a 2-coloring of its vertices. Thus, a *flag* is a partially labeled bigraph, that is, a pair $F = (G, \theta)$, where G is a bigraph and $\theta\colon [k] \rightarrowtail V(G)$ is an injection for some $k \in \mathbb{N}$. We use the notation $|F| \stackrel{\text{def}}{=} G$ for the *underlying bigraph* of F and the notation $\theta_F \stackrel{\text{def}}{=} \theta$ for the *labeling* of F. We will often abuse notation and write $F = (G, (\theta(1), \theta(2), \dots, \theta(k)))$, listing the values of θ. In fact, we will abuse the notation even more and write $F = (G, U)$ for some set $U \subseteq V(G)$ to be understood as $F = (G, \theta)$ for some $\theta\colon [|U|] \rightarrowtail V(G)$ with $\operatorname{im}(\theta) = U$, whenever the exact ordering is either clear from the context or unimportant.

An *isomorphism* between flags $F_1 = (G_1, \theta_1)$ and $F_2 = (G_2, \theta_2)$ is an isomorphism f between G_1 and G_2 that preserves the partial labeling in the sense that $f \circ \theta_1 = \theta_2$; when such an isomorphism exists, we say that F_1 and F_2 are *isomorphic* and denote it by $F_1 \cong F_2$.

If $F_1 = (G_1, \theta_1)$ and $F_2 = (G_2, \theta_2)$ are flags such that $\theta_2 \circ \theta_1^{-1}$ is an isomorphism between $G_1|_{\operatorname{im}(\theta_1)}$ and $G_2|_{\operatorname{im}(\theta_2)}$ (that is, in the terminology of flag algebras, these flags are of the same type), we let $F_1 \sqcup F_2$ be the flag obtained from the disjoint union of F_1 and F_2 by identifying vertices with the same label[1]. For $k \in \mathbb{N}_+$, we further let $F^{\sqcup k}$ be defined recursively as

[1]We avoid using F_1F_2 here to not conflict with the product as defined in flag algebras.

$F^{\sqcup 1} \stackrel{\text{def}}{=} F$ and $F^{\sqcup(k+1)} \stackrel{\text{def}}{=} F^{\sqcup k} \sqcup F$.

A *left* 1*-flag* (*right* 1*-flag*, respectively) is a flag $F = (G, \theta)$ in which $\operatorname{im}(\theta)$ is a single vertex in $V_1(G)$ ($V_2(G)$, resp.). We let $e_1 \stackrel{\text{def}}{=} (\rho, 1)$ and $K^L_{1,d} \stackrel{\text{def}}{=} (K_{1,d}, 0)$ be the unique left 1-flags such that $|e_1| = \rho$ and $|K^L_{1,d}| \stackrel{\text{def}}{=} K_{1,d}$ (thus $e_1 \cong K^L_{1,1}$). We also let $e_2 \stackrel{\text{def}}{=} (\rho, 2)$ be the unique right 1-flag such that $|e_2| = \rho$.

2.3 Bigraphons

Given probability spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$, a *bigraphon* over Ω and Λ is a bounded measurable function $W\colon X \times Y \to \mathbb{R}_+$, where $X \times Y$ is equipped with the product σ-algebra and the product measure $\mu \otimes \nu$; we will denote bigraphons by $W\colon \Omega \times \Lambda \to \mathbb{R}_+$.

The *dual bigraphon* of W is the bigraphon $W^*\colon \Lambda \times \Omega \to \mathbb{R}_+$ defined by $W^*(y, x) \stackrel{\text{def}}{=} W(x, y)$. Given two bigraphons $W_1\colon \Omega_1 \times \Lambda_1 \to \mathbb{R}_+$ and $W_2\colon \Omega_2 \times \Lambda_2 \to \mathbb{R}_+$, their *tensor product* is the bigraphon $W_1 \otimes W_2\colon (\Omega_1 \times \Omega_2) \times (\Lambda_1 \times \Lambda_2) \to \mathbb{R}_+$ given by $(W_1 \otimes W_2)((x_1, x_2), (y_1, y_2)) \stackrel{\text{def}}{=} W_1(x_1, y_1) \cdot W_2(x_2, y_2)$. For $k \in \mathbb{N}_+$, the *kth tensor power* $W^{\otimes k}$ of a bigraphon W is defined inductively by $W^{\otimes 1} \stackrel{\text{def}}{=} W$ and $W^{\otimes(k+1)} \stackrel{\text{def}}{=} W^{\otimes k} \otimes W$.

For a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ and measurable sets $X' \subseteq X$, $Y' \subseteq Y$ of positive measure, we let $W|_{X' \times Y'}\colon \Omega|_{X'} \times \Lambda|_{Y'} \to \mathbb{R}_+$ be the bigraphon that is the restriction of W to $X' \times Y'$ over the conditional probability spaces $\Omega|_{X'} \stackrel{\text{def}}{=} (X', \mu|_{X'})$ and $\Lambda|_{Y'} \stackrel{\text{def}}{=} (Y', \nu|_{Y'})$ (that is, their underlying measures are given by $\mu|_{X'}(A) \stackrel{\text{def}}{=} \mu(A)/\mu(X')$ and $\nu|_{Y'}(B) \stackrel{\text{def}}{=} \nu(B)/\nu(Y')$).

When taking integrals, our functions will always be bounded and hence Fubini's Theorem will apply and we will be omitting explicit references to it. If V is a set, we let $\Omega^V = (X^V, \mu^V)$ be the product probability space of $|V|$ copies of Ω; we will usually abuse notation and denote μ^V simply by μ. Given $x \in X^V$ and $S \subseteq V$, we let $x_S \in X^S$ be the projection of x to the coordinates in S.

For a bigraph G and a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$, we let $t(G, W) \in \mathbb{R}_+$ be given by (2). More generally, for a flag $F = (G, \theta)$ and a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$, we let the function $t(F, W)\colon \Omega^{V_1(G) \cap \operatorname{im}(\theta)} \times \Lambda^{V_2(G) \cap \operatorname{im}(\theta)} \to \mathbb{R}_+$

be given by

$$t(F,W)(x,y) \stackrel{\text{def}}{=} \int_{X^{V_1(G)\setminus \mathrm{im}(\theta)} \times Y^{V_2(G)\setminus \mathrm{im}(\theta)}} \prod_{(v,w)\in E(G)} W(x''_v, y''_w)\ d(\mu\otimes\nu)(x',y'),$$

where

$$x''_v \stackrel{\text{def}}{=} \begin{cases} x_v, & \text{if } v \in V_1(G)\cap \mathrm{im}(\theta),\\ x'_v, & \text{if } v\in V_1(G)\setminus \mathrm{im}(\theta); \end{cases} \qquad y''_w \stackrel{\text{def}}{=} \begin{cases} y_w, & \text{if } w \in V_2(G)\cap \mathrm{im}(\theta),\\ y'_w, & \text{if } w\in V_2(G)\setminus \mathrm{im}(\theta). \end{cases}$$

When $V_1(G)\cap \mathrm{im}(\theta) = \varnothing$, we will simplify the notation to $t(F,W)(y)$, and likewise for $V_2(G)\cap \mathrm{im}(\theta)=\varnothing$. We define further

$$\delta(F,W) \stackrel{\text{def}}{=} \text{ess inf}\{t(F,W)(x,y) \mid (x,y) \in X^{V_1(G)\cap \mathrm{im}(\theta)} \times Y^{V_2(G)\cap \mathrm{im}(\theta)}\};$$
$$\Delta(F,W) \stackrel{\text{def}}{=} \text{ess sup}\{t(F,W)(x,y) \mid (x,y) \in X^{V_1(G)\cap \mathrm{im}(\theta)} \times Y^{V_2(G)\cap \mathrm{im}(\theta)}\}.$$

A bigraphon W is called *F-regular* if $\delta(F,W) = \Delta(F,W) = t(|F|,W)$ (of course, equality between any two of these implies that all of them are equal). For the particular cases of e_1-regular and e_2-regular we use the names *left regular* and *right regular*, respectively. A bigraphon is *biregular* if it is both left regular and right regular.

