Left-cut-percolation and induced-Sidorenko bigraphs

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Abstract

A Sidorenko bigraph is one whose density in a bigraphon W is minimized precisely when W is constant. Several techniques of the literature to prove the Sidorenko property consist of decomposing (typically in a tree decomposition) the bigraph into smaller building blocks with stronger properties. One prominent such technique is that of N-decompositions of Conlon–Lee, which uses weakly Hölder (or weakly norming) bigraphs as building blocks. In turn, to obtain weakly Hölder bigraphs, it is typical to use the chain of implications reflection bigraph $\implies$ cut-percolating bigraph $\implies$ weakly Hölder bigraph. In an earlier result by the author with Razborov, we provided a generalization of N-decompositions, called reflective tree decompositions, that uses much weaker building blocks, called induced-Sidorenko bigraphs, to also obtain Sidorenko bigraphs.

In this paper, we show that "left-sided" versions of the concepts of reflection bigraph and cut-percolating bigraph yield a similar chain of implications: left-reflection bigraph $\implies$ left-cut-percolating bigraph $\implies$ induced-Sidorenko bigraph. We also show that under mild hypotheses, the "left-sided" analogue of the weakly Hölder property (which is also obtained via a similar chain of implications) can be used to improve bounds on another result of Conlon–Lee that roughly says that bigraphs with enough vertices on the right side of each realized degree have the Sidorenko property.

1 Introduction

In [Sid91] (see also [Sid93]), Sidorenko conjectured that if $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ are probability spaces, $W\colon X \times Y \to \mathbb{R}_+$ is a bounded measurable function (a *bigraphon*), and $G = (V_1, V_2, E)$ is a bipartite graph with a given bipartition (V_1, V_2) (a *bigraph*), then

$$t(G, W) \geq t(\rho, W)^{e(G)}, \tag{1}$$

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where ρ denotes the edge bigraph, $e(G) \stackrel{\text{def}}{=} |E(G)|$ is the number of edges of G and

$$t(G,W) \stackrel{\text{def}}{=} \int_{X^{V_1}\times Y^{V_2}} \prod_{(v,w)\in E(G)} W(x_v,y_w)\ d(\mu\otimes\nu)(x,y) \tag{2}$$

is the non-induced (labeled) density of G in W. Bigraphs G that satisfy (1) for every W are called *Sidorenko bigraphs*. In fact, Sidorenko's Conjecture is often studied under the further assumption that W is symmetric (i.e., $\Omega = \Lambda$ and $W(x,y) = W(y,x)$ for every $x, y \in X$) and let us point out right away that even though some of the literature results cited in this introduction were proved under this further assumption, they all extend straightforwardly to the asymmetric setting.

Quite a few of the known results on Sidorenko's Conjecture concern deducing that a bigraph is Sidorenko if it can be decomposed into small parts that are all Sidorenko bigraphs. An alternative way of viewing such results is that each of them recursively defines a subclass of Sidorenko bigraphs containing some base bigraphs (typically only the edge bigraph) and that is closed under some "valid" amalgamations. One is then interested in the case when "valid" amalgamations allow for highly non-trivial bigraphs in the common set of the amalgamation.

Arguably, one of the richest such recursively defined subclasses of Sidorenko bigraphs is provided by Szegedy in [Sze15a]. However, to properly describe which amalgamations are "valid" in Szegedy's framework, extra information needs to be carried around about the bigraphs G: essentially a family of probability distributions on homomorphisms from G to all possible target bigraphons, and "validness" of an amalgamation of G_1 and G_2 along a set V is determined by compatibility of the marginals of the distributions on V and a relative entropy inequality. To make the result concrete, Szegedy then shows that the second condition is satisfied if we instead recursively enforce the common part $G_1|_V \cong G_2|_V$ to be a forest (and start from the edge bigraph).

In a different but similar flavor, Conlon–Kim–Lee–Lee [CKLL18b] showed that that the class of Sidorenko graphs contains all *strongly tree decomposable* graphs, that is, graphs G containing a tree decomposition $T = (V(T), E(T))$ such that

i. For each tree vertex $U \in V(T)$, $G|_U$ is a tree.

ii. For each tree edge $\{U_1, U_2\} \in E(T)$, the intersection $U_1 \cap U_2$ induces a forest $G|_{U_1\cap U_2}$.

iii. For each tree edge $\{U_1, U_2\} \in E(T)$ there is an isomorphism f between the minimal subtrees of $G|_{U_1}$ and $G|_{U_2}$ that contain $U_1 \cap U_2$ such that f fixes $U_1 \cap U_2$ pointwisely.

This result was also recursively extended to higher-order strong tree decomposable graphs in [CKLL18a], but even then the common sets in the amalgamations are still required to be forests.

Moving away from the case when the common sets in the amalgamations are required to be forests, Conlon–Lee [CL17] generalized the result above by connecting it to the theory of *weakly norming bigraphs* (a.k.a. *weakly Hölder bigraphs*) of Hatami [Hat10], i.e., bigraphs

G such that $W \mapsto t(G, |W|)^{1/e(G)}$ is a norm in the space of bounded measurable functions $X \times Y \to \mathbb{R}$ (up to a.e. equivalence). Namely, they showed that if N is weakly norming, then the same result holds for N-*decomposable* graphs, which are graphs G containing a tree decomposition T such that

i. For each tree vertex $U \in V(T)$, $G|_U$ is isomorphic to N.

ii. For each tree edge $\{U_1, U_2\} \in E(T)$ there is an isomorphism between $G|_{U_1}$ and $G|_{U_2}$ that fixes $U_1 \cap U_2$ pointwisely.

However, the main disadvantage of this result is that very few bigraphs are weakly norming: Hatami himself showed [Hat10, Theorem 2.10(ii)] that any weakly norming bigraph is necessarily biregular after removing isolated vertices (in [DGH$^+$18], it is shown that weakly norming bigraphs are precisely those that satisfy the step Sidorenko property studied in [KMPW19] and implicitly in [Lov12, §14.2]). Nevertheless, in the same paper, Conlon–Lee showed that many non-trivial examples of weakly norming bigraphs can be obtained through a chain of implications: reflection bigraph $\implies$ cut-percolating bigraph $\implies$ weakly norming bigraph (see Sections 2.4 and 2.5 below for formal definitions of these concepts). Perhaps one of the most interesting examples of weakly norming bigraphs arising from this chain of implications are the incidence bigraphs of the complete k-uniform hypergraphs $K_n^{(k)}$ (see Figure 1).

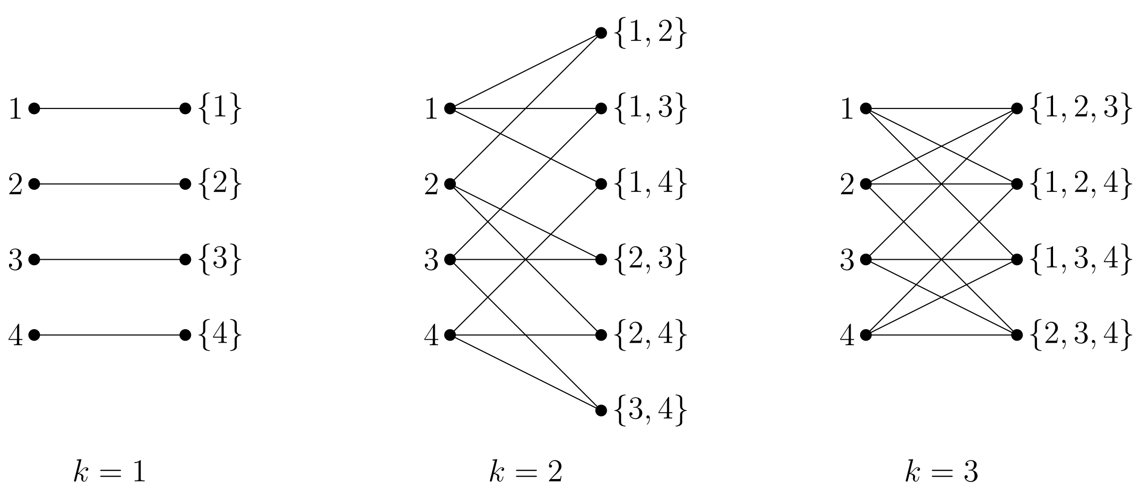

Figure 1: Incidence bigraphs of complete k-uniform hypergraphs $K_4^{(k)}$ on 4 vertices.

In a recent work with Razborov [CR21, Theorem 3.5], we showed that the strong tree decompositions and N-decompositions results can be both unified and generalized under the weaker notion of reflective tree decompositions. The formal definition[1] of reflective

[1]For the reader's convenience, this definition and the associated result are included in Appendix A, but these are not necessary for our results.

tree decompositions is more technical than that of N-decompositions, but the result in particular implies that all N-decomposable bigraphs are Sidorenko bigraph even when N is only required to be an *induced-Sidorenko* bigraph, that is, when

$$\frac{t(N,W)}{t(\rho,W)^{e(N)}} \geq \frac{t(H,W)}{t(\rho,W)^{e(H)}}$$

for every induced subgraph H of N and every bigraphon $W\colon \Omega\times\Lambda\to\mathbb{R}_+$ that is *biregular*, i.e., it satisfies

$$\int_X W(x',y)\ d\mu(x') = \int_Y W(x,y')\ d\nu(y') = t(\rho,W)$$

for almost every $x\in X$ and almost every $y\in Y$ (every weakly norming bigraph is induced-Sidorenko, see Remark 2.11).

It was already observed in [CR21] that there are several induced-Sidorenko bigraphs that are not biregular (hence not weakly norming): the class of induced-Sidorenko bigraph is closed under amalgamations with trees along a single vertex and the amalgamation of k copies of the 4-cycle along the same edge (a.k.a. the *k-book bigraph*, see Figure 2) is induced-Sidorenko.

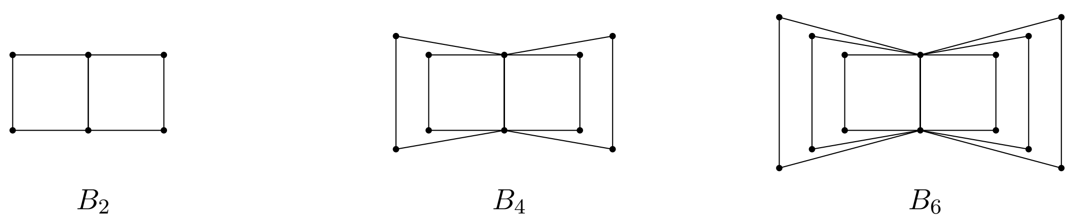

Figure 2: Book bigraphs.

Our first main result is to show (Theorems 3.1 and 3.2) how "left-sided" versions of the concepts of reflection bigraphs and cut-percolation of Conlon–Lee yield a similar chain of implications to obtain several examples of induced-Sidorenko bigraphs: left-reflection bigraph $\implies$ left-cut-percolating bigraph $\implies$ induced-Sidorenko bigraph. Let us point out right away that even though the natural analogue of weakly norming, called *left-weakly Hölder*, also follows from left-cut-percolation (Theorem 3.3, see also Theorem 3.4), it is a different notion from induced-Sidorenko (and our proof requires the full power of left-cut-percolation to get induced-Sidorenko). See also Figure 3 for a summary of the implications between these properties of bigraphs. The most important example of left-reflection bigraph (Theorem 3.5) is the incidence bigraph of the complete hypergraph $K_n^{k_1,\ldots,k_t}$ on n vertices and in uniformities $k_1,\ldots,k_t$ (see Figure 4).

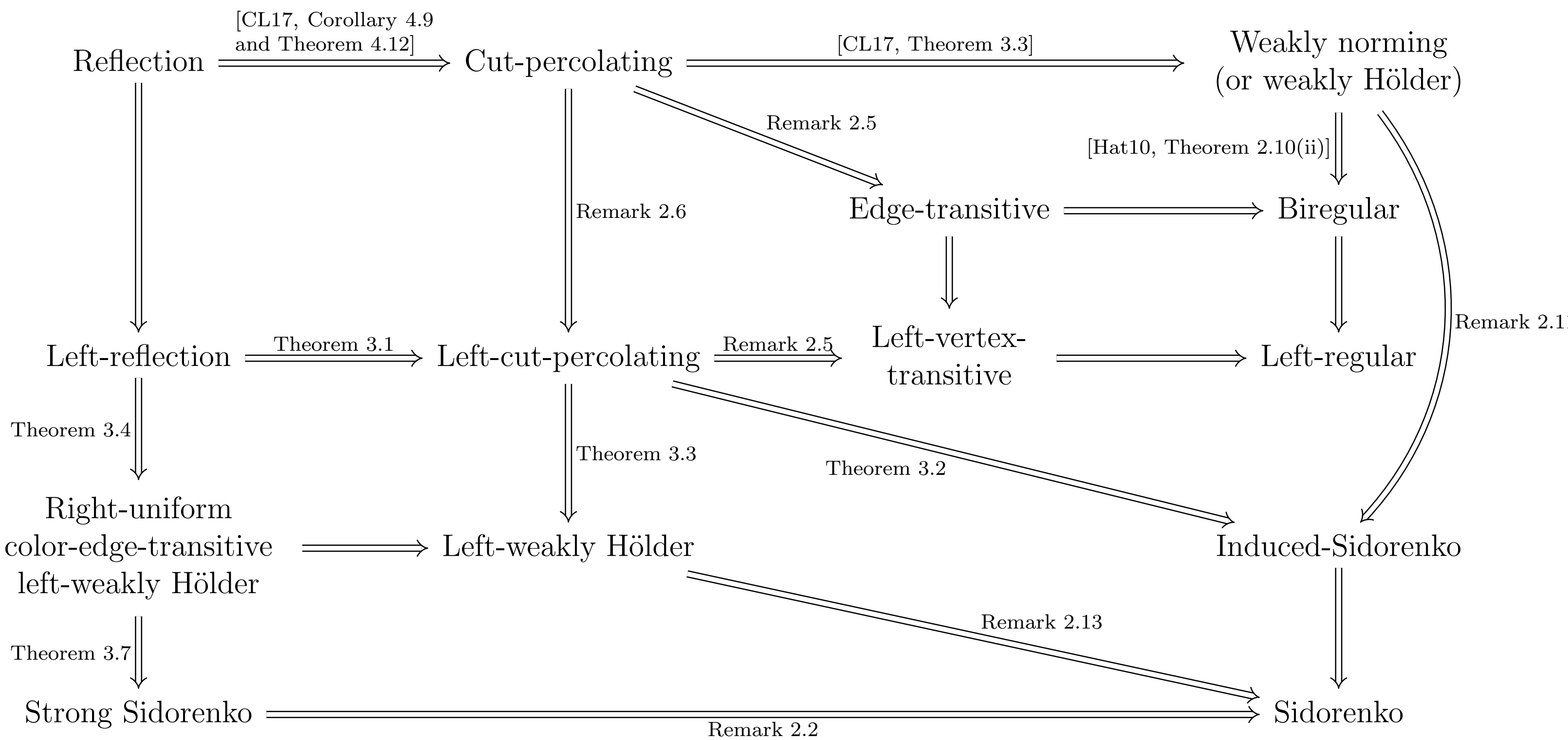

Figure 3: Implications between properties considered under the assumption that the bigraph is non-trivial and does not have any isolated vertices. Arrows are labeled with the location of their proofs and arrows with easy proofs are either labeled by remarks or unlabeled (when the proof is trivial). In this diagram, the properties about colored bigraphs should be read as some coloring of the bigraph turns it into a colored bigraph with that property.

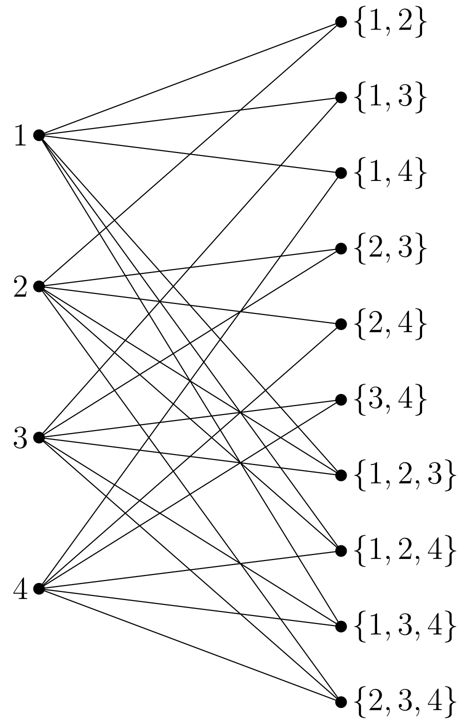

Figure 4: Incidence bigraph of the complete hypergraph $K_4^{2,3}$ on 4 vertices in uniformities 2 and 3. This is the amalgamation of the incidence bigraph of the complete 2-uniform hypergraph $K_4^{(2)}$ and the incidence bigraph of the complete 3-uniform hypergraph $K_4^{(3)}$ over their left side (see also Figure 1).