A *graphon* is a symmetric bigraphon W in the sense that $\Omega = \Lambda$ and $W(x,y) = W(y,x)$ for every $x, y \in X$. As mentioned in the introduction, a *Sidorenko bigraph* G is a bigraph such that $t(G,W) \geq t(\rho,W)^{e(G)}$ for every bigraphon W. A *symmetrically Sidorenko bigraph* G is a bigraph such that $t(G,W) \geq t(\rho,W)^{e(G)}$ for every graphon W (in this case, one can think of G as of a bipartite graph since the choice of bipartition does not affect this inequality). Clearly, every Sidorenko bigraph is also symmetrically Sidorenko but whether the converse is true is unknown.

2.4 Weak domination and reflective tree decompositions

Definitions in this section are needed only for Theorem 3.5.

Inspired by [CL17], we give the following definition of weak domination between bigraphs.

Definition 2.1. Let G_1 and G_2 be bigraphs. We say that G_1 *weakly dominates* G_2 if

$$\frac{t(G_1, W)}{t(\rho, W)^{e(G_1)}} \geq \frac{t(G_2, W)}{t(\rho, W)^{e(G_2)}}$$

for every *biregular* non-zero bigraphon W. We say that a bigraph G is *induced-Sidorenko* if it weakly dominates all of its induced subgraphs.

Remark 1. In [CL17, §5.2], domination between bigraphs G_1 and G_2 is defined by the requirement $t(G_1, W)^{1/e(G_1)} \geq t(G_2, W)^{1/e(G_2)}$ for every bigraphon W. It is easy to see that as long as $e(G_1) \geq e(G_2)$ and G_2 is Sidorenko (which is the case we are mostly interested in), domination implies weak domination. That explains our choice of the terminology. Let us also note that our main result, Theorem 3.1, readily implies that if G_1 weakly dominates G_2 and G_2 is Sidorenko then G_1 is Sidorenko as well.

Recall that for a bigraph G, the 2-*core* of G is a maximal connected subgraph in which all vertices have degree at least 2. When G is connected, it contains only one 2-core, which we denote $C_2(G)$. It can be obtained by progressively removing, in an arbitrary order, vertices of degree less than 2 until no such vertices remain.

For a flag $F = (G, \theta)$ with G connected, we define the 2-*core* $C_2(F)$ as the flag of the form $F' = (G', \theta)$, where G' is the maximal subgraph in which all vertices that are not in $\operatorname{im}(\theta)$ have degree at least two; this can of course be obtained by progressively removing vertices of degree less than 2 that are not in $\operatorname{im}(\theta)$.

Remark 2. Since in a biregular bigraphon W, we have $t(G, W) = t(G - v, W)t(\rho, W)^{d_G(v)}$ whenever $d_G(v) \leq 1$, it follows that G weakly dominates H if and only if $C_2(G)$ weakly dominates $C_2(H)$.

We now define a generalization of the notions of strong tree decompositions [CKLL18b, §1] and N-decompositions [CL17, §5.3], which themselves are generalizations of the usual notion of tree decompositions [Hal76, RS84].

Definition 2.2. Given a connected non-trivial bigraph G, a reflective tree decomposition of G is a tree T such that

i. We have $V(T) \subseteq 2^{V(G)}$ and $V(G) = \bigcup_{U \in V(T)} U$.

ii. For every $(v, w) \in E(G)$, there exists $U \in V(T)$ such that $v, w \in U$.

iii. For every $U_1, U_2 \in V(T)$ and every $U_3 \in V(T)$ in the unique path from U_1 to U_2 in T, we have $U_1 \cap U_2 \subseteq U_3$.

iv. For every $\{U_1, U_2\} \in E(T)$ we have $C_2(F_{U_1U_2}) \cong C_2(F_{U_2U_1})$, where $F_{U_iU_j} \stackrel{\text{def}}{=} (G|_{U_i}, U_1 \cap U_2)$ (we assume that each vertex of $U_1 \cap U_2$ receives the same label in $F_{U_1U_2}$ as in $F_{U_2U_1}$).

Condition (iv) above in particular implies that for every $U_1, U_2 \in V(T)$, we have $C_2(G|_{U_1}) \cong C_2(G|_{U_2})$ (since $C_2(|F|) = C_2(|C_2(F)|)$); this common 2-core bigraph is called the *core* of the decomposition.

Remark 3. The fact that G is connected implies that each $|F_{U_1U_2}|$ for $\{U_1, U_2\} \in E(T)$ and each $G|_U$ for $U \in V(T)$ is connected. Furthermore, condition (iv) is equivalent to the same condition obtained by replacing $F_{U_iU_j}$ with $F'_{U_iU_j} \stackrel{\text{def}}{=} (G|_{U_i} - E(G|_{U_1 \cap U_2}), U_1 \cap U_2)$ and it also equivalent to the existence of an automorphism of the flag $F \stackrel{\text{def}}{=} (C_2(G|_{U_1 \cup U_2}), U_1 \cap U_2)$ that maps $U_1 \cap V(|F|)$ to $U_2 \cap V(|F|)$.

Items (i), (ii) and (iii) alone say that T is a usual tree decomposition. Strong tree decompositions of [CKLL18b, §1] are precisely reflective tree decompositions whose core is empty (i.e., $G|_U$ is a tree for every $U \in V(T)$) and N-decompositions of [CL17, §5.3] are obtained by replacing the requirement $C_2(F_{U_1U_2}) \cong C_2(F_{U_2U_1})$ in (iv) with $F_{U_1U_2} \cong F_{U_2U_1}$ instead (this forces all $G|_U$ for $U \in V(T)$ to be isomorphic to a fixed bigraph N).

3 Main results

In this section we present our main results.

Theorem 3.1. *Let G be a bigraph. If there exists $c_G > 0$ such that $t(G, W) \geq c_G \cdot t(\rho, W)^{e(G)}$ for every biregular bigraphon W, then G is a Sidorenko bigraph.*

Theorem 3.2. *If F_1 and F_2 are left (or right) 1-flags such that $|F_1|$ and $|F_2|$ are Sidorenko bigraphs, then $|F_1 \sqcup F_2|$ is a Sidorenko bigraph.*

The next theorem can be seen as a partial converse to Theorem 3.2.

Theorem 3.3. *Let F be a left 1-flag and $k \in \mathbb{N}_+$. Then $|F|$ is a Sidorenko bigraph if and only if $|F^{\sqcup k}|$ is a Sidorenko bigraph.*

As we mentioned in the introduction, Theorems 3.2 and 3.3 also follow from [Sze15b] (but our proofs are simpler).

Theorem 3.4. *If G is a Sidorenko bigraph with $\delta_1(G) \geq d$, then $t(G, W) \geq t(K_{1,d}, W)^{e(G)/d}$ for every bigraphon W.*

Theorem 3.5. *If T is a reflective tree decomposition of a connected non-trivial bigraph G whose core H weakly dominates $G|_{U_1 \cap U_2}$ for every $\{U_1, U_2\} \in E(T)$, then G weakly dominates H. In particular, if H is a Sidorenko bigraph, then G is also a Sidorenko bigraph.*

Note that in Theorem 3.5 above, if the core H is an induced-Sidorenko bigraph, then both the condition that it weakly dominates $G|_{U_1 \cap U_2}$ for every $\{U_1, U_2\} \in E(T)$ and the fact that H is a Sidorenko bigraph follow (see Remark 2). Hence in that case we can conclude that G is a Sidorenko bigraph.

As we mentioned in Remark 3, the notions of strong tree decompositions and N-decompositions are particular cases of reflective tree decompositions. The corresponding results can be retrieved from Theorem 3.5 above as follows. For [CKLL18b, Theorem 1.2], any two forests weakly dominate each other for obvious reasons, which implies that strongly tree decomposable bigraphs are Sidorenko bigraphs. For [CL17, Theorem 5.12], by [Hat10, Theorem 2.14], every weakly norming bigraph N dominates any of its (not necessarily induced) subgraphs, so it is an induced-Sidorenko bigraph, hence any N-decomposable bigraph for a weakly norming bigraph N is a Sidorenko bigraph.

However, let us note that there are many induced-Sidorenko bigraphs that are not weakly norming bigraphs. For example, any weakly norming bigraph without isolated vertices is necessarily biregular [Hat10, Theorem 2.10(ii)], but the induced-Sidorenko property is trivially preserved under amalgamations with trees along a single vertex, which will destroy biregularity. For a less trivial example, let B_k be the *k-book bigraph* (see Figure 1), that is, the graph obtained by gluing k copies of 4-cycles along the same edge; since we can also see B_k as the amalgamation of two copies of B_{k-1} along a B_{k-2} (with the convention $B_0 \stackrel{\text{def}}{=} \rho$), by inductive application of Theorem 3.5 above and Remark 2, all B_k are induced-Sidorenko bigraphs.

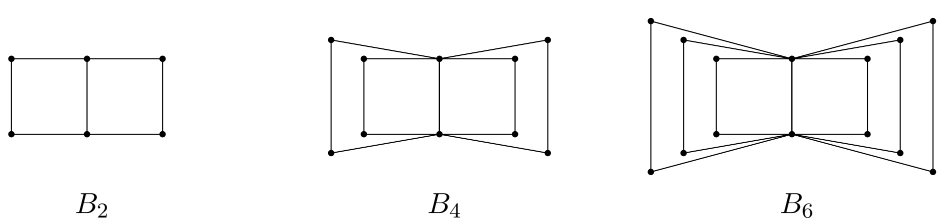

Figure 1: Book bigraphs.

4 The tensor power trick and the main lemma

We start with a slightly more general version of the tensor power trick of [Sid91, Remark 2] that we will need later.

Lemma 4.1 (Tensor power trick)**.** *Let $\mathcal{W}$ be a class of bigraphons that is closed under tensor powers, let $G_1, \ldots, G_n, H_1, \ldots, H_m$ be bigraphs and $r_1, \ldots, r_n, s_1, \ldots, s_m \in \mathbb{R}_+$. If there exists $c > 0$ such that*

$$\prod_{i=1}^{n} t(G_i, W)^{r_i} \geq c \cdot \prod_{j=1}^{m} t(H_j, W)^{s_j}$$

for every $W \in \mathcal{W}$, then the same inequality holds with c replaced by 1.

Proof. Since $\mathcal{W}$ is closed under tensor powers, for $W \in \mathcal{W}$ and $k \in \mathbb{N}_+$, we have

$$\begin{aligned}\prod_{i=1}^{n} t(G_i, W)^{r_i} &= \left(\prod_{i=1}^{n} t(G_i, W^{\otimes k})^{r_i}\right)^{1/k} \\ &\geq \left(c \cdot \prod_{j=1}^{m} t(H_j, W^{\otimes k})^{s_j}\right)^{1/k} = c^{1/k} \cdot \prod_{j=1}^{m} t(H_j, W)^{s_j}\end{aligned}$$

and letting $k \to \infty$ gives the result. □

The proof of the biregularity result, Theorem 3.1, consists of the construction of a biregular bigraphon W' from a bigraphon W with the following properties:

i. $t(\rho, W') = t(\rho, W)$;

ii. for every bigraph G there exists a constant $C_G > 0$ depending only on G such that for any bigraphon W, $t(G, W') \leq C_G \cdot t(G, W)$.

This construction will actually be a chain of steps so that we obtain progressively better "regularity properties". Namely, the steps are:

1. W_1 satisfies $\Delta(e_1, W_1) \leq 2 \cdot t(\rho, W_1)$.

2. W_2 satisfies $\max\{\Delta(e_1, W_2), \Delta(e_2, W_2)\} \leq 2 \cdot t(\rho, W_2)$.

3. W_3 satisfies $\min\{\delta(e_1, W_3), \delta(e_2, W_3)\} \geq 2^{-10} \cdot t(\rho, W_3)$.

4. W_4 is left regular and satisfies $\delta(e_2, W_4) \geq 2^{-10} \cdot t(\rho, W_4)$.

5. W_5 is biregular.

It turns out that all steps except for step (3) can be performed by the same construction in Lemma 4.2 below that improves the "quality of regularity" of its input. In fact, we will state this construction in a slightly more general setting so that we can also use it for Theorems 3.3 and 3.4 (for Theorem 3.1 we will take $F = e_1$, for Theorem 3.3 we will take F as in its statement and for Theorem 3.4, we will take $F = K^L_{1,d}$).

Lemma 4.2. *Let $d \in \mathbb{N}_+$, let $F = (G, \theta)$ be a left 1-flag such that G is left d-regular and let $\epsilon > 0$.*

Then for every bigraphon $W \colon \Omega \times \Lambda \to \mathbb{R}_+$ over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$, there exists a bigraphon $W' \colon \Omega' \times \Lambda \to \mathbb{R}_+$ such that the following hold.

i. We have $\Delta(F, W') \leq (1 + \epsilon) \cdot t(G, W')$.

ii. We have

$$\delta(F, W') \geq \min\left\{t(G, W'), \frac{\delta(F, W)}{\epsilon}\right\}.$$

iii. For every right 1-flag $F' = (G', \theta')$ such that G' is left d-regular, we have

$$t(F', W')(y) = t(F', W)(y)$$

for every $y \in Y$.

iv. For every bigraph G' with $\Delta_1(G') \leq d$, we have

$$t(G', W') \geq \left(1 + \frac{1}{\epsilon}\right)^{e(G')/d - v_1(G')} \cdot t(G', W).$$

v. *For every bigraph G' with $\delta_1(G') \geq d$, we have*

$$t(G',W') \leq \left(1 + \frac{1}{\epsilon}\right)^{e(G')/d - v_1(G')} \cdot t(G',W).$$

vi. *For every left d-regular bigraph G', we have $t(G',W') = t(G',W)$.*

Proof. If $t(G,W) = 0$, then we can simply take $\Omega' \stackrel{\text{def}}{=} \Omega$ and $W' \stackrel{\text{def}}{=} W$, so suppose that $t(G,W) > 0$.

Define the function $f\colon X \to \mathbb{R}_+$ by

$$f(x) \stackrel{\text{def}}{=} \max\{t(F,W)(x), \epsilon \cdot t(G,W)\} \geq \epsilon \cdot t(G,W) > 0 \tag{6}$$

and let $Z \stackrel{\text{def}}{=} \int_X f(x)\ d\mu(x)$. Let $\Omega' \stackrel{\text{def}}{=} (X, \mu')$, where μ' is the probability measure such that $d\mu'(x) = (f(x)/Z)\ d\mu(x)$. Since $t(F,W)(x) \leq f(x) \leq t(F,W)(x) + \epsilon \cdot t(G,W)$, we have

$$0 < t(G,W) \leq Z \leq (1+\epsilon) \cdot t(G,W). \tag{7}$$

Define now the bigraphon $W'\colon \Omega' \times \Lambda \to \mathbb{R}_+$ by

$$W'(x,y) \stackrel{\text{def}}{=} \left(\frac{Z}{f(x)}\right)^{1/d} \cdot W(x,y).$$

(Note that (6) and the upper bound of (7) imply that W' is bounded.)

We start by showing that W' satisfies the last three items. Indeed, if G' is a bigraph, then we have

$$\begin{aligned} t(G',W') &= \int_{X^{V_1(G')} \times Y^{V_2(G')}} \prod_{(v,w) \in E(G')} W'(x_v, y_w)\ d(\mu' \otimes \nu)(x', y) \\ &= Z^{e(G')/d - v_1(G')} \int_{X^{V_1(G')} \times Y^{V_2(G')}} \prod_{(v,w) \in E(G')} W(x_v, y_w) \\ &\qquad \cdot \prod_{v \in V_1(G')} f(x_v)^{1 - d_{G'}(v)/d}\ d(\mu \otimes \nu)(x,y). \end{aligned} \tag{8}$$

If $\Delta_1(G') \leq d$, then the exponent of Z in the above is non-positive and the exponent of $f(x_v)$ is non-negative, hence (6) and the upper bound of Z

in (7) imply

$$t(G',W') \geq ((1+\epsilon)\cdot t(G,W))^{e(G')/d-v_1(G')} \cdot t(G',W) \cdot \prod_{v\in V_1(G')} (\epsilon \cdot t(G,W))^{1-d_{G'}(v)/d}$$
$$= \left(1+\frac{1}{\epsilon}\right)^{e(G')/d-v_1(G')} t(G',W).$$

Thus, item (iv) follows.