One of the most interesting applications of the theory of reflection bigraphs of Conlon–Lee (in fact, it uses a hypergraph analogue of it) is the following theorem.

Theorem 1.1 (Conlon–Lee [CL21, Theorem 1.1])**.** *Let $G = (V_1, V_2, E)$ be a bigraph, let r be the maximum degree of a vertex in V_2 and for each $k \in \{2, \dots, r\}$, let d_k be the number of vertices in V_2 that have degree k.*

If $\binom{|V_1|}{r}\binom{r}{k}$ divides d_k for every $k \in \{2, \dots, r\}$, then G is a Sidorenko bigraph.

As our second main result, we generalize the theorem above by weakening the divisibility condition to the condition that d_k is either zero or at least $\binom{|V_1|}{k}$ (Theorem 3.6) note that this also improves the first non-zero value of d_k that is valid. In fact, we show that such bigraphs satisfy a stronger inequality (see Definition 2.1) that was initially used by Sidorenko [Sid91, Equation (2)] to study his conjecture. This result is derived from the aforementioned fact that the incidence bigraph of $K_n^{k_1,\dots,k_t}$ is a left-reflection bigraph and we also show how the left-weakly Hölder property (along with some extra mild properties) yields a similar result based on the symmetries of the underlying bigraph (Theorem 3.7).

The paper is organized as follows. In Section 2, we give definitions and establish the notation necessary to state our main results. In Section 3, we state our main results. In Section 4, we prove all theorems that do not directly involve Sidorenko's Conjecture, that is, Theorems 3.1, 3.3 and 3.5 (these theorems are mostly direct analogues of Conlon–Lee [CL21]). In Section 5, we prove Theorem 3.2 that establishes the connection with the induced-Sidorenko property. In Section 6, we prove Theorem 3.6 on Sidorenko bigraphs with enough vertices of each degree and its symmetries-based generalization, Theorem 3.7. We finish the paper with a brief discussion and some open problems in Section 7

2 Preliminaries

Throughout the text, we will use the notation $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, \ldots\}$ for non-negative integers and $\mathbb{N}_+ \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$ for positive integers. For $n \in \mathbb{N}$, we let $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. We also let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ the set of non-negative real numbers. Given a set V, we denote its power set by $2^V \stackrel{\text{def}}{=} \{W \mid W \subseteq V\}$.

2.1 Bigraphs

A *bigraph* is a triple $G = (V_1, V_2, E)$, where V_1 and V_2 are disjoint finite sets and $E \subseteq V_1 \times V_2$. We will also use the following notation ($i = 1, 2$):

$$\begin{aligned} V_i(G) &\stackrel{\text{def}}{=} V_i, \quad v_i(G) \stackrel{\text{def}}{=} |V_i|, \quad V(G) \stackrel{\text{def}}{=} V_1 \cup V_2, \\ E(G) &\stackrel{\text{def}}{=} E, \quad e(G) \stackrel{\text{def}}{=} |E|, \quad v(G) \stackrel{\text{def}}{=} |V_1| + |V_2|. \end{aligned} \tag{3}$$

For $v \in V(G)$, we denote its *neighborhood* by

$$N_G(v) \stackrel{\text{def}}{=} \{w \in V(G) \mid (v, w) \in E(G) \lor (w, v) \in E(G)\}.$$

and its *degree* by $d_G(v) \stackrel{\text{def}}{=} |N_G(v)|$.

We say that G is *left d-regular* (*right d-regular*, respectively) if $d_G(v) = d$ for every $v \in V_1(G)$ ($v \in V_2(G)$, resp.). We say that G is *biregular* if it is both left d_1-regular and right d_2-regular for some $d_1, d_2 \in \mathbb{N}$. An *isomorphism* between bigraphs G_1 and G_2 is a bijection $f\colon V(G_1) \rightarrowtail V(G_2)$ such that $f(V_i(G_1)) = V_i(G_2)$ ($i = 1, 2$) and $(v, w) \in E(G_1) \iff (f(v), f(w)) \in E(G_2)$ ($(v, w) \in V_1(G_1) \times V_2(G_1)$); when such an isomorphism exists, we say that G_1 and G_2 are *isomorphic*, which is denoted $G_1 \cong G_2$. An *automorphism* of G is an isomorphism of G to itself and we denote the set of automorphisms of G by $\operatorname{Aut}(G)$. A *homomorphism* from a bigraph G_1 to a bigraph G_2 is a (not necessarily injective) map $f\colon V(G_1) \to V(G_2)$ such that $f(V_i(G_1)) \subseteq f(V_i(G_2))$ ($i = 1, 2$) and $f(E(G_1)) \subseteq E(G_2)$. An *endomorphism* of G is a homomorphism of G to itself. The set of endomorphisms of G is denoted $\operatorname{End}(G)$.

For $U \subseteq V(G)$, we let $G|_U$ be the *subgraph induced by U in G*, that is, we let

$$V_i(G|_U) \stackrel{\text{def}}{=} V_i(G) \cap U, \qquad E(G|_U) \stackrel{\text{def}}{=} E(G) \cap ((U \cap V_1(G)) \times (U \cap V_2(G))).$$

We also let $G - U \stackrel{\text{def}}{=} G|_{V(G)\setminus U}$. Furthermore, for a set of edges $E \subseteq E(G)$, we let $G - E \stackrel{\text{def}}{=} (V_1(G), V_2(G), E(G) \setminus E)$ be the subgraph obtained from G by removing the edges in E.

We denote the *edge bigraph* by $\rho \stackrel{\text{def}}{=} (\{1\}, \{2\}, \{(1,2)\})$, the *d-star bigraph* by $K_{1,d} \stackrel{\text{def}}{=} (\{0\}, [d], \{(0,i) \mid i \in [d]\})$ and the *dual d-star bigraph* by $K_{d,1} \stackrel{\text{def}}{=} ([d], \{0\}, \{(i,0) \mid i \in [d]\})$.

2.2 Flags

It will be convenient to also work with partially labeled bigraphs and for this purpose we will borrow some terminology from the theory of flag algebras [Raz07].

More specifically, we work in the theory T^2_{Graph} of graphs augmented with a 2-coloring of its vertices. Thus, a *flag* is a partially labeled bigraph, that is, a pair $F = (G, \theta)$, where G is a bigraph and $\theta\colon [k] \rightarrowtail V(G)$ is an injection for some $k \in \mathbb{N}$. We use the notation $|F| \stackrel{\text{def}}{=} G$ for the *underlying bigraph* of F and the notation $\theta_F \stackrel{\text{def}}{=} \theta$ for the *labeling* of F. We will often abuse notation and write $F = (G, (\theta(1), \theta(2), \ldots, \theta(k)))$, listing the values of θ. In fact, we will abuse the notation even more and write $F = (G, U)$ for some set $U \subseteq V(G)$ to be understood as $F = (G, \theta)$ for some $\theta\colon [|U|] \rightarrowtail V(G)$ with $\operatorname{im}(\theta) = U$, whenever the exact ordering is either clear from the context or unimportant.

An *isomorphism* between flags $F_1 = (G_1, \theta_1)$ and $F_2 = (G_2, \theta_2)$ is an isomorphism f between G_1 and G_2 that preserves the partial labeling in the sense that $f \circ \theta_1 = \theta_2$; when such an isomorphism exists, we say that F_1 and F_2 are *isomorphic* and denote it by $F_1 \cong F_2$.

If $F_1 = (G_1, \theta_1)$ and $F_2 = (G_2, \theta_2)$ are flags such that $\theta_2 \circ \theta_1^{-1}$ is an isomorphism between $G_1|_{\operatorname{im}(\theta_1)}$ and $G_2|_{\operatorname{im}(\theta_2)}$ (that is, in the terminology of flag algebras, these flags are of the same type), we let $F_1 \sqcup F_2$ be the flag obtained from the disjoint union of F_1 and F_2 by identifying vertices with the same label[2]. For $i \in [2]$, we let $e_i \stackrel{\text{def}}{=} (\rho, i)$. For a bigraph G, we let $G^L \stackrel{\text{def}}{=} (G, V_1(G))$ be the flag in which all left vertices of G are labelled.

2.3 Bigraphons

Given probability spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$, a *bigraphon* over Ω and Λ is a bounded measurable function $W\colon X \times Y \to \mathbb{R}_+$, where $X \times Y$ is equipped with the product σ-algebra and the product measure $\mu \otimes \nu$; we will denote bigraphons by $W\colon \Omega \times \Lambda \to \mathbb{R}_+$.

When taking integrals, our functions will always be bounded and hence Fubini's Theorem will apply and we will be omitting references to it. If V is a set, we let $\Omega^V = (X^V, \mu^V)$ be the product probability space of $|V|$ copies of Ω; we will usually abuse notation and denote μ^V simply by μ. Given $x \in X^V$ and $S \subseteq V$, we let $x_S \in X^S$ be the projection of x to the coordinates in S.

For a bigraph G and a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$, we let $t(G, W) \in \mathbb{R}_+$ be given by (2). More generally, for a flag $F = (G, \theta)$ and a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$, we let the function

[2] We avoid using F_1F_2 here to not conflict with the product as defined in flag algebras.

$t(F,W)\colon \Omega^{V_1(G)\cap \operatorname{im}(\theta)} \times \Lambda^{V_2(G)\cap\operatorname{im}(\theta)} \to \mathbb{R}_+$ be given by

$$t(F,W)(x,y) \stackrel{\text{def}}{=} \int_{X^{V_1(G)\setminus\operatorname{im}(\theta)}\times Y^{V_2(G)\setminus\operatorname{im}(\theta)}} \prod_{(v,w)\in E(G)} W(x''_v, y''_w)\ d(\mu\otimes\nu)(x',y'),$$

where

$$x''_v \stackrel{\text{def}}{=} \begin{cases} x_v, & \text{if } v \in V_1(G)\cap\operatorname{im}(\theta),\\ x'_v, & \text{if } v\in V_1(G)\setminus\operatorname{im}(\theta); \end{cases} \qquad y''_w \stackrel{\text{def}}{=} \begin{cases} y_w, & \text{if } w \in V_2(G)\cap\operatorname{im}(\theta),\\ y'_w, & \text{if } w\in V_2(G)\setminus\operatorname{im}(\theta). \end{cases}$$

When $V_1(G)\cap\operatorname{im}(\theta) = \varnothing$, we will simplify the notation to $t(F,W)(y)$, and likewise for $V_2(G)\cap\operatorname{im}(\theta)=\varnothing$.

A bigraphon $W\colon \Omega\times\Lambda\to\mathbb{R}_+$ is called *left-regular* if it satisfies

$$t(e_1,W)(x) = t(\rho,W)$$

for almost every $x\in X$. Dually, it is called *right-regular* if

$$t(e_2,W)(y) = t(\rho,W)$$

for almost every $y\in Y$. Finally, it is called *biregular* if it is both left regular and right regular.

As mentioned in the introduction, a *Sidorenko bigraph* G is a bigraph such that $t(G,W)\geq t(\rho,W)^{e(G)}$ for every bigraphon W. While studying Sidorenko bigraphs, Sidorenko considered a stronger inequality [Sid91, Equation (2)] that yields the class of strong Sidorenko bigraphs defined below.

Definition 2.1. A bigraph G is a *strong Sidorenko bigraph*[3] if for every bigraphon $W\colon\Omega\times\Lambda\to\mathbb{R}_+$ and all sequences $f=(f_v)_{v\in V_1(G)}$ and $g=(g_w)_{w\in V_2(G)}$ of bounded measurable functions $f_v\colon\Omega\to\mathbb{R}_+$, $g_w\colon\Lambda\to\mathbb{R}_+$, we have

$$t(G;f,g;W)\geq t\left(\rho;\prod_{v\in V_1(G)} f_v^{1/e(G)}, \prod_{w\in V_2(G)} g_w^{1/e(G)}; W\right)^{e(G)},$$

where

$$t(G;f,g;W)\stackrel{\text{def}}{=}\int_{X^{V_1(G)}\times Y^{V_2(G)}} \prod_{v\in V_1(G)} f_v(x_v)\cdot\prod_{w\in V_2(G)} g_w(y_w)\cdot\prod_{(v,w)\in E(G)} W(x_v,y_w)\ d(\mu\otimes\nu)(x,y)$$

and

$$t\left(\rho;\prod_{v\in V_1(G)} f_v^{1/e(G)}, \prod_{w\in V_2(G)} g_w^{1/e(G)}; W\right) \stackrel{\text{def}}{=} \int_{X\times Y}\prod_{v\in V_1(G)} f_v(x)^{1/e(G)}\cdot\prod_{w\in V_2(G)} g_w(y)^{1/e(G)}\cdot W(x,y)\ d(\mu\otimes\nu)(x,y).$$

[3] In fact, Sidorenko's condition in [Sid91, Equation (2)] also involves global weight functions f and g and allows for arbitrary measure spaces (that are not necessarily probability spaces), but it is not hard to see that the condition stated here is equivalent (see Appendix B).

Remark 2.2. By simply taking all functions f_v and g_w to be constant equal to 1, we deduce that strong Sidorenko bigraphs are Sidorenko bigraphs. However, it is not hard to see that taking W and g_w to be constant equal to 1, we retrieve an instance of Hölder's inequality for the f_v functions that only holds if $e(G) \geq v_1(G)$. Other than examples that satisfy $e(G) < \min\{v_1(G), v_2(G)\}$ (or are indirectly obtained from such examples), it is not known whether the strong Sidorenko property is strictly stronger than the Sidorenko property.

We say that a bigraph G_1 *weakly dominates* G_2 if

$$\frac{t(G_1, W)}{t(\rho, W)^{e(G_1)}} \geq \frac{t(G_2, W)}{t(\rho, W)^{e(G_2)}}$$

for every biregular non-zero bigraphon W.

An *induced-Sidorenko* bigraph G is a bigraph that weakly dominates all of its induced subgraphs.

2.4 Cut-percolation

We start by recalling the definitions of [CL17, §3] pertaining cut-percolation.

A *cut-involution* of a bigraph G is an automorphism $\phi \in \operatorname{Aut}(G)$ that satisfies:

i. ϕ is an involution, i.e., $\phi = \phi^{-1}$.

ii. The set $\operatorname{Fix}(\phi)$ of points that are fixed by ϕ is a vertex-cut in the bigraph G, i.e., $G - \operatorname{Fix}(G)$ is disconnected.

(If G is already disconnected, then the empty set is declared to be a vertex-cut.)

The subgroup of $\operatorname{Aut}(G)$ generated by cut-involutions of G is called the *cut-involution group of* G.

A *fold*[4] of G is a pair (ϕ, L), where ϕ is a cut-involution of G and $L \subseteq V(G)$ is such that

i. $G|_L$ is a union of connected components of $G - \operatorname{Fix}(\phi)$.

ii. $(L, \operatorname{Fix}(\phi), \phi(L))$ is a partition of $V(G)$.

The set L is called the *left side* of the fold (ϕ, L) (but note that it is not necessarily contained in $V_1(G)$).

Remark 2.3. Not every cut-involution can be completed to a fold. A simple counter-example is the bigraph G of Figure 5 given by

$$V_1(G) \stackrel{\text{def}}{=} \{0, 2, 4\}, \quad V_2(G) \stackrel{\text{def}}{=} \{1, 3, 5, 6\}, \quad E(G) \stackrel{\text{def}}{=} \{(0,1), (0,3), (0,5), (0,6), (2,1), (2,3), (4,1), (4,3)\}$$

and the cut-involution ϕ that maps $(1, 2, 5)$ to $(3, 4, 6)$ and fixes 0. In fact, it is straightforward to check that a necessary and sufficient condition for the existence of some L so that (ϕ, L) is a fold is that no connected component of $G - \operatorname{Fix}(\phi)$ is fixed by ϕ as a set.

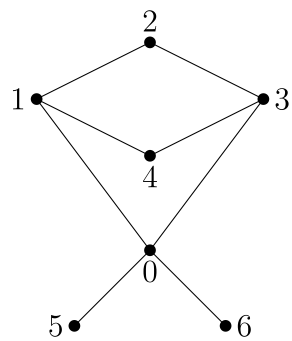

Figure 5: Example of Remark 2.3 of a bigraph with a cut-involution that does not yield a fold.

Given a fold (ϕ, L), the *left-folding* and *right-folding* maps are the maps $\phi_L, \phi_L^*\colon V(G) \to V(G)$, respectively, defined by

$$\phi_L(v) \stackrel{\text{def}}{=} \begin{cases} \phi(v), & \text{if } v \notin L; \\ v, & \text{otherwise} \end{cases} \qquad \phi_L^*(v) \stackrel{\text{def}}{=} \begin{cases} \phi(v), & \text{if } v \in L; \\ v, & \text{otherwise.} \end{cases}$$

Note that these are endomorphisms of G.