On the other hand, if $\delta_1(G') \geq d$ instead, then the exponent of Z is non-negative and the exponent of $f(x_v)$ is non-positive, so the same bounds on $f(x)$ and Z flip the inequality above to give item (v). Item (vi) follows by combining items (iv) and (v) when G' is left d-regular.

For items (i) and (ii), since G is left d-regular, a calculation analogous to the one in (8) for $t(F,W')(x)$ has the exponent of Z being 1 (since the labeled vertex is not integrated out) and exponents of all $f(x_v)$ being 0 except for the one corresponding to the labeled vertex, which has exponent -1 instead (as the labeled vertex is not integrated), so we get

$$t(F,W')(x) = \frac{Z}{f(x)} \cdot t(F,W)(x) \leq Z \leq (1+\epsilon)\cdot t(G,W),$$

where the inequalities follow from (6) and the upper bound in (7), respectively. Thus, item (i) follows.

On the other hand, by using the full definition of $f(x)$ from (6) and the lower bound in (7) instead, we get

$$t(F,W')(x) \geq \min\left\{t(G,W), \frac{t(F,W)(x)}{\epsilon}\right\},$$

and since $t(G,W) = t(G,W')$ by item (vi), we conclude that item (ii) holds.

Finally, for item (iii), since G' is left d-regular, a calculation analogous to the one in (8) has exponents of Z and the $f(x_v)$ all zero (as the labeled vertex is on the right), so we get $t(F',W')(y) = t(F',W)(y)$. □

5 Biregularity

In this section we prove our biregularity result, Theorem 3.1. Let us first extract from Lemma 4.2 its partial case $d = 1$, $F = e_1$, $F' = e_2$ needed for that purpose.

Corollary 5.1. *For every $\epsilon > 0$ and every bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$, there exists a bigraphon $W'\colon \Omega' \times \Lambda \to \mathbb{R}_+$ such that the following hold.*

i. *We have $\Delta(e_1, W') \leq (1 + \epsilon) \cdot t(\rho, W')$.*

ii. *We have*

$$\delta(e_1, W') \geq \min\left\{t(\rho, W'), \frac{\delta(e_1, W)}{\epsilon}\right\}.$$

iii. *For every $y \in Y$, $t(e_2, W')(y) = t(e_2, W)(y)$. Therefore, $t(\rho, W') = t(\rho, W)$.*

iv. *For every bigraph G we have*

$$t(G, W') \leq \left(1 + \frac{1}{\epsilon}\right)^{e(G)} \cdot t(G, W).$$

Proof. The only thing to be explained here is the absence of the condition $\delta_1(G) \geq 1$ in item (iv) (that corresponds to item (v) in Lemma 4.2). This is simply because removing all isolated vertices in $V_1(G)$ does not change any of the three quantities in this inequality. □

As we will see below, Corollary 5.1 will take care of all steps in our program, except for $W_2 \implies W_3$. This remaining step is performed by Lemma 5.2, which can be seen as a limit, non-symmetric version of the argument in [CKLL18b, Lemma 3.4].

Lemma 5.2. *If $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ is a bigraphon over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ such that*

$$\max\{\Delta(e_1, W), \Delta(e_2, W)\} \leq 2 \cdot t(\rho, W), \tag{9}$$

then there exists a bigraphon $W'\colon \Omega' \times \Lambda' \to \mathbb{R}_+$ such that the following hold.

i. *We have $t(\rho, W') = t(\rho, W)$.*

ii. *We have $\min\{\delta(e_1, W'), \delta(e_2, W')\} \geq 2^{-10} \cdot t(\rho, W')$.*

iii. For every bigraph G, we have $t(G, W') \leq 2^{3v(G)+e(G)} \cdot t(G, W)$.

Proof. If $t(\rho, W) = 0$, then we can simply take $\Omega' \stackrel{\text{def}}{=} \Omega$, $\Lambda' \stackrel{\text{def}}{=} \Lambda$ and $W' \stackrel{\text{def}}{=} W$, so suppose $t(\rho, W) > 0$.

Without loss of generality, let us assume that the probability spaces Ω and Λ are atomless (if not, we can simply replace each atom of the space by a copy of an interval of appropriate length equipped with Lebesgue measure).

Let $\alpha \in (0, 1)$, to be specified later. We define sequences of measurable sets $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ and $Y = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ using the following algorithm.

1. Let $X_0 \stackrel{\text{def}}{=} X$ and $Y_0 \stackrel{\text{def}}{=} Y$.
2. Given X_i and Y_i,
   a. if there exists a measurable $R_i^1 \subseteq X_i$ with $\mu|_{X_i}(R_i^1) = \alpha$ and $t(e_1, W|_{X_i \times Y_i})(x) < t(\rho, W|_{X_i \times Y_i})/10$ for every $x \in R_i^1$, then let $X_{i+1} \stackrel{\text{def}}{=} X_i \setminus R_i^1$ and $Y_{i+1} \stackrel{\text{def}}{=} Y_i$;
   b. otherwise, if there exists a measurable $R_i^2 \subseteq Y_i$ with $\nu|_{Y_i}(R_i^2) = \alpha$ and $t(e_2, W|_{X_i \times Y_i})(y) < t(\rho, W|_{X_i \times Y_i})/10$ for every $y \in R_i^2$, then let $X_{i+1} \stackrel{\text{def}}{=} X_i$ and $Y_{i+1} \stackrel{\text{def}}{=} Y_i \setminus R_i^2$;
   c. otherwise, stop the construction.

The first order of business is to show that the construction above stops in finitely many steps. To do so, note first that if X_i gets changed at some stage i, then we have

$$\begin{aligned} t(\rho, W|_{X_{i+1} \times Y_{i+1}}) &= \frac{t(\rho, W|_{X_i \times Y_i}) - \int_{R_i^1} t(e_1, W|_{X_i \times Y_i})(x)\ d\mu|_{X_i}(x)}{\mu|_{X_i}(X_{i+1})} \\ &\geq \frac{1 - \alpha/10}{1 - \alpha} \cdot t(\rho, W|_{X_i \times Y_i}). \end{aligned}$$

By symmetry, the same conclusion holds when Y_i gets changed. Thus, by induction, we conclude that whenever the algorithm proceeds to the ith stage, we have

$$t(\rho, W|_{X_i \times Y_i}) \geq \left(\frac{1 - \alpha/10}{1 - \alpha}\right)^i t(\rho, W) \geq t(\rho, W). \tag{10}$$

Since $t(\rho, W|_{X_i \times Y_i}) \leq \|W\|_\infty$, it follows that the construction indeed must halt in finitely many steps. Let i_0 be the step at which this happens. The heart of the whole argument is to bound i_0 as *a function of α only*; more specifically, we are going to prove that

$$i_0 \leq \frac{1}{\log_2(1-\alpha/10) - \log_2 \sqrt{1-\alpha}}. \tag{11}$$

Towards that end, let us define

$$D_i \stackrel{\text{def}}{=} \Delta(e_1, W|_{X_i \times Y_i}) \cdot \Delta(e_2, W|_{X_i \times Y_i}).$$

We claim that for every $i \in \{0, \ldots, i_0\}$, we have

$$D_i \leq \frac{4t(\rho, W)^2}{(1-\alpha)^i};$$

we prove this by induction on i.

For $i = 0$ this immediately follows from the assumption (9).

For the inductive step, if the bigraphon gets decreased by removing R_i^j ($j = 1, 2$) then we have

$$\Delta(e_j, W|_{X_{i+1} \times Y_{i+1}}) \leq \Delta(e_j, W|_{X_i \times Y_i});$$
$$\Delta(e_{3-j}, W|_{X_{i+1} \times Y_{i+1}}) \leq \frac{\Delta(e_{3-j}, W|_{X_i \times Y_i})}{1-\alpha}.$$

This completes the inductive step.

We now conclude that

$$\begin{aligned} t(\rho, W|_{X_{i_0} \times Y_{i_0}}) &\leq \min\{\Delta(e_1, W|_{X_{i_0} \times Y_{i_0}}), \Delta(e_2, W|_{X_{i_0} \times Y_{i_0}})\} \\ &\leq \sqrt{D_{i_0}} \leq \frac{2}{(1-\alpha)^{i_0/2}} \cdot t(\rho, W). \end{aligned}$$

Comparing this with the bound (10) gives us (11), as desired.

Note also that a simple induction gives

$$\min\{\mu(X_{i_0}), \nu(Y_{i_0})\} \geq (1-\alpha)^{i_0}. \tag{12}$$

Let

$$X' \stackrel{\text{def}}{=} \{x \in X_{i_0} \mid t(e_1, W|_{X_{i_0} \times Y_{i_0}})(x) \geq t(\rho, W|_{X_{i_0} \times Y_{i_0}})/10\},$$
$$Y' \stackrel{\text{def}}{=} \{y \in Y_{i_0} \mid t(e_2, W|_{X_{i_0} \times Y_{i_0}})(y) \geq t(\rho, W|_{X_{i_0} \times Y_{i_0}})/10\}$$

and note that since the probability spaces Ω and Λ are atomless, we must have $\mu|_{X_{i_0}}(X') \geq 1-\alpha$ and $\nu|_{Y_{i_0}}(Y') \geq 1-\alpha$ since otherwise we could have continued with the algorithm. Together with (11) and (12), this gives

$$\min\{\mu(X'), \nu(Y')\} \geq (1-\alpha)^{1+1/(\log_2(1-\alpha/10)-\log_2\sqrt{1-\alpha})}.$$

Let $M = M(\alpha)$ be the right-hand side of the above; a straightforward calculation shows that

$$\lim_{\alpha\to 0} M(\alpha) = \frac{\sqrt{2}}{8}. \tag{13}$$

Now, let $\widehat{W} \stackrel{\text{def}}{=} W|_{X'\times Y'}$. Note that for every bigraph G, we have

$$t(G, \widehat{W}) \leq \frac{t(G, W)}{\mu(X')^{v_1(G)} \cdot \nu(Y')^{v_2(G)}} \leq \frac{t(G, W)}{M^{v(G)}}. \tag{14}$$

Note also that by (10),

$$\begin{aligned} t(\rho, \widehat{W}) \geq\ & t(\rho, W|_{X_{i_0}\times Y_{i_0}}) \\ & - \int_{X_{i_0}\setminus X'} t(e_1, W|_{X_{i_0}\times Y_{i_0}})(x)\ d\mu|_{X_{i_0}}(x) \\ & - \int_{Y_{i_0}\setminus Y'} t(e_2, W|_{X_{i_0}\times Y_{i_0}})(y)\ d\nu|_{Y_{i_0}}(y) \\ \geq\ & \left(1-\frac{\alpha}{5}\right) \cdot t(\rho, W|_{X_{i_0}\times Y_{i_0}}) \geq \frac{t(\rho, W)}{2}. \end{aligned} \tag{15}$$

Finally, from the definition of X' and Y', we have

$$\begin{aligned} \min\{\delta(e_1, \widehat{W}), \delta(e_2, \widehat{W})\} & \geq \frac{t(\rho, W|_{X_{i_0}\times Y_{i_0}})}{10} - \alpha \cdot \|W\|_\infty \\ & \geq \frac{t(\rho, W)}{10} - \alpha \cdot \|W\|_\infty, \end{aligned} \tag{16}$$

where the first inequality follows from $\mu(X_{i_0} \setminus X'), \nu(Y_{i_0} \setminus Y') < \alpha$ and the second inequality again follows from (10).

By (13), if we choose $\alpha \in (0, 1)$ small enough then

$$M(\alpha) \geq \frac{1}{8}, \qquad \frac{1}{10} - \alpha \cdot \frac{\|W\|_\infty}{t(\rho, W)} \geq \frac{1}{16}. \tag{17}$$

Finally, define

$$W' \stackrel{\text{def}}{=} \frac{t(\rho, W)}{t(\rho, \widehat{W})} \cdot \widehat{W}$$

so that item (i) follows trivially. Since $t(\rho, \widehat{W}) \geq t(\rho, W)/2$, the first condition in (17) along with (14) gives

$$t(G, W') \leq 2^{e(G)} \cdot t(G, \widehat{W}) \leq 2^{3v(G)+e(G)} \cdot t(G, W), \tag{18}$$

and item (iii) also follows.

For item (ii), note that (16) and the second condition in (17) imply

$$\begin{aligned}
\min\{\delta(e_1, W'), \delta(e_2, W')\} &= \frac{t(\rho, W)}{t(\rho, \widehat{W})} \cdot \min\{\delta(e_1, \widehat{W}), \delta(e_2, \widehat{W})\} \\
&\geq \frac{t(\rho, W)}{t(\rho, \widehat{W})} \cdot \frac{t(\rho, W)}{16} \\
&\geq \frac{t(\rho, W)}{t(\rho, \widehat{W})} \cdot 2^{-10} \cdot t(\rho, \widehat{W}) \\
&= 2^{-10} \cdot t(\rho, W'),
\end{aligned}$$

where the last inequality follows from (18). □

Proof of Theorem 3.1. We make the constructions $W \Longrightarrow W_1 \Longrightarrow W_2 \Longrightarrow W_3 \Longrightarrow W_4 \Longrightarrow W_5$, where the first two arrows are applications of Corollary 5.1 and its dual, respectively, with $\epsilon = 1$, the third arrow is an application of Lemma 5.2 and the last two arrows are applications of Corollary 5.1 and its dual, respectively, with $\epsilon = 2^{-10}$.

Checking all necessary conditions is straightforward, the only thing worth noticing is (bi)regularity of W_4 and W_5. It is implied by the following computation on the base of items (ii) and (iii) in Corollary 5.1:

$$\delta(e_1, W_5) = \delta(e_1, W_4) \geq \min\left\{t(\rho, W_4), \frac{\delta(e_1, W_3)}{2^{-10}}\right\} = t(\rho, W_5),$$

$$\delta(e_2, W_5) \geq \min\left\{t(\rho, W_5), \frac{\delta(e_2, W_4)}{2^{-10}}\right\} \geq \min\left\{t(\rho, W_5), \frac{\delta(e_2, W_3)}{2^{-10}}\right\} = t(\rho, W_5).$$

We also have the following chain of inequalities.

$$\begin{aligned}
t(G,W) &\geq 2^{-e(G)} \cdot t(G,W_1) \geq 2^{-2e(G)} \cdot t(G,W_2) \\
&\geq 2^{-3v(G)-3e(G)} \cdot t(G,W_3) \geq 1025^{-e(G)} \cdot 2^{-3v(G)-3e(G)} \cdot t(G,W_4) \\
&\geq 1025^{-2e(G)} \cdot 2^{-3v(G)-3e(G)} \cdot t(G,W_5) \\
&\geq 1025^{-2e(G)} \cdot 2^{-3v(G)-3e(G)} \cdot c_G \cdot t(\rho,W_5)^{e(G)} \\
&= 1025^{-2e(G)} \cdot 2^{-3v(G)-3e(G)} \cdot c_G \cdot t(\rho,W)^{e(G)}.
\end{aligned}$$

Therefore, G is a Sidorenko bigraph by Lemma 4.1. □

Theorem 3.1 has the following simple but very useful corollary (that of course can be extracted already from the approximate version in [CKLL18b]).

Corollary 5.3. *If v is a vertex of degree 1 in a bigraph G, then G is a Sidorenko bigraph if and only if $G - v$ is a Sidorenko bigraph.*

Proof. Follows from Theorem 3.1 and the fact that in a biregular bigraphon W we have $t(G,W) = t(G-v,W) \cdot t(\rho,W)$. □

6 1-flags and d-stars

In this section we show how Theorem 3.1, along with Lemma 4.2, yields Theorems 3.2, 3.3 and 3.4.

Proof of Theorem 3.2. Let $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ be a biregular bigraphon over spaces $\Omega = (X,\mu)$ and $\Lambda = (Y,\nu)$ and for each $i \in [2]$, let $f_i \in \mathbb{R}_+$ be such that

$$\mu(\{x \in X \mid t(F_i,W)(x) < f_i\}) \leq \frac{1}{3},$$
$$\mu(\{x \in X \mid t(F_i,W)(x) \leq f_i\}) \geq \frac{1}{3}.$$

Let $U_i \stackrel{\text{def}}{=} \{x \in X \mid t(F_i,W)(x) \leq f_i\}$ (so that $\mu(U_i) \geq 1/3$) and let $W_i \stackrel{\text{def}}{=} W|_{U_i \times Y}$.