A *cut-percolating sequence* of a bigraph is a sequence of sets $E_0, E_1, \ldots, E_m \subseteq E(G)$ such that $|E_0| = 1$, $E_m = E(G)$ and for every $i \in [m]$, there exists a fold (ϕ_i, L_i) of G such that $E_i = (\phi_i)_{L_i}^{-1}(E_{i-1})$. In this definition, when want to make explicit the folds used in the cut-percolating sequence, we say that it is a cut-percolating sequence with respect to $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$.

A bigraph is called *cut-percolating* if it has a cut-percolating sequence. In fact, if S is a set of folds of G, we say that G is *cut-percolating under S* if it has a cut-percolating sequence with respect to a sequence of folds in S.

The left-sided analogue of the above is defined in terms of left vertices.

Definition 2.4. A *left-cut-percolating sequence* of a bigraph is a sequence of sets $U_0, U_1, \ldots, U_m \subseteq V_1(G)$ such that $|U_0| = 1$, $U_m = V_1(G)$ and for every $i \in [m]$, there exists a fold (ϕ, L) of G such that $U_i = \phi_L^{-1}(U_{i-1})$. Again, we say that this sequence is a left-cut-percolating sequence with respect to $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ when we want to make the sequence of folds explicit.

A bigraph is called *left-cut-percolating* if it has a left-cut-percolating sequence. For a set of folds S of G, we say that G *left-cut-percolating under S* if it has a left-cut-percolating sequence with respect to a sequence of folds in S.

Remark 2.5. The cut-percolating sequence of a cut-percolating bigraph G shows that the orbit of the single edge in E_0 under the action of the cut-involution group of G is $E(G)$,

[4] Let us remark that in [CL17, §3], Conlon–Lee use the same name "cut-involution" for folds, leaving the choice of the set L implicit.

so G is edge-transitive under its cut-involution group action. By the same token, a left-cut-percolating bigraph is necessarily left-vertex-transitive under its cut-involution group action.

In fact, both these statements trivially remain true replacing the cut-involution group of G with its subgroup generated by the cut-involutions that appear in the sequence of folds used by the cut-percolating sequence.

Remark 2.6. It is easy to see that every cut-percolating bigraph without isolated vertices is left-cut-percolating by simply tracking down the left endpoints of the edges in a cut-percolating sequence.

2.5 Reflection bigraphs

Again, we start by recalling the definitions of [CL17, §4] pertaining reflection bigraphs.

Let $R \subseteq \mathrm{GL}_n(\mathbb{R})$ be a finite reflection group (i.e., a finite group generated by reflections) and let $T \subseteq R$ be the set of reflections in R. The set Φ of unit vectors that are orthogonal to some hyperplane that is fixed by an element of T is called *root system of* R and its elements are called *roots*. Given further an ordered basis $\mathcal{U} = (u_1, \ldots, u_n)$ of $\mathbb{R}^n$, a root $\alpha \in \Phi$ is called *positive* (with respect to $\mathcal{U}$) if it can be written as $\alpha = \sum_{i=1}^{n} c_i u_i$ with $c_{i_0} > 0$, where $i_0 \in [n]$ is the first index for which c_i is non-zero; otherwise, the root is called *negative*. The set Φ is then partitioned into the sets Φ^+ and Φ^- of positive and negative roots, respectively.

For $\alpha \in \Phi$, let H_α be the hyperplane (through the origin) orthogonal to α, let $s_\alpha \in T$ be the reflection on H_α and let

$$D_\alpha^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle > 0\}; \qquad\qquad D_\alpha^- \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle x, \alpha \rangle < 0\}.$$

Dually, for $t \in T$, we let $\alpha_t \in \Phi^+$ be the unique positive root such that $s_{\alpha_t} = t$. We also use the shorthands $H_t \stackrel{\text{def}}{=} H_{\alpha_t}$, $D_t^+ \stackrel{\text{def}}{=} D_{\alpha_t}^+$ and $D_t^- \stackrel{\text{def}}{=} D_{\alpha_t}^-$.

A set $\Delta \subseteq \Phi^+$ is a *simple system* if every $\alpha \in \Phi^+$ can be written as a conic combination (i.e., a linear combination with non-negative coefficients) of elements of Δ and Δ is minimal with this property. It is known (see [Hum90, §1.3 and 1.5]) that Δ is unique with respect to $\mathcal{U}$, its elements are linearly independent and $S_\Delta \stackrel{\text{def}}{=} \{s_\alpha \mid \alpha \in \Delta\}$ generates R. The elements of Δ are called *simple roots* and the elements of S_Δ are called *simple reflections*.

For $I \subseteq S_\Delta$, let

$$C(I) \stackrel{\text{def}}{=} \left(\bigcap_{s \in I} H_s \right) \cap \left(\bigcap_{s \in S_\Delta \setminus I} D_s^+ \right).$$

Given $S_1, S_2 \subseteq S_\Delta$, the $(S_1, S_2; S_\Delta)$-*reflection bigraph*[5] is the bigraph G defined by

$$V_1(G) \stackrel{\text{def}}{=} R/R_1 = \{rR_1 \mid r \in R\},$$
$$V_2(G) \stackrel{\text{def}}{=} R/R_2 = \{rR_2 \mid r \in R\},$$
$$E(G) \stackrel{\text{def}}{=} \{(rR_1, rR_2) \mid r \in R\},$$

where R_i is the subgroup of R generated by S_i. In words, the left and right vertices are the left cosets of R_1 and R_2, respectively and an edge is present whenever these left cosets intersect.

In [CL17, Corollary 4.9], Conlon–Lee showed that every reflection $t \in T$ naturally defines a fold $(\phi(t; S_1, S_2; S_\Delta), L(t; S_1, S_2; S_\Delta))$ of the $(S_1, S_2; S_\Delta)$-reflection bigraph G given by

$$\phi(t; S_1, S_2; S_\Delta)(rR_i) \stackrel{\text{def}}{=} trR_i, \qquad (r \in R, i \in [2])$$
$$L(t; S_1, S_2; S_\Delta) \stackrel{\text{def}}{=} \{rR_i \mid r(C(S_i)) \subseteq D^+_{\alpha_t}, i \in [2]\}.$$

We will call such folds *reflection folds* of the reflection bigraph G.

The left-sided analogue of reflection bigraphs uses multiple subgroups for the right vertices.

Definition 2.7. Given $S_0, S_1, \ldots, S_k \subseteq S_\Delta$, the $(S_0; S_1, \ldots, S_k; S_\Delta)$-left-reflection bigraph is the amalgamation of the $(S_0, S_i; S_\Delta)$-reflection bigraphs G_i ($i \in [k]$) over the left side, that is, it is obtained from their disjoint union by identifying the correpending left vertices. Formally, it is the bigraph G defined by

$$V_1(G) \stackrel{\text{def}}{=} R/R_0 = \{rR_0 \mid r \in R\},$$
$$V_2(G) \stackrel{\text{def}}{=} \bigsqcup_{i=1}^{k} R/R_i \stackrel{\text{def}}{=} \{(rR_i, i) \mid r \in R, i \in [k]\},$$
$$E(G) \stackrel{\text{def}}{=} \{(rR_0, (rR_i, i)) \mid r \in R, i \in [k]\}.$$

Equivalently, in flag language, we have $G \stackrel{\text{def}}{=} |\bigsqcup_{i=1}^{k} G_i^L|$.

For each $t \in T$, the *reflection fold* $(\phi(t; S_0; S_1, \ldots, S_k; S_\Delta), L(t; S_0; S_1, \ldots, S_k; S_\Delta))$ of G is the amalgamation of the reflection folds $(\phi(t; S_0, S_i; S_\Delta), L(t; S_0, S_i; S_\Delta))$ of the G_i; formally, we define

$$\phi(t; S_0; S_1, \ldots, S_k; S_\Delta)(rR_0) \stackrel{\text{def}}{=} trR_0,$$
$$\phi(t; S_0; S_1, \ldots, S_k; S_\Delta)((rR_i, i)) \stackrel{\text{def}}{=} (trR_i, i),$$
$$L(t; S_0; S_1, \ldots, S_k; S_\Delta) \stackrel{\text{def}}{=} \{rR_0 \mid r(C(S_0)) \subseteq D^+_{\alpha_t}\} \cup \bigcup_{i \in [k]} \{(rR_i, i) \mid r(C(S_i)) \subseteq D^+_{\alpha_t}\}.$$

[5]In [CL17], the group R is also included in this notation, but since it can be retrieved from S_Δ, we drop it from the notation here.

The *natural coloring* of the $(S_0; S_1, \ldots, S_k; S_\Delta)$-left-reflection-bigraph G is the function $c\colon E(G) \to [k]$ given by

$$c(rR_0, (rR_i, i)) \stackrel{\text{def}}{=} i.$$

2.6 Colored bigraphs and left-weakly Hölder bigraphs

A *colored bigraph* is a pair $H = (G, c)$, where G is a bigraph and $c\colon E(G) \to C$ is a function called *coloring*. We use the shorthand notations $c_H \stackrel{\text{def}}{=} c$, $C_H \stackrel{\text{def}}{=} C$ and $G(H) \stackrel{\text{def}}{=} G$. We also abuse notation of (3) by applying it directly to H (e.g., $E(H) \stackrel{\text{def}}{=} E(G(H))$). For each $i \in C$, we let $e_i(H) \stackrel{\text{def}}{=} |c^{-1}(i)|$ be the number of edges that have color i and for each $v \in V(G)$, we let

$$d_{H,i}(v) \stackrel{\text{def}}{=} |\{w \in V(G) \mid (v, w) \in c^{-1}(i) \lor (w, v) \in c^{-1}(i)\}|$$

denote its *i-degree in H*. We say that H is *left-color-regular* if $d_{H,i}(v) = d_{H,i}(w)$ for every $i \in C_H$ and every $v, w \in V_1(H)$ (which is equivalent to saying that $d_{H,i}(v) = e_i(H)/v_1(H)$ for every $v \in V_1(H)$).

For $U \subseteq V(H)$, we let $H|_U \stackrel{\text{def}}{=} (G(H)|_U, c_H|_{E(G(H)|_U)})$ be the *colored bigraph induced by U*. For a set of colors $C' \subseteq C_H$, we let $H_{C'} = (G_{C'}, c_{H_{C'}})$ be the colored bigraph obtained from H by keeping only edges that have color in C', that is, we have $G_{C'} \stackrel{\text{def}}{=} G(H) - c_H^{-1}(C_H \setminus C')$ and $c_{H_{C'}} \stackrel{\text{def}}{=} c_H|_{E(G_{C'})}$. We say that H is *right-uniform* if the coloring c factors as $c(v, w) = f(w)$ for some function $f\colon V_2(G) \to C$. An *isomorphism* between colored bigraphs $H_1 = (G_1, c_1)$ and $H_2 = (G_2, c_2)$ is an isomorphism f between G_1 and G_2 that preserves the coloring in the sense that $c_1(v, w) = c_2(f(v), f(w))$ for every $(v, w) \in E(G_1)$. An *automorphism* of H is an isomorphism of H to itself and we denote the set of automorphisms of H by $\mathrm{Aut}(H)$. A colored bigraph H is called *color-edge-transitive* if for every $(v_1, w_1), (v_2, w_2) \in E(H)$ with $c_H(v_1, w_1) = c_H(v_2, w_2)$, there exists an automorphism $\sigma \in \mathrm{Aut}(H)$ such that $\phi(v_1) = v_2$ and $\sigma(w_1) = w_2$.

Given a colored bigraph H and a sequence of bigraphons $W = (W_i)_{i \in C_H}$ all over the same spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$, we define

$$t(H, W) \stackrel{\text{def}}{=} \int_{X^{V_1(H)} \times Y^{V_2(H)}} \prod_{(v,w) \in E(H)} W_{c_H(v,w)}(x_v, y_w)\ d(\mu \otimes \nu)(x, y).$$

Remark 2.8. Note that if $\sigma \in \mathrm{Aut}(G)$, we have $t((G, c), W) = t((G, c \circ \sigma), W)$ (even if σ is not in $\mathrm{Aut}((G, c))$) by simply renaming the integration variables.

The notions of flags $F = (H, \theta)$ over colored bigraphs $H = (G, c)$ and the corresponding function $t(F, W)$ are defined analogously to Section 2.2.

As mentioned in the introduction, a *weakly norming bigraph*, or a *weakly Hölder bigraph* is a bigraph G such that $W \mapsto t(G, |W|)^{1/e(G)}$ defines a norm in the space of bounded functions $X \times Y \to \mathbb{R}$ up to a.e. equivalence. In [Hat10, Theorem 2.10(ii)], Hatami showed that a bigraph G is weakly norming if and only if for every coloring $c\colon E(G) \to C$ and for every sequence of bigraphons $W = (W_i)_{i\in C}$, we have the following Hölder-like inequality

$$t((G,c),W) \leq \prod_{(v,w)\in E(G)} t(G, W_{c(v,w)})^{1/e(G)}.$$

Furthermore, in [DGH$^+$18], weakly norming bigraphs are further characterized as precisely those bigraphs that have the step Sidorenko property studied in [KMPW19] and implicitly in [Lov12, §14.2].

The left-sided analogue of this is a bit more technical and is defined for colored bigraphs instead.

Definition 2.9. A *left-coloring* of a bigraph G is a function $\ell\colon V_1(G) \to C$.

Given both a coloring $c\colon E(G) \to C'$ and a left-coloring $\ell\colon V_1(G) \to C$ of the same bigraph G, we define the coloring $\ell \otimes c\colon E(G) \to C \times C'$ by

$$(\ell \otimes c)(v,w) = (\ell(v), c(v,w)).$$

A colored bigraph $H = (G,c)$ is called *left-weakly Hölder* if for every left-coloring $\ell\colon V_1(G) \to C$ of $G(H)$ and every sequence of bigraphons $W = (W_i)_{i\in C\times C_H}$, we have

$$t((G,\ell\otimes c),W) \leq \prod_{v\in V_1(G)} t((G,\ell(v)\otimes c),W)^{1/v_1(G)},$$

where $\ell(v)$ on the right-hand side is interpreted as the left-coloring $V_1(G) \to C$ that is constant equal to $\ell(v)$.

Remark 2.10. If $C' \subseteq C_H$ for a left-weakly Hölder bigraph H, then $H_{C'}$ is also left-weakly Hölder. This can be seen by setting $W_i \equiv 1$ for every $i \in C \times (C_H \setminus C')$ in the left-weak Hölder property of H.

Remark 2.11. By [Hat10, Theorem 2.14], every weakly norming bigraph G satisfies $t(G,W)^{1/e(G)} \geq t(H,W)^{1/e(H)}$ for any (not necessarily induced) subgraph H of G, which in particular implies that G is induced-Sidorenko, since (assuming G is non-empty) for an induced subgraph H of G, we have

$$t(G,W) = t(G,W)^{e(H)/e(G)} \cdot t(G,W)^{1-e(H)/e(G)} \geq t(H,W) \cdot t(\rho,W)^{e(G)-e(H)}.$$

Remark 2.12. Since by [Hat10, Theorem 2.10(ii)] every weakly norming bigraph without isolated vertices is biregular, it follows that if G is weakly norming, without any isolated vertices and c is a constant coloring of G, then (G,c) is left-weakly Hölder, as for every left-coloring ℓ, we have

$$\begin{aligned} t((G,\ell\otimes c),W) &\leq \prod_{(v,w)\in E(G)} t(G, W_{(\ell(v),c_0)})^{1/e(G)} = \prod_{v\in V_1(G)} t(G, W_{(\ell(v),c_0)})^{d_G(v)/e(G)} \\ &= \prod_{v\in V_1(G)} t(G, W_{(\ell(v),c_0)})^{1/v_1(G)}, \end{aligned}$$

where c_0 is the unique element in $\mathrm{im}(c)$.

Remark 2.13. An argument analogous to that of Remark 2.12 can be used to show that the underlying bigraph of left-weakly Hölder bigraphs $H = (G, c)$ is Sidorenko: let $v_0 \in V_1(G)$ be a non-isolated vertex, let $\ell\colon V_1(G) \to \{0,1\}$ be given by $\ell(v) \stackrel{\text{def}}{=} \mathbb{1}[v = v_0]$ and for a bigraphon W, considering the sequence $W' = (W'_{t,i})_{t\in\{0,1\},i\in C_H}$ given by $W'_{0,i} \stackrel{\text{def}}{=} 1$ and $W'_{1,i} \stackrel{\text{def}}{=} W$, we get

$$\begin{aligned} t(\rho, W)^{e(G)} &\leq t(K_{1,d_G(v_0)}, W)^{e(G)/d_G(v_0)} = t((G, \ell \otimes c), W')^{e(G)/d_G(v_0)} \\ &\leq t((G, \ell(v_0) \otimes c), W')^{e(G)/(v_1(G)d_G(v_0))} \prod_{v\in V_1(G)\setminus\{v_0\}} t((G, \ell(v) \otimes c), W')^{e(G)/(v_1(G)d_G(v_0))} \\ &= t(G, W)^{e(G)/(v_1(G)d_G(v_0))} = t(G, W), \end{aligned}$$

where the first inequality follows from Jensen's Inequality and the last equality follows from Lemma 5.1 in Section 5 below (as the derivation above holds for every bigraphon W). In fact, a similar argument in Lemma 6.2 will show that every left-weakly Hölder bigraph without isolated vertices is necessarily left-color-regular.

3 Main results

In this section we state our main results.

Theorem 3.1. *Every left-reflection bigraph is left-cut-percolating under reflection folds.*

Theorem 3.2. *Every left-cut-percolating bigraph is induced-Sidorenko.*

Theorem 3.3. *Let G be a left-cut-percolating bigraph under a set S of folds of G. If $c\colon E(G) \to C$ is a coloring of G that is invariant under the subgroup of $\operatorname{Aut}(G)$ generated by $\{\phi \mid (\phi, L) \in S\}$, then (G, c) is left-weakly Hölder.*

Theorem 3.4. *Let G be a left-reflection bigraph and let c be its natural coloring. Then (G, c) is right-uniform, color-edge-transitive and left-weakly Hölder.*

Theorem 3.5. *Let $n, t \in \mathbb{N}_+$ and let $k_1, \ldots, k_t \in [n]$. Then the incidence bigraph G of the complete hypergraph on n vertices and in uniformities $k_1, \ldots, k_t$ defined by*

$$\begin{aligned} V_1(G) &\stackrel{\text{def}}{=} [n], \\ V_2(G) &\stackrel{\text{def}}{=} \bigsqcup_{i=1}^{t} \binom{[n]}{k_i} \stackrel{\text{def}}{=} \left\{ (U, i) \;\middle|\; U \in \binom{[n]}{k_i}, i \in [t] \right\}, \\ E(G) &\stackrel{\text{def}}{=} \left\{ (v, (U, i)) \;\middle|\; v \in U, i \in [t], U \in \binom{[n]}{k_i} \right\} \end{aligned}$$

is a left-reflection bigraph.

As mentioned in the introduction, the next theorem generalizes Theorem 1.1 from [CL21, Theorem 1.1].

Theorem 3.6. *Let G be a bigraph without isolated vertices and for each $k \in \mathbb{N}$, let $d_k \stackrel{\text{def}}{=} |\{w \in V_2(G) \mid d_G(w) = k\}|$ be the number of vertices in $V_2(G)$ that have degree k.*

If for each $k \geq 2$, we have either $d_k = 0$ or $d_k \geq \binom{v_1(G)}{k}$, then G is a strong Sidorenko bigraph.

Theorem 3.7. *Let H be a non-trivial right-uniform color-edge-transitive left-weakly Hölder bigraph without isolated vertices and let G be a bigraph with $V_1(G) = V_1(H)$ and without isolated vertices. For every $U \subseteq V_1(G)$, let*

$$d_G(U) \stackrel{\text{def}}{=} |\{w \in V_2(G) \mid N_G(w) = U\}|,$$
$$d_H(U) \stackrel{\text{def}}{=} |\{w \in V_2(H) \mid N_H(w) = U\}|.$$

Suppose further that for every $U \subseteq V_1(G)$ with $|U| \geq 2$ the following hold.

i. $\sum_{\sigma\in\operatorname{Aut}(H)} d_G(\sigma(U)) = 0$ *if and only if* $\sum_{\sigma\in\operatorname{Aut}(H)} d_H(\sigma(U)) = 0$.

ii. $\sum_{\sigma\in\operatorname{Aut}(H)} d_G(\sigma(U)) \geq \sum_{\sigma\in\operatorname{Aut}(H)} d_H(\sigma(U))$.

Then G is a strong Sidorenko bigraph. In particular, $G(H)$ is a strong Sidorenko bigraph.

The most useful examples of right-uniform color-edge transitive left-weakly Hölder bigraphs to be used in the theorem above are obtained from left-reflection bigraphs through Theorem 3.4.

4 Left-sided properties

In this section, we show the theorems that do not directly involve Sidorenko's Conjecture, that is, Theorems 3.1, 3.3 and 3.5. These theorems are mostly direct analogues of Conlon–Lee [CL17].

Theorem 3.1 will be easily derived from the following property of left-cut-percolating sequences.

Lemma 4.1. *Let $G_1, \ldots, G_k$ be bigraphs with $V_1(G_1) = \cdots = V_1(G_k)$ and $V_2(G_1), \ldots, V_2(G_k)$ pairwise disjoint and let G be the amalgamation of $G_1, \ldots, G_k$ over the left side, i.e., we have*

$$V_1(G) \stackrel{\text{def}}{=} V_1(G_1), \qquad V_2(G) \stackrel{\text{def}}{=} \bigcup_{i\in[k]} V_2(G_i), \qquad E(G) \stackrel{\text{def}}{=} \bigcup_{i\in[k]} E(G_i).$$

Suppose further that G_1 has a left-cut-percolating sequence $U_0, U_1, \ldots, U_m$ with respect to $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ and for every $i \in [m]$ and $j \in \{2, \ldots, k\}$, there exists a fold (ψ_{ij}, L_{ij}) of G_j such that $\phi_i|_{V_1(G_1)} = \psi_{ij}|_{V_1(G_j)}$ and $L_i \cap V_1(G_1) = L_{ij} \cap V_1(G_j)$.

Then $U_0, U_1, \ldots, U_m$ is a left-cut-percolating sequence in G with respect to $\widehat{\Phi} = ((\widehat{\phi}_1, \widehat{L}_1), \ldots, (\widehat{\phi}_m, \widehat{L}_m))$, where $\widehat{L}_i \stackrel{\text{def}}{=} L_i \cup \bigcup_{j=2}^k L_{ij}$ and

$$\widehat{\phi}_i(v) = \begin{cases} \phi_i(v), & \text{if } v \in V(G_1),\\ \psi_{ij}(v), & \text{if } v \in V(G_j),\ j \in \{2, \ldots, k\}\end{cases}$$

is the amalgamation of $\phi_i, \psi_{2k}, \ldots, \psi_{ik}$.

Proof. Since $(\widehat{\phi_i})_{\widehat{L}_i}|_{V_1(G)} = (\phi_i)_{L_i}|_{V_1(G_1)}$, it will follow that $U_0, U_1, \ldots, U_m$ is a left-cut-percolating sequence of G with respect to $\widehat{\Phi}$ as long as we show that $\widehat{\Phi}$ is indeed a sequence of folds of G.

Fix $i \in [m]$, write $\psi_{i1} \stackrel{\text{def}}{=} \phi_i$ and $L_{i1} \stackrel{\text{def}}{=} L_i$. The fact that $\widehat{\phi_i}$ is an involution of G follows simply because the functions $\psi_{i1}, \ldots, \psi_{ik}$ are involutions of their respective bigraphs and they coincide in the common part $V_1(G_1)$.

From the same property, it also follows that $\operatorname{Fix}(\widehat{\phi_i}) = \bigcup_{j=1}^k \operatorname{Fix}(\psi_{ij})$ and $\widehat{\phi_i}(\widehat{L}_i) = \bigcup_{j=1}^k \psi_{ij}(L_{ij})$ and thus $\widehat{\phi_i}$ is a cut-involution of G and $(\widehat{L}_i, \operatorname{Fix}(\widehat{\phi_i}), \widehat{\phi_i}(\widehat{L}_i))$ forms a partition of $V(G)$.

It remains to show that $G|_{\widehat{L}_i}$ is a union of connected components of $G - \operatorname{Fix}(\widehat{\phi_i})$. To show this, it is sufficient to show that if $v_1, v_2 \in V(G) \setminus \operatorname{Fix}(\widehat{\phi_i})$ are in the same component of $G - \operatorname{Fix}(\widehat{\phi_i})$ and $v_1 \in \widehat{L}_i$, then $v_2 \in \widehat{L}_i$. But indeed, if we partition any path P from v_1 to v_2 into segments $P_1, \ldots, P_n$ such that each segment P_t is completely contained in $V(G_{j_t})$ for some $j_t \in [k]$, then P_t must be entirely contained in L_{j_t} as L_{j_t} is a union of connected components of $G_{j_t} - \operatorname{Fix}(\psi_{ij})$ (and $\operatorname{Fix}(\psi_{ij}) = \operatorname{Fix}(\widehat{\phi_i}) \cap V(G_j)$). Therefore $(\widehat{\phi_i}, \widehat{L}_i)$ is a fold of G. ■

We now prove Theorem 3.1, which says that every left-reflection bigraph is left-cut-percolating under reflection folds.

Proof of Theorem 3.1. Let G be an $(S_0; S_1, \ldots, S_k; S_\Delta)$-left-reflection bigraph relative to the reflection group R with set of reflections T and for each $j \in [k]$ let G_j be the $(S_0, S_j; S_\Delta)$-reflection bigraph.

By [CL17, Corollary 4.9 and Theorem 4.12], there exists a cut-percolating sequence $E_0, \ldots, E_m$ with respect to a sequence $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ of *reflection folds* of G_1 ($i \in [m]$).

Since G_1 does not have isolated vertices, by Remark 2.6, setting $U_i \stackrel{\text{def}}{=} E_i \cap V_1(G_1)$ ($i \in \{0, 1, \ldots, m\}$) gives a left-cut-percolating sequence of G_1 with respect to Φ.

For each $i \in [m]$, let $t_i \in T$ be a reflection defining the reflection fold (ϕ_i, L_i) of G_1, that is, we have $\phi_i = \phi(t_i; S_0, S_1; S_\Delta)$ and $L_i = L(t_i; S_0, S_1; S_\Delta)$ and for each $j \in \{2, \ldots, k\}$, let $(\psi_{ij}, L_{ij}) \stackrel{\text{def}}{=} (\phi(t_i; S_0, S_j; S_\Delta), L(t_i; S_0, S_j; S_\Delta))$ be the reflection fold of G_j defined by the same reflection t_i. Then the hypotheses of Lemma 4.1 are satisfied and we deduce that G is left-cut-percolating with respect to a sequence of reflection folds. ■

To prove Theorem 3.3, we need to recall another definition and lemma from [CL17].

Definition 4.2 (slightly adapted from Conlon–Lee [CL17, §3])**.** Let G be bigraph, let $c\colon E(G) \to C$ be a coloring of G and let $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ be a sequence of folds of G. The *Cauchy–Schwarz tree* rooted at (G, c) corresponding to Φ is the rooted complete binary tree $T(G, c, \Phi)$ of height m in which each vertex is labeled by a coloring of c so that:

i. The root of $T(G, c, \Phi)$ is labeled by c.

ii. If a node at height $i \in \{0, 1, \ldots, m-1\}$ is labeled by c', then its left and right children are labeled by $c \circ (\phi_i)_{L_i}$ and $c \circ (\phi_i)^*_{L_i}$, respectively.

Lemma 4.3 (adapted from Conlon–Lee [CL17, §3]). *Let G be a bigraph, let $c\colon E(G) \to C$ be a coloring of G, let (ϕ, L) be a fold of G and let $W = (W_j)_{j \in C}$ be a sequence of bigraphons. Then*

$$t((G, c), W) \leq t((G, c \circ \phi_L), W)^{1/2} \cdot t((G, c \circ \phi^*_L), W)^{1/2},$$

More generally, if $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ is a sequence of folds of G and $c_1, \ldots, c_{2^m}$ are the colorings (with multiplicities) that label the leaves of the Cauchy–Schwarz tree $T(G, c, \Phi)$, then

$$t((G, c), W) \leq \prod_{t=1}^{2^m} t((G, c_t), W)^{2^{-m}}.$$

Proof. For the first statement, let

$$\begin{aligned}
G_0 &\stackrel{\text{def}}{=} G|_{\operatorname{Fix}(\phi)}, & G_1 &\stackrel{\text{def}}{=} G|_{L \cup \operatorname{Fix}(\phi)} - E(G_0), & G_2 &\stackrel{\text{def}}{=} G|_{\phi(L) \cup \operatorname{Fix}(\phi)} - E(G_0), \\
c_0 &\stackrel{\text{def}}{=} c|_{E(G_0)}, & c_1 &\stackrel{\text{def}}{=} c|_{E(G_1)}, & c_2 &\stackrel{\text{def}}{=} c|_{E(G_2)}, \\
F_0 &\stackrel{\text{def}}{=} ((G_0, c_0), \operatorname{Fix}(\phi)), & F_1 &\stackrel{\text{def}}{=} ((G_1, c_1), \operatorname{Fix}(\phi)), & F_2 &\stackrel{\text{def}}{=} ((G_2, c_2), \operatorname{Fix}(\phi)).
\end{aligned}$$

By Cauchy–Schwarz Inequality, we have

$$\begin{aligned}
t((G, c), W) &= \int_{X^{V_1} \cap Y^{V_2}} t(F_0, W)(x, y) \cdot t(F_1, W)(x, y) \cdot t(F_2, W)(x, y)\ d(\mu \otimes \nu)(x, y) \\
&\leq \left(\int_{X^{V_1} \cap Y^{V_2}} t(F_0, W)(x, y) \cdot t(F_1, W)(x, y)^2\ d(\mu \otimes \nu)(x, y) \right)^{1/2} \\
&\qquad \cdot \left(\int_{X^{V_1} \cap Y^{V_2}} t(F_0, W)(x, y) \cdot t(F_2, W)(x, y)^2\ d(\mu \otimes \nu)(x, y) \right)^{1/2} \\
&= t((G, c \circ \phi_L), W)^{1/2} \cdot t((G, c \circ \phi^*_L), W)^{1/2},
\end{aligned}$$

where $V_i \stackrel{\text{def}}{=} V_i(G) \cap \operatorname{Fix}(\phi)$.

The statement for Cauchy–Schwarz trees follows by induction. ■

We now show Theorem 3.3, whose statement is repeated below.

Theorem 3.3. *Let G be a left-cut-percolating bigraph under a set S of folds of G. If $c\colon E(G) \to C$ is a coloring of G that is invariant under the subgroup of $\operatorname{Aut}(G)$ generated by $\{\phi \mid (\phi, L) \in S\}$, then (G, c) is left-weakly Hölder.*

Proof of Theorem 3.3. Let $U_0, \ldots, U_m$ be a left-cut-percolating sequence of G with respect to a sequence $\Phi \stackrel{\text{def}}{=} ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ of folds in S and for each $t \in \mathbb{N}_+$, let Φ^t be the concatenation of Φ with itself t times. Let also $\widehat{S}$ be the subgroup of $\operatorname{Aut}(G)$ generated by $\{\phi \mid (\phi, L) \in S\}$.

Let us call a coloring of G left-constant if it is of the form $\ell \otimes c$ for some *constant* left-coloring $\ell\colon V_1(G) \to C'$.

We claim that for any left-coloring $\ell\colon V_1(G) \to C'$, at least a $1 - (1 - 2^{-m})^t$ fraction of the leaves of the Cauchy–Schwarz tree $T(G, \ell \otimes c, \Phi^t)$ are labeled by left-constant colorings of G.

For $t = 1$, since $U_0, \ldots, U_m$ is a left-cut-percolating sequence, the label of the leftmost leaf of $T(G, \ell \otimes c, \Phi)$ is

$$c_0 \stackrel{\text{def}}{=} (\ell \otimes c) \circ (\phi_1)_{L_1} \circ \cdots \circ (\phi_m)_{L_m} = (\ell \circ (\phi_1)_{L_1} \circ \cdots \circ (\phi_m)_{L_m}) \otimes c,$$

where the equality follows from the fact that c is $\widehat{S}$-invariant. This means that if v_0 is the unique element of U_0, then for every $(v, w) \in E(G)$, we have

$$c_0(v, w) = ((\ell \circ (\phi_1)_{L_1} \circ \cdots (\phi_m)_{L_m})(v), c(v, w)) = (\ell(v_0), c(v, w)),$$

so $c_0 = \ell(v_0) \otimes c(v, w)$. Therefore at least a $2^{-m} = 1 - (1 - 2^{-m})^1$ fraction of leaves are labeled by constant left-colorings

Suppose now that $t \geq 2$ and note that if a node of $T(G, \ell \otimes c, \Phi^t)$ is labeled by a left-constant coloring c_0, then all of its descendants must also be labeled by c_0. By induction, we also know that at least a $1 - (1 - 2^{-m})^{t-1}$ fraction of the nodes at level $(t-1)m$ are labeled by left-constant colorings. On the other hand, the case $t = 1$ above also guarantees that for each node at level $(t-1)m$, at least one of its descendant leaves is labeled by a left-constant coloring. This means that the fraction of leaves of $T(G, \ell \otimes c, \Phi^t)$ labeled by left-constant colorings is at least

$$(1 - (1 - 2^{-m})^{t-1}) + \frac{(1 - 2^{-m})^{t-1}}{2^m} = 1 - (1 - 2^{-m})^t,$$

as desired.