Since W is left regular, it follows that W_i is also left regular and hence $t(\rho,W) = t(\rho,W_i)$. On the other hand, since $|F_i|$ is a Sidorenko bigraph, we have

$$\begin{aligned} t(\rho, W)^{e(|F_i|)} &= t(\rho, W_i)^{e(|F_i|)} \leq t(|F_i|, W_i) \\ &\leq \frac{1}{\mu(U_i)^{v_1(|F_i|)}} \int_{U_i} t(F_i, W)(x)\ d\mu(x) \leq \frac{f_i}{\mu(U_i)^{v_1(|F_i|)-1}} \\ &\leq 3^{v_1(|F_i|)-1} \cdot f_i. \end{aligned}$$

Let now $X' \stackrel{\text{def}}{=} \{x \in X \mid t(F_1, W)(x) \geq f_1 \land t(F_2, W)(x) \geq f_2\}$ and note that $\mu(X') \geq 1/3$, so we have

$$\begin{aligned} t(|F_1 \sqcup F_2|, W) &\geq \int_{X'} t(F_1, W)(x) \cdot t(F_2, W)(x)\ d\mu(x) \geq \frac{1}{3} \cdot f_1 \cdot f_2 \\ &\geq \frac{1}{3^{v_1(|F_1|)+v_1(|F_2|)-1}} \cdot t(\rho, W)^{e(|F_1|)+e(|F_2|)} \\ &= \frac{1}{3^{v_1(|F_1 \sqcup F_2|)}} \cdot t(\rho, W)^{e(|F_1 \sqcup F_2|)}. \end{aligned}$$

Hence $|F_1 \sqcup F_2|$ is a Sidorenko bigraph by Theorem 3.1. □

Proof of Theorem 3.3. The forward direction follows by inductive application of Theorem 3.2.

For the reverse direction, let us first prove the case in which $|F|$ is left d-regular for some $d \in \mathbb{N}_+$. Given a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$, we apply Lemma 4.2 with $\epsilon = 1$ to get a bigraphon W' such that $\Delta(F, W') \leq 2 \cdot t(|F|, W') = 2 \cdot t(|F|, W)$. In particular, we have

$$t(|F^{\sqcup k}|, W') = \int_X t(F, W')(x)^k\ d\mu(x) \leq 2^k \cdot t(|F|, W)^k.$$

Since $|F^{\sqcup k}|$ is a Sidorenko bigraph, we conclude

$$t(|F|, W) \geq \frac{1}{2} \cdot t(|F^{\sqcup k}|, W')^{1/k} \geq \frac{1}{2} \cdot t(\rho, W')^{e(|F^{\sqcup k}|)/k} = \frac{1}{2} \cdot t(\rho, W')^{e(|F|)},$$

so $|F|$ is a Sidorenko bigraph by Lemma 4.1.

Let us now show the general case. Let $d \stackrel{\text{def}}{=} \Delta_1(|F|)$ and let $\widehat{F}$ be the flag obtained from F by adding $d \cdot v_1(|F|) - e(|F|)$ vertices to $V_2(|F|)$ and connecting each of these newly added vertices to a single left vertex so that $|\widehat{F}|$ is left d-regular. By repeated application of Corollary 5.3, $|F|$ is a Sidorenko bigraph if and only if $|\widehat{F}|$ is so.

Note now that even though $|\widehat{F}^{\sqcup k}|$ is not left regular, it can also be obtained from $F^{\sqcup k}$ by adding vertices to $V_2(F^{\sqcup k})$ and connecting each of them to a single left vertex. Again, by Corollary 5.3, $|F^{\sqcup k}|$ is a Sidorenko bigraph if and only if $|\widehat{F}^{\sqcup k}|$ is so. Since $|\widehat{F}|$ is left d-regular, the result now follows from the previous case. □

To prove Theorem 3.4, we will use steps similar to Theorem 3.1 with the differences that this time we are only interested in the left regularity, and we focus on $K_{1,d}^L$ rather than on e_1 so that in particular the density of $K_{1,d}$ will be preserved throughout our transformations. Fortunately, since now we are only concerned with the left side, the analogue of the crucial Lemma 5.2 is much easier to prove.

Lemma 6.1. *If $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ is a bigraphon over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ such that $\Delta(K_{1,d}^L, W) \leq 2 \cdot t(K_{1,d}, W)$, then there exists a bigraphon $W'\colon \Omega' \times \Lambda \to \mathbb{R}_+$ such that the following hold.*

i. We have $t(K_{1,d}, W') = t(K_{1,d}, W)$.

ii. We have $\delta(K_{1,d}^L, W') \geq t(K_{1,d}, W')/6$.

iii. For every bigraph G, we have $t(G, W') \leq 3^{v_1(G)} \cdot t(G, W)$.

Proof. If $t(K_{1,d}, W) = 0$, we can simply take $W' \overset{\text{def}}{=} W$, so suppose $t(K_{1,d}, W) > 0$. Let

$$\widehat{X} \overset{\text{def}}{=} \{x \in X \mid t(K_{1,d}^L, W)(x) \geq t(K_{1,d}, W)/2\}$$

and note that since $\Delta(K_{1,d}^L, W) \leq 2 \cdot t(K_{1,d}, W)$, Markov's Inequality implies $\mu(\widehat{X}) \geq 1/3$.

Let $\widehat{W} \overset{\text{def}}{=} W|_{\widehat{X} \times Y}$ and note that

$$\begin{aligned} t(G, \widehat{W}) &\leq \frac{1}{\mu(\widehat{X})^{v_1(G)}} \cdot t(G, W) \leq 3^{v_1(G)} \cdot t(G, W); \\ t(K_{1,d}, \widehat{W}) &\geq t(K_{1,d}, W); \\ \delta(K_{1,d}^L, \widehat{W}) &\geq \frac{1}{2} \cdot t(K_{1,d}, W) \geq \frac{1}{6} \cdot t(K_{1,d}, \widehat{W}). \end{aligned}$$

Thus, defining $W' \overset{\text{def}}{=} (t(K_{1,d}, W)/t(K_{1,d}, \widehat{W}))^{1/d} \cdot \widehat{W}$ gives the desired result. □

Proof of Theorem 3.4. We make the constructions $W \Longrightarrow W_1 \Longrightarrow W_2 \Longrightarrow W_3$, where the first and third arrows are applications of Lemma 4.2 both with $F = K_{1,d}^L$ but with $\epsilon = 1$ and $\epsilon = 1/6$, respectively, and the second arrow is an application of Lemma 6.1.

Our constructions ensure that $t(K_{1,d}, W_i) = t(K_{1,d}, W)$ for every $i \in [3]$. Note also that

$$\delta(K_{1,d}^L, W_3) \geq \min\left\{t(K_{1,d}, W_3), \frac{\delta(K_{1,d}^L, W_2)}{1/6}\right\} = t(K_{1,d}, W_3),$$

so W_3 is $K_{1,d}^L$-regular. Since $t(e_1, W_3)(x) = t(K_{1,d}^L, W_3)(x)^{1/d}$ for every x, it follows that W_3 is left regular, which in particular implies $t(K_{1,d}, W_3) = t(\rho, W_3)^d$.

Then we can deduce the following chain of inequalities.

$$\begin{aligned}
t(G, W) &\geq 2^{v_1(G)-e(G)/d} \cdot t(G, W_1) \\
&\geq 2^{v_1(G)-e(G)/d} \cdot 3^{-v_1(G)} \cdot t(G, W_2) \\
&\geq 2^{v_1(G)-e(G)/d} \cdot 3^{-v_1(G)} \cdot 7^{v_1(G)-e(G)/d} \cdot t(G, W_3) \\
&\geq 2^{v_1(G)-e(G)/d} \cdot 3^{-v_1(G)} \cdot 7^{v_1(G)-e(G)/d} \cdot t(\rho, W_3)^{e(G)} \\
&= 2^{v_1(G)-e(G)/d} \cdot 3^{-v_1(G)} \cdot 7^{v_1(G)-e(G)/d} \cdot t(K_{1,d}, W_3)^{e(G)/d} \\
&= 2^{v_1(G)-e(G)/d} \cdot 3^{-v_1(G)} \cdot 7^{v_1(G)-e(G)/d} \cdot t(K_{1,d}, W)^{e(G)/d}.
\end{aligned}$$

Since this is true for every bigraphon W, by Lemma 4.1 we conclude that $t(G, W) \geq t(K_{1,d}, W)^{e(G)/d}$, again for every W. □

7 Reflective tree decompositions

In this section we prove Theorem 3.5 on reflective tree decompositions.

Proof of Theorem 3.5. Let $\mathcal{W}$ be the class of biregular bigraphons that are bounded away from zero. We claim that it is sufficient to show that

$$\frac{t(G, W)}{t(\rho, W)^{e(G)}} \geq \frac{t(H, W)}{t(\rho, W)^{e(H)}} \tag{19}$$

only for $W \in \mathcal{W}$. Indeed, if W is an arbitrary non-zero biregular bigraphon, then it can be approximated by $W_\epsilon \stackrel{\text{def}}{=} W + \epsilon \in \mathcal{W}$ ($\epsilon > 0$) and (19) for

W follows by applying the Dominated Convergence Theorem to the same inequality for W_ϵ as $\epsilon \to 0$.

Since $\mathcal{W}$ is closed under tensor powers, by Lemma 4.1, it is enough to prove (19) for every $W \in \mathcal{W}$ up to a multiplicative constant that does not depend on W.

For convenience of notation, by possibly replacing $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ with $W'\colon (\Omega \times \Lambda) \times (\Omega \times \Lambda) \to \mathbb{R}_+$ given by $W'((x_1, y_1), (x_2, y_2)) = W(x_1, y_2)$, we may assume that $\Omega = \Lambda$ (but W is still *not* necessarily symmetric!).

Given a subtree T' of T, let $V_{T'} \stackrel{\text{def}}{=} \bigcup_{U\in V(T')} U$, $G_{T'} \stackrel{\text{def}}{=} G|_{V_{T'}}$ and

$$d_{T'} \stackrel{\text{def}}{=} e(G_{T'}) - \sum_{\{U_1,U_2\}\in E(T')} e(|C_2(F'_{U_1U_2})|),$$

where $F'_{U_1U_2} \stackrel{\text{def}}{=} (G|_{U_1} - E(G|_{U_1\cap U_2}), U_1 \cap U_2)$ is as in Remark 3, which guarantees that the summand does not depend on the orientation of the edge $\{U_1, U_2\}$.

Given further a bigraphon $W\colon \Omega\times\Omega \to \mathbb{R}_+$ in $\mathcal{W}$ over a space $\Omega = (X, \mu)$, let $f_{T'}\colon \Omega^{V_{T'}} \to \mathbb{R}_+$ be given by

$$f_{T'}(x) \stackrel{\text{def}}{=} \frac{t((G_{T'}, V_{T'}), W)(x)}{\prod_{\{U_1,U_2\}\in E(T')} t(C_2(F'_{U_1U_2}), W)(x_{U_1\cap U_2})}.$$

Let us remark that the flag $(G_{T'}, V_{T'})$ is trivial (totally labeled) hence the expression $t((G_{T'}, V_{T'}), W)(x)$ is simply equal to $\prod_{(v,w)\in E(G_{T'})} W(x_v, x_w)$, where we assume that $v \in V_1,\ w \in V_2$ and no integration takes place. For the sake of uniformity, however, we stick to the former notation.

Note that since W is bounded away from zero, all functions $f_{T'}$ are bounded.

Claim 7.1. *For every $U_0 \in V(T)$ and every $x \in X^{U_0}$, we have*

$$\int_{X^{V(G)\setminus U_0}} f_T(x, x')\ d\mu(x') = t(\rho, W)^{d_T - e(G|_{U_0})} \cdot t((G|_{U_0}, U_0), W)(x). \quad (20)$$

Proof. We will show by induction on $v(T) - v(T')$ that if T' is a subtree of T with $U_0 \in V(T')$, then

$$\int_{X^{V(G)\setminus U_0}} f_T(x, x')\ d\mu(x') = t(\rho, W)^{d_T - d_{T'}} \cdot \int_{X^{V_{T'}\setminus U_0}} f_{T'}(x, x')\ d\mu(x'). \quad (21)$$

Once this is proved then (20) follows by taking T' as the subtree of T with $V(T') = \{U_0\}$.

If $T' = T$, then (21) holds trivially. If T' is a proper subtree of T containing U_0, then let T'' be a subtree of T containing T' as a subtree and having exactly one more vertex U_1 than T', which must necessarily be a leaf of T'', so we can let U_2 be its unique neighbor in T'. By inductive hypothesis, we have

$$\int_{X^{V(G)\setminus U_0}} f_T(x, x')\ d\mu(x') = t(\rho, W)^{d_T - d_{T''}} \cdot \int_{X^{V_{T''}\setminus U_0}} f_{T''}(x, x')\ d\mu(x').$$

But note that in the expression for $f_{T''}(x, x')$, variables indexed by $U_1 \setminus V_{T'}$ appear only in the numerator, so integrating these in the right-hand side of the above gives

$$\begin{aligned}&\int_{X^{V(G)\setminus U_0}} f_T(x, x')\ d\mu(x')\\ &\quad = t(\rho, W)^{d_T - d_{T''}} \cdot \int_{X^{V_{T'}\setminus U_0}} f_{T'}(x, x') \cdot \frac{t(F'_{U_1U_2}, W)(x_{U_1\cap U_2})}{t(C_2(F'_{U_1U_2}), W)(x_{U_1\cap U_2})}\ d\mu(x').\end{aligned}$$

Since W is biregular, the fraction under the integral is equal to $t(\rho, W)^{e(|F'_{U_1U_2}|) - e(|C_2(F'_{U_1U_2})|)}$, and (21) follows. □

Let now

$$Z \stackrel{\text{def}}{=} \int_{X^{V(G)}} f_T(x)\ d\mu(x).$$

By picking $U_0 \in V(T)$ arbitrarily and integrating (20) over x, we similarly get

$$Z = t(\rho, W)^{d_T - e(H)} \cdot t(H, W). \tag{22}$$

We now let η be the probability measure such that $d\eta(x) = (f_T(x)/Z)\ d\mu(x)$ and for each $\{U_1, U_2\} \in E(T)$, we let

$$\begin{aligned}D_{U_1U_2} &\stackrel{\text{def}}{=} \Bigg\{x \in X^{U_1\cap U_2} \ \Bigg|\ t(C_2(F'_{U_1U_2}), W)(x)\\ &\qquad \le \frac{t(H, W)}{2v(T) \cdot t(G|_{U_1\cap U_2}, W) \cdot t(\rho, W)^{e(H) - e(G|_{U_1\cap U_2}) - e(|C_2(F'_{U_1U_2})|)}}\Bigg\};\\ D'_{U_1U_2} &\stackrel{\text{def}}{=} \{x \in X^{V(G)} \mid x_{U_1\cap U_2} \in D_{U_1U_2}\}.\end{aligned}$$

Then we have

$$\begin{aligned}
\eta(D'_{U_1U_2}) &= \frac{1}{Z}\int_{X^{V(G)}} \mathbb{1}[x_{U_1\cap U_2}\in D_{U_1U_2}]\cdot f_T(x)\ d\mu(x)\\
&= \frac{t(\rho,W)^{e(H)-e(G|_{U_1})}}{t(H,W)}\int_{X^{U_1}} \mathbb{1}[x_{U_1\cap U_2}\in D_{U_1U_2}]\cdot t((G|_{U_1},U_1),W)(x)\ d\mu(x)\\
&= \frac{t(\rho,W)^{e(H)-e(G|_{U_1})+e(|F'_{U_1U_2}|)-e(|C_2(F'_{U_1U_2})|)}}{t(H,W)}\\
&\qquad\cdot\int_{D_{U_1U_2}} t((G|_{U_1\cap U_2},U_1\cap U_2),W)(x)\cdot t(C_2(F'_{U_1U_2}),W)(x)\ d\mu(x)\\
&\leq \frac{1}{2v(T)},
\end{aligned}$$

where the second equality follows from Claim 7.1 with $U_0=U_1$ and (22), the third equality follows since W is biregular and the inequality follows from the definition of $D_{U_1U_2}$ and the fact that $e(G|_{U_1})-e(|F'_{U_1U_2}|)=e(G|_{U_1\cap U_2})$.