Note now that the definition of $T(G, \ell \otimes c, \Phi^t)$ ensures that each of the left-constant colorings that appears in the leaves must be of the form $\ell(v) \otimes c$ for some $v \in V_1(G)$. This means that if $\mathcal{C}$ is the set of colorings $c'\colon E(G) \to C' \times C$ that are *not* left-constant, then applying Lemma 4.3 to $T(G, \ell \otimes c, \Phi^t)$ and using the claim above, we get that

$$t((G, \ell \otimes c), W) \leq \prod_{v \in V_1(G)} t((G, \ell(v) \otimes c), W)^{\alpha_v^t} \cdot \prod_{c' \in \mathcal{C}} t((G, c'), W)^{\beta_{c'}^t}$$

for some $\alpha_v^t \in [0, 1]$ and $\beta_c^t \in [0, 1]$ satisfying

$$\sum_{v \in V_1(G)} \alpha_v^t + \sum_{c' \in \mathcal{C}} \beta_{c'}^t = 1, \qquad \sum_{v \in V_1(G)} \alpha_v^t \geq 1 - (1 - 2^{-m})^t.$$

Letting $t \to \infty$ along some subsequence such that $(\alpha_v^t)_t$ is convergent for every $v \in V_1(G)$, we get

$$t((G, \ell \otimes c), W) \leq \prod_{v \in V_1(G)} t((G, \ell(v) \otimes c), W)^{\alpha_v}$$

for some $\alpha_v \geq 0$ such that $\sum_{v \in V_1(G)} \alpha_v = 1$, since we have $\beta_t^{c'} \leq (1 - 2^{-m})^t \xrightarrow{t \to \infty} 0$ for every $c' \in \mathcal{C}$.

Recall from Remark 2.5 that G is left-vertex-transitive under the action of $\widehat{S}$. Then, by Remark 2.8 we get

$$\begin{aligned} t((G, \ell \otimes c), W) &= \prod_{\sigma \in \widehat{S}} t((G, (\ell \otimes c) \circ \sigma), W)^{1/\lvert\widehat{S}\rvert} \\ &\leq \prod_{\sigma \in \widehat{S}} \prod_{v \in V_1(G)} t((G, (\ell(v) \otimes c) \circ \sigma), W)^{\alpha_v/\lvert\widehat{S}\rvert} \\ &= \prod_{\sigma \in \widehat{S}} \prod_{v \in V_1(G)} t((G, (\ell(\sigma(v)) \otimes c)), W)^{\alpha_v/\lvert\widehat{S}\rvert} \\ &= \prod_{v \in V_1(G)} t((G, \ell(v) \otimes c), W)^{1/v_1(G)}, \end{aligned}$$

where the second equality follows since c is $\widehat{S}$-invariant and the third equality follows since G is left-vertex-transitive under the action of $\widehat{S}$. Therefore (G, c) is left-weakly Hölder. ■

Finally, for Theorem 3.5, which says that the incidence bigraph of the complete hypergraph on n vertices and in uniformities $k_1, \ldots, k_t$ is a left-reflection bigraph, we will heavily rely on the fact that the incidence bigraph of complete k_i-uniform hypergraph is a reflection bigraph.

Proof of Theorem 3.5. Recall from [CL17, Example 4.4] that the symmetric group $\mathfrak{S}_n$ on n points with its natural embedding in $\mathrm{GL}_n(\mathbb{R})$ is generated by the transpositions, which are the reflections of $\mathfrak{S}_n$. By using the canonical ordered basis $U \stackrel{\text{def}}{=} (e_1, \ldots, e_n)$, the set of positive roots of $\mathfrak{S}_n$ is precisely

$$\Phi^+ = \left\{ \frac{e_i - e_j}{\sqrt{2}} \;\middle|\; 1 \leq i < j \leq n \right\},$$

the set of simple roots is

$$\Delta = \left\{ \frac{e_i - e_{i+1}}{\sqrt{2}} \;\middle|\; i \in [n-1] \right\},$$

and the set of simple reflections is

$$S_\Delta = \{ t_{i,i+1} \mid i \in [n-1] \},$$

where $t_{i,j}$ is the transposition that swaps i and j.

For $k \in [n]$, let $S_k \stackrel{\text{def}}{=} S_\Delta \setminus \{t_{k,k+1}\}$, which generates the naturally embedded subgroup $R_k \stackrel{\text{def}}{=} \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. The left-cosets of R_k can be identified with $\binom{[n]}{k}$ naturally via $\sigma R_k \mapsto \sigma([k])$, which means that the $(S_1, S_k; S_\Delta)$-reflection bigraph is isomorphic to the incidence bigraph of the complete k-uniform hypergraph H on n vertices given by

$$V_1(H) \stackrel{\text{def}}{=} [n], \qquad V_2(H) \stackrel{\text{def}}{=} \binom{[n]}{k}, \qquad E(H) \stackrel{\text{def}}{=} \left\{(i, A) \in [n] \times \binom{[n]}{k} \;\middle|\; i \in A\right\},$$

and the isomorphism is given by

$$\mathfrak{S}_n/R_1 \ni \sigma R_1 \mapsto \sigma(1) \in V_1(H),$$
$$\mathfrak{S}_n/R_k \ni \sigma R_k \mapsto \sigma([k]) \in V_1(H).$$

Thus the $(S_1; S_{k_1}, \ldots, S_{k_t}; S_\Delta)$-left-reflection bigraph is isomorphic to the incidence bigraph G of the complete hypergraph on n vertices and in uniformities $k_1, \ldots, k_t$, with the isomorphism given by

$$\mathfrak{S}_n/R_1 \ni \sigma R_1 \mapsto \sigma(1) \in V_1(G),$$
$$\mathfrak{S}_n/R_{k_i} \ni \sigma R_{k_i} \mapsto (\sigma([k]), j) \in V_2(G).$$ ■

We conclude this section proving Theorem 3.4, which says that left-reflection bigraphs become right-uniform color-edge-transitive left-weakly Hölder bigraphs when equipped with their natural coloring.

Proof of Theorem 3.4. The fact that G is left-weakly Hölder follows from Theorems 3.1 and 3.3 by noting that the natural coloring of a left-reflection bigraph is invariant under the subgroup generated by the cut-involutions coming from reflection folds.

It is also obvious from the definition of the natural coloring that it is right-uniform. Finally, since each of the color classes of G yields a reflection bigraph, which is cut-percolating under reflection folds by [CL17, Corollary 4.9 and Theorem 4.12], hence edge-transitive with respect to cut-involutions coming from reflection folds (see Remark 2.5), it follows that G is color-edge-transitive under the action of the group generated by cut-involutions coming from reflection folds. ■

5 Induced-Sidorenko

In this section, we prove Theorem 3.2 that says that every left-cut-percolating bigraph is induced-Sidorenko.

We start with a lemma that is often viewed as an obstacle for inequalities concerning densities in bigraphons. However, this lemma can also be used positively to deduce equalities of exponents (see for example its use in Remark 2.13).

Lemma 5.1. *Let W be a non-zero bigraphon, let $G_1, \ldots, G_n$ be bigraphs and let $r_1, \ldots, r_n \in \mathbb{R}$. If for every $\lambda > 0$, we have*

$$\prod_{i=1}^{n} t(G_i, \lambda W)^{r_i} \geq 1,$$

then $\sum_{i=1}^{n} r_i \cdot e(G_i) = 0$.

Proof. Since $t(G_i, \lambda W) = \lambda^{e(G_i)} \cdot t(G_i, W)$, we must have

$$\lambda^{\sum_{i=1}^{n} r_i \cdot e(G_i)} \prod_{i=1}^{n} t(G_i, W)^{r_i} \geq 1$$

for every $\lambda > 0$. Since W is non-zero, we have $t(G_i, W) > 0$. If the exponent of λ is positive, then making λ small enough violates the inequality above. If the exponent of λ is negative, then making λ large enough violates the inequality above. ■

We now introduce the notion of 2-threshold subgraphs.

Definition 5.2. Let G be a bigraph and let $f\colon V(G) \to \{0, 1, 2\}$, the 2*-threshold subgraph of G corresponding to f* is the spanning subgraph G_f of G defined by

$$V_1(G_f) \stackrel{\text{def}}{=} V_1(G), \qquad V_2(G_f) \stackrel{\text{def}}{=} V_2(G),$$
$$E(G_f) \stackrel{\text{def}}{=} \{(v, w) \in E(G) \mid f(v) + f(w) \geq 2\}.$$

We let $T_2(G)$ be the set of all 2-threshold subgraphs of G.

Given an endomorphism $\phi \in \text{End}(G)$ of G and a spanning subgraph G' of G, we let $\phi^{-1}(G')$ be the spanning subgraph of G defined by

$$V_1(\phi^{-1}(G')) \stackrel{\text{def}}{=} V_1(G), \qquad V_2(\phi^{-1}(G')) \stackrel{\text{def}}{=} V_2(G),$$
$$E(\phi^{-1}(G')) \stackrel{\text{def}}{=} \phi^{-1}(E(G')) = \{(v, w) \in E(G) \mid (\phi(v), \phi(w)) \in E(G')\}.$$

Remark 5.3. It is easy to see that if $f = \mathbb{1}_V$ for some $V \subseteq V(G)$, then $G_f = G - (E(G) \setminus E(G|_V))$, that is, 2-threshold subgraphs corresponding to $\{0, 1\}$-valued functions essentially capture induced subgraphs (except for the presence of extra isolated vertices to make them spanning).

It is also easy to see that for an endomorphism $\phi \in \text{End}(G)$ of G and $f\colon V(G) \to \{0, 1, 2\}$, we have $\phi^{-1}(G_f) = G_{f \circ \phi}$.

Lemma 5.4. *Let G be a left-cut-percolating bigraph and let*

$$T \stackrel{\text{def}}{=} \{G_f \in T_2(G) \mid f\colon V(G) \to \{0, 1, 2\}, f^{-1}(2) \subseteq V_1(G)\}.$$

Then there exists $m \in \mathbb{N}_+$ such that for every $H \in T$, there exist $\ell_H \in \mathbb{N}$ with $\ell_H \leq e(G)/v_1(G)$ and a function $r_H \colon T \to \mathbb{R}_+$ such that $\sum_{H' \in T} r_H(H') = 1 - 2^{-m}$ and

$$t(H, W) \leq \frac{t(G, W)^{1/2^m}}{t(\rho, W)^{\ell_H}} \cdot \prod_{H' \in T} t(H', W)^{r_H(H')}.$$

for every biregular non-zero bigraphon W.

Proof. Let $U_0, \ldots, U_m$ be a left-cut-percolating sequence of G with respect to a sequence of folds $\Phi = ((\phi_1, L_1), \ldots, (\phi_m, L_m))$ and let v_0 be the unique element of U_0.

Fix an element $H = G_f$ ($f \colon V(G) \to \{0, 1, 2\}$ with $f^{-1}(2) \subseteq V_1(G)$) of T and let us first prove the case when $f(v_0) = 2$ and in this case, we will show that we can take $\ell_H = 0$.

Note that there is a natural one-to-one correspondence between colorings $c \colon E(G) \to \{0, 1\}$ of G and spanning subgraphs of G in which a coloring c corresponds to the spanning subgraph G^c given by $E(G^c) \stackrel{\text{def}}{=} c^{-1}(1)$ and a spanning subgraph G' of G corresponds to the coloring $c_{G'} \stackrel{\text{def}}{=} \mathbb{1}_{E(G')}$. Consider the sequence $W' = (W'_0, W'_1)$ of bigraphons, where $W'_0 \stackrel{\text{def}}{=} 1$ and $W'_1 \stackrel{\text{def}}{=} W$ and note that $t((G, c), W') = t(G^c, W)$.

Consider now the Cauchy–Schwarz tree $T(G, c_H, \Phi)$ and since the folding maps are endomorphisms of G, by Remark 5.3, all of the nodes of this tree are labeled by colorings of the form $c_{H'}$ for some $H' \in T$. Note further that by the same remark, the leftmost leaf of Φ has label $c_0 \stackrel{\text{def}}{=} c_{G_g}$, where $g \colon V(G) \to \{0, 1\}$ is given by

$$g \stackrel{\text{def}}{=} f \circ (\phi_1)_{L_1} \circ \cdots (\phi_m)_{L_m},$$

and since $U_0, \ldots, U_m$ is left-cut-percolating, we have $g(v) = g(v_0) = 2$ for every $v \in V_1(G)$, which implies $G^{c_0} = G_g = G$. By Lemma 4.3, it follows that

$$t(H, W) = t((G, c_H), W) \leq t(G, W)^{1/2^m} \cdot \prod_{H' \in T} t(H', W)^{r_H(H')},$$

where $r_H(H')$ is the number of non-leftmost leaves of $T(G, c_H, \Phi)$ that are labeled by $c_{H'}$ divided by 2^m (note that c_G can also appear as the label of non-leftmost leaves) and thus $\sum_{H' \in T} r_H(H') = 1 - 2^{-m}$.

We now consider the case when $f(v) = 2$ for some $v \in V_1(G)$ (but not necessarily $v = v_0$) and we will also show that in this case we can take $\ell_H = 0$. By Remark 2.5, we know that G is left-vertex-transitive, so letting $\psi \in \operatorname{Aut}(G)$ be an automorphism with $\psi(v_0) = v$, from Remarks 2.8 and 5.3, we get

$$\begin{aligned} t(H, W) &= t((G, c_H), W') = t((G, c_H \circ \psi), W') \\ &= t(\psi^{-1}(G_f), W) = t(G_{f \circ \psi}, W) \end{aligned}$$

so the result follows from the previous case as $(f \circ \psi)(v_0) = 2$.

Let us now show the case when $\operatorname{im}(f) \subseteq \{0,1\}$ and $f(v_0) = 1$. Let $f'\colon V(G) \to \{0,1,2\}$ be the function obtained from f by changing the value of v_0 to 2 and let

$$\ell_H \stackrel{\text{def}}{=} |N_G(v_0) \cap f^{-1}(0)| \leq d_G(v_0) = \frac{e(G)}{v_1(G)}$$

be the number of neighbors of v_0 that have value 0 under f (the last equality in the above follows from left-vertex-transitivity). Note that $G_{f'}$ is obtained from G_f by precisely adding the ℓ_H edges from v_0 to its neighbors in G whose value under f is 0, and since $\operatorname{im}(f) \subseteq \{0,1\}$, the right endpoint of these newly added edges were isolated vertices in G_f, hence in $G_{f'}$ they have only v_0 as their neighbor. Since W is biregular, it follows that

$$t(G_{f'}, W) = t(G_f, W) \cdot t(\rho, W)^{\ell_H}.$$

Since $f'(v_0) = 2$, the result now follows from the first case (note that the upper bound on ℓ_H for this case is $e(G)/v_1(G)$ since the first case was shown with $\ell_H = 0$).

The case when $\operatorname{im}(f) \subseteq \{0,1\}$ and there exists $v \in V_1(G)$ with $f(v) = 1$ follows by left-vertex-transitivity from the previous case.

The final case is when $f|_{V_1(G)} = 0$. But since $f(V_2(G)) \subseteq \{0,1\}$, it follows that H is empty and since G is a Sidorenko bigraph (by Theorem 3.3 and Remark 2.13), we get

$$t(H,W) = 1 \leq \frac{t(G,W)^{1/2^m}}{t(\rho,W)^{e(G)/2^m}} = \frac{t(G,W)^{1/2^m}}{t(\rho,W)^{e(G)/2^m}} \cdot t(H,W)^{1-2^{-m}},$$

which means that the result will follow by setting $r_H(H') \stackrel{\text{def}}{=} \mathbb{1}[H' = H](1 - 2^{-m})$ and $\ell_H \stackrel{\text{def}}{=} e(G)/2^m$ as long as we prove the bound $\ell_H \leq e(G)/v_1(G)$, that is, we need to show that $v_1(G) \leq 2^m$. But note that from the definition of left-cut-percolating sequence, we have $|U_i| \leq 2|U_{i-1}|$ for every $i \in [m]$, hence a simple induction gives $v_1(G) = |U_m| \leq 2^m|U_0| = 2^m$, as desired. ■

We conclude this section proving Theorem 3.2, which says that every left-cut-percolating bigraph is induced-Sidorenko.

Proof of Theorem 3.2. Let G be a left-cut-percolating bigraph and G' be an induced bigraph of G and let us show that G weakly dominates G'.

Let T and m be as in Lemma 5.4 and for each $H \in T$, let ℓ_H and r_H also be as in the same lemma. Let also $p \stackrel{\text{def}}{=} 1 - 2^{-m}$.