Define then $D\stackrel{\text{def}}{=}X^{V(G)}\setminus\bigcup_{\{U_1,U_2\}\in E(T)}D'_{U_1U_2}$ and note that

$$\eta(D)\geq 1-\frac{e(T)}{2v(T)}\geq\frac{1}{2}.$$

We have

$$\begin{aligned}
t(G,W) &= Z\int_{X^{V(G)}}\prod_{\{U_1,U_2\}\in E(T)} t(C_2(F'_{U_1U_2}),W)(x_{U_1\cap U_2})\ d\eta(x)\\
&\geq Z\cdot\eta(D)\cdot\prod_{\{U_1,U_2\}\in E(T)}\frac{t(H,W)}{2v(T)\cdot t(G|_{U_1\cap U_2},W)\cdot t(\rho,W)^{e(H)-e(G|_{U_1\cap U_2})-e(|C_2(F'_{U_1U_2})|)}}\\
&\geq t(\rho,W)^{d_T-e(H)}\cdot t(H,W)\cdot\frac{\eta(D)}{(2v(T))^{e(T)}}\prod_{\{U_1,U_2\}\in E(T)} t(\rho,W)^{e(|C_2(F'_{U_1U_2})|)}\\
&\geq\frac{1}{2^{e(T)+1}\cdot v(T)^{e(T)}}\cdot t(\rho,W)^{e(G)-e(H)}\cdot t(H,W),
\end{aligned}$$

where the second inequality follows from (22) and since H weakly dominates each $G|_{U_1\cap U_2}$. Therefore (19) holds by Lemma 4.1, so G weakly dominates H.

Finally, if further H is a Sidorenko bigraph, then by Theorem 3.1, G must also be a Sidorenko bigraph as it weakly dominates H. □

8 The symmetric setting

In this section, we briefly sketch how to adapt the results from Sections 4, 5 and 7 to the symmetric setting. First we note that the tensor power trick of Lemma 4.1 still holds in the symmetric setting. For Lemma 4.2, we need to make some adjustments.

Lemma 8.1 (Symmetric version of Lemma 4.2)**.** *Let $d \in \mathbb{N}_+$, let $F = (G, \theta)$ be a left 1-flag such that G is both left and right d-regular and let $\epsilon > 0$.*

Then for every graphon $W\colon \Omega \times \Omega \to \mathbb{R}_+$ over $\Omega = (X, \mu)$, there exists a graphon $W'\colon \Omega' \times \Omega' \to \mathbb{R}_+$ such that the following hold.

i. We have $\Delta(F, W') \leq (1 + \epsilon) \cdot t(G, W')$.

ii. We have

$$\delta(F, W') \geq \min\left\{t(G, W'), \frac{\delta(F, W)}{\epsilon}\right\}.$$

iii. For every bigraph G' with $\max\{\Delta_1(G'), \Delta_2(G')\} \leq d$, we have

$$t(G', W') \geq \left(1 + \frac{1}{\epsilon}\right)^{2e(G')/d - v(G')} \cdot t(G', W).$$

iv. For every bigraph G' with $\min\{\delta_1(G'), \delta_2(G')\} \geq d$, we have

$$t(G', W') \leq \left(1 + \frac{1}{\epsilon}\right)^{2e(G')/d - v(G')} \cdot t(G', W).$$

v. For every bigraph G' that is both left and right d-regular, we have $t(G', W') = t(G', W)$.

Proof (sketch). Analogous to that of Lemma 4.2 but using the definition

$$W'(x, y) \stackrel{\text{def}}{=} \left(\frac{Z^2}{f(x)f(y)}\right)^{1/d} \cdot W(x, y)$$

that ensures that W' is symmetric. □

Even though it is possible to adapt the proof of Lemma 5.2 to the symmetric setting, we can instead simply use the finite version [CKLL18b, Lemma 3.4] that inspired it to prove the symmetric version of the biregularity result, Theorem 3.1.

Theorem 8.2. *Let G be a bigraph. If there exists $c_G > 0$ such that $t(G, W) \geq c_G \cdot t(\rho, W)^{e(G)}$ for every regular graphon W, then G is a symmetrically Sidorenko bigraph.*

Proof (sketch). By [CKLL18b, Lemmas 3.3 and 3.4], it is sufficient to show that $t(G, H) \geq c'_G \cdot t(\rho, H)^{e(G)}$ for some constant $c'_G > 0$ depending only on G and every *graph* H whose degrees are all between $d_{\mathrm{ave}}(H)/8$ and $2 \cdot d_{\mathrm{ave}}(H)$, where $d_{\mathrm{ave}}(H)$ is the average degree of H. By considering the step graphon associated with H, it follows that it is sufficient to prove that $t(G, W) \geq c'_G \cdot t(\rho, W)^{e(G)}$ for every graphon W such that

$$\frac{t(\rho, W)}{8} \leq \delta_1(e_1, W) \leq \Delta_1(e_1, W) \leq 2 \cdot t(\rho, W).$$

We then apply Lemma 8.1 with $\epsilon = 1/8$ and $F = e_1$ to get a graphon W' that satisfies

$$\delta_1(e_1, W') \geq \min\left\{t(\rho, W'), \frac{\delta(e_1, W)}{1/8}\right\} = t(\rho, W),$$

that is W' is regular. Hence,

$$\begin{aligned} t(G, W) &\geq 9^{2e(G)-v(G)} \cdot t(G, W') \geq 9^{2e(G)-v(G)} \cdot c_G \cdot t(\rho, W')^{e(G)} \\ &\geq 9^{2e(G)-v(G)} \cdot c_G \cdot t(\rho, W)^{e(G)}, \end{aligned}$$

so G is symmetrically Sidorenko. □

Finally, the symmetric analogue of Theorem 3.5 (i.e., once we also replace weak domination by its symmetric version) can be shown with the same proof, replacing Theorem 3.1 with Theorem 8.2 for the final statement on symmetrically Sidorenko bigraphs.

9 Conclusion and open problems

In this paper, we have shown how to reduce Sidorenko's Conjecture to biregular bigraphons (or regular graphons in the symmetric case). We have also shown that this reduction and the construction of Lemma 4.2 can be used to obtain simple proofs of some properties of the class of Sidorenko bigraphs.

The proofs in Section 6 heavily rely on the fact that the two sides of the bigraphs and bigraphons can be manipulated independently. It is then natural to ask if Theorem 3.4 holds in the symmetric setting as well (the symmetric analogues of Theorem 3.2 and 3.3 follow from a symmetric analogue of [Sze15b, Theorem 4]).

In another direction, Conlon–Kim–Lee–Lee [CKLL18a] also provided a higher-order version of their strong tree decompositions, which is reminiscent (but yields a completely different class of symmetrically Sidorenko bigraphs) of Szegedy's conditionally independent coupling constructions [Sze15a]. While we believe that a higher-order version of the reflective tree decompositions result should also hold (more specifically by using the same definition of higher-order decompositions and simply replacing level 0 with reflective tree decompositions), these higher-order decompositions have the restriction that $G|_{U_1 U_2}$ is a forest for each $\{U_1, U_2\} \in E(T)$ and we would like to ask instead if this restriction can be replaced by some weak domination restriction as in reflective tree decompositions. One stepping stone toward this goal is the following natural generalization of Theorem 3.5.

Conjecture 1. If T is a reflective tree decomposition of a connected non-trivial bigraph G whose core H weakly dominates $G|_{U_1 \cap U_2}$ for every $\{U_1, U_2\} \in E(T)$, then for every non-empty $V \subseteq V(T)$, G weakly dominates $G|_{\bigcup_{U \in V} U}$.

Theorem 3.5 is the particular case of the conjecture above when V consists of a single vertex of T.

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