Let $H \stackrel{\text{def}}{=} G_{\mathbb{1}_{V(G')}}$ be the 2-threshold subgraph of G corresponding to the indicator function $\mathbb{1}_{V(G')}$ of $V(G')$. Note that $H \in T$ and since $E(H) = E(G')$, for every biregular non-zero bigraphon W, we have $t(G', W) = t(H, W)$, so to show that G weakly dominates G', we can show that it weakly dominates H (see Remark 5.3).

We construct a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of non-negative reals and a sequence $(r_n)_{n\in\mathbb{N}}$ of functions $r_n\colon T \to \mathbb{R}_+$ such that the following hold.

i. For every $n \in \mathbb{N}$ and every biregular non-zero bigraphon W, we have

$$t(H,W) \leq \frac{t(G,W)^{1-p^n}}{t(\rho,W)^{\alpha_n}} \prod_{H' \in T} t(H',W)^{r_n(H')}. \tag{4}$$

ii. For every $n \in \mathbb{N}$, we have

$$\sum_{H' \in T} r_n(H') = p^n.$$

iii. For every $n \in \mathbb{N}$, we have

$$\alpha_n \leq \alpha_{n+1} \leq \alpha_n + p^n \cdot \frac{e(G)}{v_1(G)}.$$

The construction is by induction: we start with $\alpha_0 = 0$ and $r_0(H') = \mathbb{1}[H = H']$ and for $n \in \mathbb{N}_+$, we define

$$\alpha_n \stackrel{\text{def}}{=} \alpha_{n-1} + \sum_{H' \in T} r_{n-1}(H') \cdot \ell_{H'}, \qquad r_n(H') \stackrel{\text{def}}{=} \sum_{H'' \in T} r_{H''}(H') \cdot r_{n-1}(H'').$$

The three items follow by induction when we apply Lemma 5.4 to all $H' \in T$ on the right-hand side of (4).

Note that the second item ensures that $\lim_{n\to\infty} r_n(H') = 0$ for every $H' \in T$ and the third item ensures that the limit $\alpha \stackrel{\text{def}}{=} \lim_{n\to\infty} \alpha_n$ exists as the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is non-decreasing and upper bounded by $e(G)/((1-p)v_1(G))$. By letting $n \to \infty$ in (4), we conclude that for every biregular non-zero bigraphon W, we have

$$t(H,W) \leq \frac{t(G,W)}{t(\rho,W)^{\alpha}}$$

and since this holds for every such W, by Lemma 5.1, it follows that $\alpha = e(G) - e(H)$, so G weakly dominates H as desired. ■

6 Symmetrizations and fractional powers of colored bigraphs

In this section we prove Theorems 3.6 and 3.7. We start by showing how the former (which is restated below) is a particular case of the latter.

Theorem 3.6. *Let G be a bigraph without isolated vertices and for each $k \in \mathbb{N}$, let $d_k \stackrel{\text{def}}{=} |\{w \in V_2(G) \mid d_G(w) = k\}|$ be the number of vertices in $V_2(G)$ that have degree k.*

If for each $k \geq 2$, we have either $d_k = 0$ or $d_k \geq \binom{v_1(G)}{k}$, then G is a strong Sidorenko bigraph.

Proof of Theorem 3.6. Without loss of generality, suppose that $V_1(G) = [n]$. Let

$$K \stackrel{\text{def}}{=} \{d_G(w) \mid w \in V_2(G)\} \setminus \{0,1\} = \{k \in \{2,\ldots,n\} \mid d_k \neq 0\}$$

and enumerate its elements as $k_1,\ldots,k_t$. Let G' be the incidence bigraph of the complete hypergraph on n vertices and in uniformities $k_1,\ldots,k_t$. By Theorem 3.5, we know that G' is a left-reflection bigraph. It is straightforward to see that its natural coloring $c\colon E(G') \to [t]$ is the unique function such that $d_{G'}(w) = k_{c(v,w)}$ for every $(v,w) \in E(G')$. By Theorem 3.4, we know that $H \stackrel{\text{def}}{=} (G,c)$ is left-weakly Hölder. Note also that $\mathrm{Aut}(H) = \mathrm{Aut}(G')$ (and is isomorphic to the symmetric group $\mathfrak{S}_n$ on n points). Furthermore, note that for $U \subseteq [n]$, we have

$$\sum_{\sigma\in\mathrm{Aut}(H)} d_G(\sigma(U)) = \frac{|\mathrm{Aut}(H)| \cdot d_{|U|}}{\binom{n}{|U|}},$$

$$\sum_{\sigma\in\mathrm{Aut}(H)} d_H(\sigma(U)) = \begin{cases} |\mathrm{Aut}(H)|, & \text{if } |U| \in K,\\ 0, & \text{otherwise.}\end{cases}$$

Since $d_k \geq \binom{n}{k}$ for every $k \in K$, the result now follows from Theorem 3.7. ■

To prove Theorem 3.7, we will need to work with fractional colored bigraphs and a notion that we call color-Sidorenko defined below. The intuition is that a fractional colored bigraph encodes the left side of a right-uniform colored bigraph as its vertex set V and counts how many vertices on the right side have neighborhood in color $i \in C$ exactly equal to a set $U \subseteq V$, except that we allow this count to be fractional.

Definition 6.1. A *colored fractional bigraph* is a function $h\colon 2^V \times C \to \mathbb{R}_+$, where V and C are sets, called *vertex set* and *color set* of h, respectively. We use the shorthand notations $V_h \stackrel{\text{def}}{=} V$ and $C_h \stackrel{\text{def}}{=} C$ and we let

$$v(h) \stackrel{\text{def}}{=} |V_h|,$$
$$e_i(h) \stackrel{\text{def}}{=} \sum_{U\subseteq V_h} |U| \cdot h(U,i) \qquad (i \in C),$$
$$e(h) \stackrel{\text{def}}{=} \sum_{i\in C} e_i(h).$$

For each $i \in C_h$ and each $v \in V_h$, the *i-degree of v in h* is defined as

$$d_{h,i}(v) \stackrel{\text{def}}{=} \sum_{\substack{U\subseteq V_h\\ v\in U}} h(U,i).$$

We say that h is *color-regular* if for every $i \in C_h$ and every $v_1, v_2 \in V_h$, we have $d_{h,i}(v_1) = d_{h,i}(v_2)$ (which is equivalent to saying that $d_{h,i}(v) = e_i(h)/v(h)$ for every $v \in V_h$ and every $i \in C_h$).

Given a right-uniform colored bigraph H without isolated vertices, its corresponding colored fractional bigraph $h_H \colon 2^{V_1(H)} \times C_H \to \mathbb{R}_+$ is defined by

$$h_H(U,i) \stackrel{\text{def}}{=} |\{w \in V_2(H) \mid N_H(w) = U \wedge \forall v \in N_H(w), c_H(v,w) = i\}|.$$

Given a colored fractional bigraph h and a sequence $W = (W_i)_{i \in C_h}$ of bigraphons over the same spaces $\Omega = (X,\mu)$ and $\Lambda = (Y,\nu)$, we let

$$t(h,W) \stackrel{\text{def}}{=} \int_{X^{V_h}} \prod_{U \subseteq V_h} \prod_{i \in C_h} t(K^L_{|U|,1}, W_i)(x_U)^{h(U,i)} \, d\mu(x).$$

(Note that this ensures that $t(H,W) = t(h_H,W)$ for right-uniform colored bigraphs H.)

Given a colored fractional bigraph h and a tuple $\vec{p} \stackrel{\text{def}}{=} (p_i)_{i \in C_h} \in \mathbb{R}_+^{C_h}$, the $\vec{p}$ *color-power* of h is the colored fractional bigraph $h^{\vec{p}} \colon 2^{V_h} \times C_h \to \mathbb{R}_+$ given by

$$h^{\vec{p}}(U,i) \stackrel{\text{def}}{=} h(U,i) \cdot p_i$$

We extend this definition to right-uniform colored bigraphs H without isolated vertices as $H^{\vec{p}} \stackrel{\text{def}}{=} h_H^{\vec{p}}$.

Given a set of colors C, the C*-rainbow star* is the connected colored bigraph ρ_C with one vertex on the left side and one edge of each color in C. Formally, it is given by

$$V_1(\rho_C) \stackrel{\text{def}}{=} \{1\}, \qquad V_2(\rho_C) \stackrel{\text{def}}{=} C, \qquad E(\rho_C) \stackrel{\text{def}}{=} \{1\} \times C,$$

and $c_{\rho_C} \colon E(\rho_C) \to C$ is defined by

$$c_{\rho_C}(1,i) \stackrel{\text{def}}{=} i \qquad (i \in C).$$

Given a colored fractional bigraph h with $e(h) > 0$, we let $\rho_h \stackrel{\text{def}}{=} \rho_{C_h}^{\vec{p}}$, where $\vec{p} = (p_i)_{i \in C_h}$ is given by $p_i \stackrel{\text{def}}{=} e_i(h)/e(h)$. We extend this definition to right-uniform colored bigraphs H by letting $\rho_H \stackrel{\text{def}}{=} \rho_{h_H}$.

A colored fractional bigraph h with $e(h) > 0$ is called *color-Sidorenko* if for every sequence $W = (W_i)_{i \in C_h}$ of bigraphons over the same spaces we have

$$t(h,W) \geq t(\rho_h, W)^{e(h)}.$$

A right-uniform colored bigraph H is called *color-Sidorenko* if its corresponding colored fractional bigraph h_H is color-Sidorenko.

We start by showing that right-uniform left-weakly Hölder bigraphs H are left-color regular and color-Sidorenko.

Lemma 6.2. *If $H = (G,c)$ is a non-trivial left-weakly Hölder bigraph without isolated vertices, then H is left-color-regular. Furthermore, if H is also right-uniform, then H is color-Sidorenko.*

Proof. Fix $v_0 \in V_1(H)$ and a sequence $W = (W_i)_{i\in C_H}$ of bigraphons on the same spaces $\Omega = (X,\mu)$ and $\Lambda = (Y,\nu)$. Let $\ell\colon V_1(H) \to \{0,1\}$ be given by $\ell(v) \stackrel{\text{def}}{=} \mathbb{1}[v = v_0]$ and define the sequence $W' = (W'_{t,i})_{t\in\{0,1\},i\in C_H}$ by

$$W'_{t,i} \stackrel{\text{def}}{=} \begin{cases} W_i, & \text{if } t = 1,\\ 1, & \text{if } t = 0.\end{cases}$$

Then we have

$$t((G, 0 \otimes c), W') = 1, \qquad\qquad t((G, 1 \otimes c), W') = t(H, W).$$

Let further $H' \stackrel{\text{def}}{=} H|_{v_0\cup N_H(v_0)}$ be the restriction of H to v_0 and its neighbors and note that $t((G, \ell \otimes c), W') = t(H', W)$, so the left-weak Hölder property gives $t(H', W) \leq t(H, W)^{1/v_1(H)}$, which can be rewritten as

$$t(H, W) \geq t(H', W)^{v_1(H)}.$$

If we instantiate the above to the case where $W_{i_0} = \widehat{W}$ for some fixed $i_0 \in C_H$ and $W_i = 1$ every $i \neq i_0$, we get

$$t(H_{\{i_0\}}, \widehat{W}) \geq t(K_{1,d_{H,i_0}(v_0)}, \widehat{W})^{v_1(H)},$$

which by Lemma 5.1 implies that $e_{i_0}(H) = d_{H,i_0}(v_0) \cdot v_1(H)$, that is, H is left-color-regular.

Going back to the general sequence W, if H is also right-uniform, then we conclude that

$$\begin{aligned} t(H, W) &\geq t(H', W)^{v_1(H)}\\ &= \left(\int_X \prod_{i\in C_H} \left(\int_Y W_i(x,y)\ d\nu(y)\right)^{d_{H,i}(v_0)}\ d\mu(x)\right)^{v_1(H)}\\ &\geq t(\rho_H, W)^{e(H)}, \end{aligned}$$

where the last inequality follows from Jensen's Inequality for the convex function $z \mapsto z^{e(H)/v_1(H)}$ and the fact that H is left-color-regular. Therefore H is color-Sidorenko. ∎

Our next objective is to prove that color-powers $H^{\vec{p}}$ of left-weakly Hölder bigraphs are also color-Sidorenko under the further assumptions that H is right-uniform and $p_i \geq 1$ for every $i \in C_H$. To do so we need to establish several lemmas. We start with one that says that it is sufficient to check the color-Sidorenko property only for sequences of bigraphons in which all but one are left-regular. The trick employed here is similar to the one in [CR21], except that since h is color-regular, we can perform a much simpler construction.

Lemma 6.3. *Let h be a color-regular colored fractional bigraph and let $i_0 \in C_h$ be such that $e_{i_0}(h) \neq 0$. Suppose that $t(h, W) \geq t(\rho_h, W)^{e(h)}$ for every sequence $W = (W_i)_{i\in C_h}$ of positive bigraphons over the same spaces such that W_i is left-regular and non-zero for every $i \in C_h \setminus \{i_0\}$. Then h is color-Sidorenko.*

Proof. Without loss of generality, we may suppose that $e_i(h) \neq 0$ for every $i \in C_h$.

By possibly replacing each W_i with $W_i^\epsilon \stackrel{\text{def}}{=} \epsilon + W_i$ and applying the Dominated Convergence Theorem letting $\epsilon \to 0$, it is sufficient to show that $t(h, W) \geq t(\rho_h, W)^{e(h)}$ holds for every sequence of bigraphons over the same spaces that are bounded away from 0. Fix one such sequence W of bigraphons over spaces $\Omega = (X, \mu)$ and $\Lambda = (Y, \nu)$ and define the sequence $W' = (W'_i)_{i \in C_h}$ by

$$W'_i(x, y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{W_i(x, y)}{t(e_1, W_i)(x)}, & \text{if } i \neq i_0, \\ W_{i_0}(x, y) \cdot \displaystyle\prod_{j \in C_h \setminus \{i_0\}} t(e_1, W_j)(x)^{e_j(h)/e_{i_0}(h)}, & \text{if } i = i_0. \end{cases}$$

Note that the fact that each W_i is bounded away from 0 ensures that the functions above are bounded. It is also trivial that W'_i is left-regular for each $i \neq i_0$.

Note now that

$$\begin{aligned} t(\rho_h, W') = \int_X \left(t(e_1, W_{i_0})(x) \cdot \prod_{j \in C_h \setminus \{i_0\}} t(e_1, W_j)(x)^{e_j(h)/e_{i_0}(h)} \right)^{e_{i_0}(h)/e(h)} & \\ \cdot \prod_{i \in C_h \setminus \{i_0\}} \left(\frac{t(e_1, W_i)}{t(e_1, W_i)} \right)^{e_i(h)/e(h)} & \, d\mu(x) \\ = t(\rho_h, W). & \end{aligned}$$

On the other hand, since h is color-regular, we have $d_{h,i}(v) = e_i(h)/v(h)$ for every $i \in C_h$ and every $v \in V_h$, so we get

$$\begin{aligned} t(h, W') &= \int_{X^{V_h}} \prod_{U \subseteq V_h} \prod_{i \in C_h} t(K^L_{|U|,1}, W_i)(x_U)^{h(U,i)} \\ &\quad \cdot \prod_{v \in V_h} \prod_{i \in C_h \setminus \{i_0\}} t(e_1, W_i)(x)^{e_i(h) \cdot d_{h,i_0}(v)/e_{i_0}(h) - d_{h,i}(v)} \\ &= t(h, W) \end{aligned}$$

Therefore, we conclude that

$$t(h, W) = t(h, W') \geq t(\rho_h, W')^{e(h)} = t(\rho_h, W)^{e(h)},$$

so h is color-Sidorenko. ■

The next lemma can be seen as a color version of the induced-Sidorenko property for right-uniform left-weakly Hölder bigraphs.

Lemma 6.4. *Let $H = (G, c)$ be a right-uniform left-weakly Hölder bigraph, let $C \subseteq C_H$ be a subset of colors. Let also $W = (W_i)_{i \in C_H}$ be a sequence of bigraphons on the same space such that for each $i \in C_H \setminus C$, the bigraphon W_i is left-regular and non-zero. Then*

$$t(H_C, W) \leq \frac{t(H, W)}{\prod_{i \in C_H \setminus C} t(\rho, W_i)^{e_i(H)}}.$$

Proof. Without loss of generality, let us suppose that H does not have isolated vertices and is not trivial.

Fix a vertex $v_0 \in V_1(H)$, let $\ell \colon V_1(G) \to \{0, 1\}$ be the left-coloring of G defined by $\ell(v) \stackrel{\text{def}}{=} \mathbb{1}[v = v_0]$ and let $W' = (W'_{t,i})_{t \in \{0,1\}, i \in C_H}$ be given by

$$W'_{t,i} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } t = 0 \text{ and } i \notin C, \\ W_i, & \text{if } t = 1 \text{ or } i \in C. \end{cases}$$

Then we have

$$t((G, 0 \otimes c), W') = t(H_C, W), \qquad\qquad t((G, 1 \otimes c), W') = t(H, W).$$

Let now $H' = (G', c')$ be the colored bigraph obtained from G by removing all edges with colors in $C_H \setminus C$ that are not adjacent to v_0, that is, we have $G' \stackrel{\text{def}}{=} G - \{(v, w) \in c^{-1}(C_H \setminus C) \mid v \neq v_0\}$ and $c' \stackrel{\text{def}}{=} c|_{E(G')}$. Note that $t((G, \ell \otimes c), W') = t(H', W)$. Note also that since H is right-uniform, all edges (v, w) in $E(H') \setminus E(H_C)$ satisfy $v = v_0$, $d_{G'}(w) = 1$ and $c(v, w) \in C_H \setminus C$, so since each W_i with $i \in C_H \setminus C$ is left-regular, we get

$$t(H', W) = t(H_C, W) \cdot \prod_{i \in C_H \setminus C} t(\rho, W_i)^{d_{H,i}(v)}.$$

Recalling from Lemma 6.2 that $d_{H,i}(v) = e_i(H)/v_1(H)$, the left-weak Hölder property gives

$$\begin{aligned} t(H_C, W) \cdot \prod_{i \in C_H \setminus C} t(\rho, W_i)^{e_i(H)/v_1(H)} &= t(H', W) = t((G, \ell \otimes c), W') \\ &\leq t((G, 0 \otimes c), W')^{(v_1(H)-1)/v_1(H)} \cdot t((G, 1 \otimes c), W')^{1/v_1(H)} \\ &= t(H_C, W)^{(v_1(H)-1)/v_1(H)} \cdot t(H, W)^{1/v_1(H)}, \end{aligned}$$

so the result follows by taking the $v_1(H)$th power. ■

We now prove an inductive form of Jensen's inequality for moments (this can also be seen as inductive form of Hölder's Inequality).

Proposition 6.5. *Let $p_1 \geq p_2 \geq \cdots \geq p_n \geq 1$ and let $f_1, \ldots, f_n, g \colon \Omega \to \mathbb{R}_+$ be bounded positive measurable functions in a probability space $\Omega = (X, \mu)$. Then*

$$\int_X g(x) \cdot \prod_{i=1}^n f_i(x)^{p_i} \; d\mu(x) \geq \frac{\left(\int_X g(x) \cdot \prod_{i=1}^n f_i(x) \; d\mu(x)\right)^{p_1}}{\prod_{i=1}^n \left(\int_X g(x) \cdot \prod_{j=i+1}^n f_j(x) \; d\mu(x)\right)^{p_i - p_{i+1}}},$$

where $p_{n+1} \stackrel{\text{def}}{=} 1$ and products of the form $\prod_{j=t}^{t-1}$ are interpreted as 1.

Proof. The proof is by induction in n. For $n = 0$, the result is trivial. The case $n = 1$ follows directly from Jensen's Inequality for the convex function $z \mapsto z^{p_1}$.

Suppose then that $n \geq 2$ and that the result holds for $n - 1$. Then by the $n = 1$ case with the same g but taking the whole product as a single function and using the exponent p_n, we have

$$\int_X g(x) \cdot \prod_{i=1}^n f_i(x)^{p_i} \; d\mu(x) \geq \frac{\left(\int_X g(x) \cdot \prod_{i=1}^n f_i(x)^{p_i/p_n} \; d\mu(x)\right)^{p_n}}{\left(\int_X g(x) \; d\mu(x)\right)^{p_n - p_{n+1}}}.$$

The result now follows by applying the inductive hypothesis to the integral in the numerator above using $g \cdot f_n$ in place of g and exponents $p_1/p_n \geq \cdots \geq p_{n-1}/p_n$ for the functions $f_1, \ldots, f_{n-1}$, respectively. ■

We can now prove that color-powers of right-uniform left-weakly Hölder bigraphs are color-Sidorenko.

Lemma 6.6. *Let H be a non-trivial right-uniform left-weakly Hölder bigraph without isolated vertices and let $\vec{p} = (p_i)_{i \in C_H} \in \mathbb{R}_+^{C_H}$ be such that $p_i \geq 1$ for every $i \in C_H$. Then $H^{\vec{p}}$ is color-Sidorenko.*

Proof. By possibly removing unused colors and renaming them, we may assume without loss of generality that $\operatorname{im}(c_H) = C_H = [n]$ for some $n \in \mathbb{N}_+$ and that $p_1 \geq \cdots \geq p_n \geq 1$.

Let $h \stackrel{\text{def}}{=} H^{\vec{p}}$. By Lemma 6.3, it is sufficient to show that $t(h, W) \geq t(\rho_h, W)^{e(h)}$ only for sequences $W = (W_i)_{i=1}^n$ of positive bigraphons on the same spaces such that all W_i are left-regular except possibly for W_n.

For every $i \in [n]$, let $C_i \stackrel{\text{def}}{=} \{i+1, \ldots, n\}$. We now apply Proposition 6.5 to get

$$\begin{aligned} t(h, W) &= \int_{X^{V_1(H)}} \prod_{i=1}^n t(H^L_{\{i\}}, W)(x)^{p_i} \; d\mu(x) \\ &\geq \frac{\left(\int_{X^{V_1(H)}} \prod_{i=1}^n t(H^L_{\{i\}}, W)(x) \; d\mu(x)\right)^{p_1}}{\prod_{i=1}^n \left(\int_{X^{V_1(H)}} \prod_{j=i+1}^n t(H^L_{\{j\}}, W)(x) \; d\mu(x)\right)^{p_i - p_{i+1}}} \\ &= \frac{t(H, W)^{p_1}}{\prod_{i=1}^{n-1} t(H_{C_i}, W)^{p_i - p_{i+1}}}, \end{aligned}$$

where $p_{n+1} \stackrel{\text{def}}{=} 1$ (note that the nth term of the final product is omitted because $t(H_{C_n}, W) = 1$ as $C_n = \varnothing$).

Recall now that all W_i except possibly for W_n are left-regular and non-zero and since $n \in C_1 \cap \cdots \cap C_{n-1}$, all W_j with $j \in C_H \setminus C_i$ for some $i \in [n-1]$ are left-regular and non-zero, so by Lemma 6.4 (and recalling that $p_i \geq p_{i+1}$), we have

$$
\begin{aligned}
t(h, W) &\geq \frac{t(H, W)^{p_1}}{\prod_{i=1}^{n-1} t(H_{C_i}, W)^{p_i - p_{i+1}}} \\
&\geq t(H, W)^{p_1} \cdot \prod_{i=1}^{n-1} \left(\frac{\prod_{j \in C_H \setminus C_i} t(\rho, W_j)^{e_j(H)}}{t(H, W)} \right)^{p_i - p_{i+1}} \\
&= t(H, W)^{p_n} \cdot \prod_{i=1}^{n} t(\rho, W_i)^{e_i(H) \cdot (p_i - p_n)} \\
&\geq t(\rho_H, W)^{p_n \cdot e(H)} \cdot \prod_{i=1}^{n} t(\rho, W_i)^{e_i(H) \cdot (p_i - p_n)}
\end{aligned}
\tag{5}
$$

where the last inequality follows since H is color-Sidorenko by Lemma 6.2.

Note now that since $p_n \cdot e(H) + \sum_{i=1}^{n} e_i(H) \cdot (p_i - p_n) = e(h)$, by Hölder's Inequality, we have

$$
\begin{aligned}
t(\rho_h, W) &= \int_X \prod_{i=1}^{n} t(e_1, W_i)^{p_i \cdot e_i(H)/e(h)} \, d\mu(x) \\
&= \int_X \left(\prod_{i=1}^{n} t(e_1, W_i)^{p_n \cdot e_i(H)/e(h)} \right) \cdot \prod_{i=1}^{n} t(e_1, W_i)^{e_i(H) \cdot (p_i - p_n)/e(h)} \, d\mu(x) \\
&\leq \left(\int_X \prod_{i=1}^{n} t(e_1, W_i)^{e_i(H)/e(H)} \right)^{p_n \cdot e(H)/e(h)} \cdot \prod_{i=1}^{n} t(\rho, W_i)^{e_i(H) \cdot (p_i - p_n)/e(h)} \\
&= t(\rho_H, W)^{p_n \cdot e(H)/e(h)} \cdot \prod_{i=1}^{n} t(\rho, W_i)^{e_i(H) \cdot (p_i - p_n)/e(h)}.
\end{aligned}
$$

Plugging the $e(h)$th power of this inequality in (5) then gives $t(h, W) \geq t(\rho_h, W)^{e(h)}$, that is, $h = H^{\vec{p}}$ is color-Sidorenko. ■

We can finally prove Theorem 3.7 (restated below).

Theorem 3.7. *Let H be a non-trivial right-uniform color-edge-transitive left-weakly Hölder bigraph without isolated vertices and let G be a bigraph with $V_1(G) = V_1(H)$ and without isolated vertices. For every $U \subseteq V_1(G)$, let*

$$
\begin{aligned}
d_G(U) &\stackrel{\text{def}}{=} |\{w \in V_2(G) \mid N_G(w) = U\}|, \\
d_H(U) &\stackrel{\text{def}}{=} |\{w \in V_2(H) \mid N_H(w) = U\}|.
\end{aligned}
$$

Suppose further that for every $U \subseteq V_1(G)$ with $|U| \geq 2$ the following hold.

i. $\sum_{\sigma \in \mathrm{Aut}(H)} d_G(\sigma(U)) = 0$ *if and only if* $\sum_{\sigma \in \mathrm{Aut}(H)} d_H(\sigma(U)) = 0$.

ii. $\sum_{\sigma \in \mathrm{Aut}(H)} d_G(\sigma(U)) \geq \sum_{\sigma \in \mathrm{Aut}(H)} d_H(\sigma(U))$.

Then G is a strong Sidorenko bigraph. In particular, $G(H)$ is a strong Sidorenko bigraph.

Proof of Theorem 3.7. First, by [Sid91, Theorem 2], the class of strong Sidorenko bigraphs is closed under amalgamation with a single edge along a vertex; thus, by an inductive application of this result, it is sufficient to prove the case when all vertices on the right of G have degree at least 2, or equivalently, we have $d_G(U) = 0$ whenever $|U| \leq 1$.

Without loss of generality, let us assume $C_H = \operatorname{im}(c_H)$.

Since H is right-uniform and does not have isolated vertices, we can define a coloring $r_H \colon V_2(H) \to C_H$ of the right side of H by letting $r_H(w)$ be the color of any (equivalently, all) edges incident to w. Given a set $U \subseteq V_1(H)$, let us also define

$$D_G(U) \stackrel{\text{def}}{=} \{w \in V_2(G) \mid N_G(w) = U\}, \qquad D_H(U) \stackrel{\text{def}}{=} \{w \in V_2(H) \mid N_H(w) = U\},$$

so that $d_G(U) = |D_G(U)|$ and $d_H(U) = |D_H(U)|$. For each $i \in C_H$, let

$$\mathcal{U}_i \stackrel{\text{def}}{=} \{U \subseteq V_1(H) \mid \exists w \in D_H(U), r_H(w) = i\}.$$

Note that for $U \in \mathcal{U}_i$, if $\operatorname{Aut}(H) \cdot U$ is the orbit of U under the action of $\operatorname{Aut}(H)$, then color-edge-transitivity of H implies that $\operatorname{Aut}(H) \cdot U = \mathcal{U}_i$.

Claim 6.7. *For every $U_1, U_2 \subseteq V_1(H)$ and every $i \in C_H$, if $D_H(U_1) \cap r_H^{-1}(i)$ and $D_H(U_2) \cap r_H^{-1}(i)$ are non-empty, then $|U_1| = |U_2|$ and $|D_H(U_1) \cap r_H^{-1}(i)| = |D_H(U_2) \cap r_H^{-1}(i)|$.*

Proof. For $j \in [2]$, let $u_j \in U_j$ and let $w_j \in D_H(U_j) \cap r_H^{-1}(i)$ so that $c_H(u_j, w_j) = i$. Since H is color-edge-transitive, there exists $\sigma \in \operatorname{Aut}(H)$ such that $\sigma(u_1) = u_2$ and $\sigma(w_1) = w_2$ and since $N_H(w_j) = U_j$ ($j \in [2]$), we must have $\sigma(U_1) = U_2$, hence $|U_1| = |U_2|$. In turn, since $\sigma(U_1) = U_2$, it also follows that $|D_H(U_1) \cap r_H^{-1}(i)| = |D_H(U_2) \cap r_H^{-1}(i)|$ as σ is an automorphism. ■

Let us start by proving the case in which for every $U \subseteq V_1(H)$, we have $|r_H(D_H(U))| \leq 1$, that is, all vertices of $D_H(U)$ have the same color. Note that this hypothesis implies that the $\mathcal{U}_i$ are pairwise disjoint. In fact, this along with the hypothesis (i) of the theorem gives that each $w \in V_2(G)$ belongs to exactly one set of the form $N_G(U)$ for some $U \in \mathcal{U}_i$ and some $i \in C_H$ (recall that the degree of w is at least 2). For each $i \in C_H$, define

$$\begin{aligned}
d_i(H) &\stackrel{\text{def}}{=} \frac{|\mathcal{U}_i|}{|\operatorname{Aut}(H)|} \cdot \sum_{\sigma \in \operatorname{Aut}(H)} d_H(\sigma(U)),\\
d_i(G) &\stackrel{\text{def}}{=} \frac{|\mathcal{U}_i|}{|\operatorname{Aut}(H)|} \cdot \sum_{\sigma \in \operatorname{Aut}(H)} d_G(\sigma(U)),\\
p_i &\stackrel{\text{def}}{=} d_i(G)/d_i(H),\\
m_i &\stackrel{\text{def}}{=} |U|,
\end{aligned}$$

where U is any set in $\mathcal{U}_i$. Claim 6.7 implies that the definitions above do not depend on the choice of U. Note also that hypothesis (ii) of the theorem gives $p_i \geq 1$. Finally, define

$$\mathcal{V}_i \stackrel{\text{def}}{=} \{w \in V_2(G) \mid \exists U \in \mathcal{U}_i, N_G(w) = U\}$$

so that $|\mathcal{V}_i| = d_i(G)$.

Fix a bigraphon $W\colon \Omega \times \Lambda \to \mathbb{R}_+$ and sequences $f = (f_v)_{v \in V_1(G)}$ and $g = (g_w)_{w \in V_2(G)}$ of bounded measurable functions $f_v\colon \Omega \to \mathbb{R}_+$ and $g_w\colon \Lambda \to \mathbb{R}_+$. For each $i \in C_H$, we define the function $\widehat{g}_i\colon \Lambda \to \mathbb{R}_+$ by

$$\widehat{g}_i(y) = \prod_{w \in \mathcal{V}_i} g_w(y)^{1/d_i(G)},$$

that is, $\widehat{g}_i$ is the geometric average of the sequence $(g_w \mid w \in \mathcal{V}_i)$.

Note now that by renaming the variables, for every $\sigma \in \operatorname{Aut}(H)$, we have

$$\begin{aligned} t(G; f, g; W) &= \int_{X^{V_1(G)} \times Y^{V_2(G)}} \prod_{v \in V_1(G)} f_v(x_{\sigma(v)}) \prod_{w \in V_2(G)} g_w(y_{\sigma(w)}) \\ &\qquad \cdot \prod_{(v,w) \in E(G)} W(x_{\sigma(v)}, y_{\sigma(w)})\, d(\mu \otimes \nu)(x, y) \\ &= \int_{X^{V_1(G)}} \prod_{v \in V_1(G)} f_v(x_{\sigma(v)}) \cdot \prod_{i \in C_H} \prod_{U \in \mathcal{U}_i} \prod_{w \in D_G(U)} t(K_{m_i,1}^L, W_w)(x_{\sigma(U)})\, d\mu(x), \end{aligned}$$

where $W_w(x, y) \stackrel{\text{def}}{=} W(x, y) g_w(y)$ and the second equality follows from the fact that each $w \in V_2(G)$ belongs to exactly one set of the form $N_G(U)$ for some $U \in \mathcal{U}_i$ and some $i \in C_H$. Then Hölder's Inequality implies

$$\begin{aligned} t(G; f, g; W) \geq \int_{X^{V_1(G)}} \prod_{\sigma \in \operatorname{Aut}(H)} \Bigg(&\prod_{v \in V_1(G)} f_v(x_{\sigma(v)}) \\ &\cdot \prod_{i \in C_H} \prod_{U \in \mathcal{U}_i} \prod_{w \in D_G(U)} t(K_{m_i,1}^L, W_w)(x_{\sigma(U)}) \Bigg)^{1/|\operatorname{Aut}(H)|} d\mu(x). \end{aligned} \tag{6}$$

Let us analyze each of the terms under the integral above.

For the part corresponding to the family of functions f, by Lemma 6.2, we know that H is left-color-regular and since it is also color-edge-transitive, it follows that it is left-vertex-transitive. Finally, since $V_1(G) = V_1(H)$, we get

$$\prod_{\sigma \in \operatorname{Aut}(H)} \prod_{v \in V_1(G)} f_v(x_{\sigma(v)})^{1/|\operatorname{Aut}(H)|} = \prod_{v_1, v_2 \in V_1(G)} f_{v_1}(x_{v_2})^{1/v_1(G)}. \tag{7}$$

The other part is more involved: fix $i \in C_H$ and note that

$$\prod_{\sigma\in\mathrm{Aut}(H)} \prod_{U\in\mathcal{U}_i} \prod_{w\in D_G(U)} t(K^L_{m_i,1}, W_w)(x_{\sigma(U)})^{1/|\mathrm{Aut}(H)|}$$
$$= \prod_{\sigma\in\mathrm{Aut}(H)} \prod_{U\in\mathcal{U}_i} \prod_{w\in D_G(\sigma(U))} t(K^L_{m_i,1}, W_w)(x_U)^{1/|\mathrm{Aut}(H)|}.$$

Since for each $U \in \mathcal{U}_i$, we have $\mathrm{Aut}(H) \cdot U = \mathcal{U}_i$, it follows that for any given $(U, w) \in \mathcal{U}_i \times \mathcal{V}_i$, the factor $t(K^L_{m_i,1}, W_w(x_U))^{1/|\mathrm{Aut}(H)|}$ appears exactly $|\mathrm{Aut}(H)|/|\mathcal{U}_i|$ times: exactly when $\sigma(U) = N_G(w)$, so we deduce

$$\prod_{\sigma\in\mathrm{Aut}(H)} \prod_{U\in\mathcal{U}_i} \prod_{w\in D_G(U)} t(K^L_{m_i,1}, W_w)(x_{\sigma(U)})^{1/|\mathrm{Aut}(H)|} = \prod_{U\in\mathcal{U}_i} \prod_{w\in\mathcal{V}_i} t(K^L_{m_i,1}, W_w)(x_U)^{1/|\mathcal{U}_i|}. \tag{8}$$

On the other hand, by Hölder's Inequality, for each $U \in \mathcal{U}_i$, we have

$$\begin{aligned}
\prod_{w\in\mathcal{V}_i} t(K^L_{m_i,1}, W_w)(x_U)^{1/|\mathcal{U}_i|} &= \prod_{w\in\mathcal{V}_i} \left(\int_Y \prod_{u\in U} W(x_u, y) \cdot g_w(y)\ d\nu(y)\right)^{1/|\mathcal{U}_i|} \\
&\geq \left(\int_Y \prod_{u\in U} W(x_u, y) \cdot \prod_{w\in\mathcal{V}_i} g_w(y)^{1/d_i(G)}\ d\nu(y)\right)^{d_i(G)/|\mathcal{U}_i|} \\
&= t(K^L_{m_i,1}, \widehat{W}_i)(x_U)^{d_i(G)/|\mathcal{U}_i|},
\end{aligned} \tag{9}$$

where $\widehat{W}_i(x, y) \overset{\text{def}}{=} W(x, y) \cdot \widehat{g}_i(y)^{1/m_i}$.

Putting (6), (7), (8) and (9) together, we get

$$\begin{aligned}
t(G; f, g; W) &\geq \int_{X^{V_1(G)}} \prod_{v_1,v_2\in V_1(G)} f_{v_1}(x_{v_2})^{1/v_1(G)} \prod_{i\in C_H} \prod_{U\in\mathcal{U}_i} t(K^L_{m_i,1}, \widehat{W}_i)(x_U)^{d_i(G)/|\mathcal{U}_i|}\ d\mu(x) \\
&= \int_{X^{V_1(G)}} \prod_{i\in C_H} \prod_{U\in\mathcal{U}_i} t(K^L_{m_i,1}, W'_i)(x_U)^{d_i(G)/|\mathcal{U}_i|}\ d\mu(x) \\
&= \int_{X^{V_1(G)}} \prod_{i\in C_H} \prod_{U\in\mathcal{U}_i} t(K^L_{m_i,1}, W'_i)(x_U)^{p_i\cdot d_i(H)/|\mathcal{U}_i|}\ d\mu(x) \\
&= t(H^{\vec{p}}, W'),
\end{aligned}$$

where $W'_i(x, y) \overset{\text{def}}{=} \widehat{W}_i(x, y) \cdot \prod_{v\in V_1(G)} f_v(x)^{1/e(G)}$, the first equality follows since for each $i \in C_H$, each $v \in V_1(G)$ belongs to exactly $|\mathcal{U}_i| \cdot m_i/v_1(G)$ sets $U \in \mathcal{U}_i$ (as H is left-vertex-transitive and $v_1(H) = v_1(G)$) and we have

$$\sum_{i\in C_H} m_i \cdot d_i(G) = e(G),$$

and the second equality follows since $p_i = d_i(G)/d_i(H) \geq 1$. Since $H^{\vec{p}}$ is color-Sidorenko by Lemma 6.6, we conclude that

$$t(G; f, g; W) \geq t(H^{\vec{p}}, W') \geq t(\rho_{H^{\vec{p}}}, W')^{e(H^{\vec{p}})}.$$

Note now that

$$e_i(H^{\vec{p}}) = p_i \cdot m_i \cdot d_i(H) = m_i \cdot d_i(G),$$
$$e(H^{\vec{p}}) = \sum_{i \in C_H} m_i \cdot d_i(G) = e(G).$$

This means that

$$\begin{aligned} t(\rho_{H^{\vec{p}}}, W') &= \int_{X \times Y} \prod_{v \in V_1(G)} f_v(x)^{1/e(G)} \prod_{i \in C_H} \widehat{g}_i(y)^{d_i(G)/e(G)} \cdot W(x, y)\ d(\mu \otimes \nu)(x, y) \\ &= \int_{X \times Y} \prod_{v \in V_1(G)} f_v(x)^{1/e(G)} \prod_{i \in C_H} \prod_{w \in \mathcal{V}_i} g_w(y)^{1/e(G)} \cdot W(x, y)\ d(\mu \otimes \nu)(x, y) \\ &= t\left(\rho; \prod_{v \in V_1(G)} f_v^{1/e(G)}, \prod_{w \in V_2(G)} g_w^{1/e(G)}; W\right). \end{aligned}$$

Therefore G is a strong Sidorenko bigraph.

We will now show how the general case reduces to the previous case. More specifically, we will show that if G satisfies the hypotheses of the theorem for a general H, then it also satisfies the same hypotheses for some H' satisfying $|r_{H'}(D_{H'}(U))| \leq 1$ for every $U \subseteq V_1(H')$.

Given a general H let $C \subseteq C_H$ be a minimal set of colors such that if $U \subseteq V_1(H)$ satisfies $d_H(U) \geq 1$, then there exists $w \in D_H(U)$ with $r_H(w) \in C$. Let then H' be the colored bigraph obtained from H_C by removing all of its isolated vertices. Note that only vertices in $V_2(H)$ can be removed in this procedure, so $V_1(H') = V_1(H) = V_1(G)$. It is also clear that H' is right-uniform and by Remark 2.10, it follows that H_C is left-weakly Hölder, which implies that H' is also left-weakly Hölder.

Note further that if $\sigma \in \operatorname{Aut}(H)$, then $\sigma|_{V(H')} \in \operatorname{Aut}(H')$, which in particular implies that H' is color-edge transitive.

Conversely, we claim that every element of $\operatorname{Aut}(H')$ is of the form $\sigma|_{V(H')}$ for some $\sigma \in \operatorname{Aut}(H)$. Indeed, to extend $\sigma \in \operatorname{Aut}(H')$ to an automorphism of H, we note that Claim 6.7 implies that for each $i \in C_H$ and each $U_1, U_2 \in \mathcal{U}_i$, there exists a bijection $\theta^i_{U_1,U_2}$ between $D_H(U_1) \cap r_H^{-1}(i)$ and $D_H(U_2) \cap r_H^{-1}(i)$, so defining

$$\sigma(w) \stackrel{\text{def}}{=} \theta^i_{N_H(w),\sigma(N_H(w))}(w)$$

for every $w \in V_2(H) \setminus V_2(H')$ with $r_H(w) = i$ gives an extension of σ to an automorphism of H.

In particular, this means that the orbits of the actions of $\operatorname{Aut}(H)$ and $\operatorname{Aut}(H')$ on $V_1(H) = V_1(H')$ are the same.

We claim that for every $U \subseteq V_1(H')$, we have $|r_{H'}(D_{H'}(U))| \leq 1$. Suppose not and let $U_0 \subseteq V_1(H')$ be such that $D_{H'}(U_0) \cap r_{H'}^{-1}(i_1) \cap r_{H'}^{-1}(i_2)$ is non-empty for some $i_1 \neq i_2$. Note that we must necessarily have $i_1, i_2 \in C$, which along with Claim 6.7 implies that for every $U \subseteq V_1(H')$, we have

$$D_H(U) \cap r_H^{-1}(i_1) \neq \varnothing \iff D_H(U) \cap r_H^{-1}(i_2) \neq \varnothing. \tag{10}$$

Let us now show that $C \setminus \{i_2\}$ contradicts the minimality of C. From the choice of C, we know that if $U \subseteq V_1(H)$ is such that $d_H(U) \geq 1$, then there exists $w \in D_H(U)$ with $w \in C$, but from (10), we conclude that there exists $w' \in D_H(U)$ with $w' \in C \setminus \{i_2\}$, thus $C \setminus \{i_2\}$ also satisfies the same property defining C, contradicting its minimality. This concludes the proof of the claim. Thus, the previous case of the theorem can be applied to H'.

It remains to show that the hypotheses (i) and (ii) of the theorem for G and H imply that the same hypotheses hold for G and H'. But indeed, from the definition of H' and since the orbits of the actions of $\operatorname{Aut}(H)$ and $\operatorname{Aut}(H')$ on $V_1(H) = V_1(H') = V_1(G)$ are the same, we get

$$d_H(U) \geq d_{H'}(U), \qquad\qquad d_H(U) = 0 \iff d_{H'}(U) = 0$$

for every $U \subseteq V_1(H)$, so the hypotheses (i) and (ii) of the theorem for G and H imply that the same hypotheses hold for G and H'. ■

7 Conclusion and open problems

In this paper, we have shown how left-sided analogues of the concepts of reflection bigraphs and cut-percolating bigraphs can be used to obtain induced-Sidorenko bigraphs. We also showed that the left-sided version of the weakly Hölder property, along with color-edge transitivity and right-uniformity, also follow from left-reflection and can be used along with a standard symmetrization technique to obtain the strong Sidorenko property.

In the proof of Theorem 3.2, we exploited heavily the fact that weak domination (hence also the induced-Sidorenko property) only uses target bigraphons that are biregular. On the other hand, Szegedy [Sze15b] showed that there is no loss in generality in studying Sidorenko's Conjecture only when the target bigraphs are both edge-vertex-transitive, so it is natural to ask what advantages this stronger assumption can yield.

As mentioned in the introduction, it was shown in [DGH+18] that the weak norming property can be equivalently restated as an extremal property called step Sidorenko property, which was studied in [KMPW19] and implicitly in [Lov12, §14.2]. Since both the induced-Sidorenko property and the left-weak Hölder-property are weaker analogues of the weak norming property, it is natural to ask if there are similar characterizations of them in terms of an extremal property.

In [CL17, Conjectures 6.1 and 6.2], Conlon–Lee conjectured that a bigraph is weakly norming if and only if it is edge-transitive under its cut-involution group. A similar but different conjecture is that a bigraph is cut-percolating if and only if it is edge-transitive under the group generated by cut-involutions coming from folds (see Remark 2.3). Analogously, one can conjecture that a bigraph is left-cut-percolating if and only if it is left-vertex-transitive under the group generated by cut-involutions coming from folds. Since there is a mismatch between the fact that the left-cut-percolation property is defined for bigraphs and left-weakly Hölder property is defined for *colored* bigraphs, we believe that a decent analogous conjecture relating the two might require some hypotheses how the coloring relates to folds (cf. Theorem 3.3).

A Reflective tree decompositions

This section contains the definition of reflective tree decompositions (and the definition of 2-cores required by it) and the associated result from [CR21].

Definition A.1 (2-cores)**.** For a connected bigraph G, the 2-*core* of G is the maximal connected subgraph $C_2(G)$ in which all vertices have degree at least 2. Alternatively, $C_2(G)$ can be obtained from G by progressively removing, in an arbitrary order, vertices of degree less than 2 until no such vertices remain.

For a flag $F = (G, \theta)$ with G connected, the 2-*core* of F is the flag of the form $F' = (G', \theta)$, where G' is the maximal subgraph in which all vertices that are *not* in $\operatorname{im}(\theta)$ have degree at least two; this can of course be obtained by progressively removing vertices of degree less than 2 that are not in $\operatorname{im}(\theta)$ until no such vertices remain.

Remark A.2. Since in the definition of weak domination the target bigraphons are biregular, it follows that G weakly dominates H if and only if $C_2(G)$ weakly dominates $C_2(H)$.

Definition A.3 (Reflective tree decompositions)**.** Given a connected non-trivial bigraph G, a *reflective tree decomposition* of G is a tree T such that

i. We have $V(T) \subseteq 2^{V(G)}$ and $V(G) = \bigcup_{U \in V(T)} U$.

ii. For every $(v, w) \in E(G)$, there exists $U \in V(T)$ with $v, w \in U$.

iii. For every $U_1, U_2 \in V(T)$ and every $U_3 \in V(T)$ in the unique path from $U_1, U_2 \in V(T)$ and every $U_3 \in V(T)$ in the unique path from U_1 to U_2 in T, we have $U_1 \cap U_2 \subseteq U_3$.

iv. For every $\{U_1, U_2\} \in E(T)$, we have $C_2(F_{U_1 U_2}) \cong C_2(F_{U_2 U_1})$, where $F_{U_i U_j} \stackrel{\text{def}}{=} (G|_{U_i}, U_1 \cap U_2)$ (we assume that each vertex of $U_1 \cap U_2$ receives the same label in $F_{U_1 U_2}$ as in $F_{U_2 U_1}$).

Condition (iv) above implies that $C_2(G|_{U_1}) \cong C_2(G|_{U_2})$ for every $U_1, U_2 \in V(T)$; this common 2-core bigraph is called the *core* of the decomposition.

Theorem A.4 ([CR21, Theorem 3.5]). *If T is a reflective tree decomposition of a non-trivial connected bigraph G whose core H weakly dominates $G|_{U_1 \cap U_2}$ for every $\{U_1, U_2\} \in E(T)$, then G weakly dominates H.*

In particular, if H is a Sidorenko bigraph, then G is also a Sidorenko bigraph.

In the theorem above, by Remark A.2, we have that H weakly dominates $G|_{U_1 \cap U_2}$ if and only if H weakly dominates $C_2(G|_{U_1 \cap U_2})$ and since the latter is a induced subgraph of H, we get the following corollary when H is an induced-Sidorenko bigraph.

Corollary A.5 ([CR21]). *If G is a non-trivial connected bigraph with a reflective tree decomposition with an induced-Sidorenko core H, then G weakly dominates H and G is a Sidorenko bigraph.*

B Strong Sidorenko bigraphs

In [Sid91, §2], Sidorenko defined the class $\mathcal{F}$ as the class of bigraphs G such that

$$\int_{X^{V_1(G)} \times Y^{V_2(G)}} \prod_{v \in V_1(G)} f_v(x_v) \cdot \prod_{w \in V_2(G)} g_w(y_w) \cdot \prod_{(v,w) \in E(G)} W(x_v, y_w) \; d(\mu \otimes \nu)(x, y)$$
$$\cdot \left(\int_X F(x) \; d\mu(x) \right)^{e(G) - v_1(G)} \cdot \left(\int_Y G(y) \; d\nu(y) \right)^{e(G) - v_2(G)}$$
$$\geq \left(\int_{X \times Y} \left(F(x)^{e(G) - v_1(G)} \cdot G(x)^{e(G) - v_2(G)} \cdot \prod_{v \in V_1(G)} f_v(x) \cdot \prod_{w \in V_2(G)} g_w(y) \right)^{1/e(G)} W(x, y) \; d(\mu \otimes \nu) \right)^{e(G)} \tag{11}$$

for all finite measure spaces (X, μ) and (Y, ν) and all bounded measurable functions $F, f_v \colon X \to \mathbb{R}_+$ $(v \in V_1(G))$, $G, g_w \colon Y \to \mathbb{R}_+$ $(w \in V_2(G))$ and $W \colon X \times Y \to \mathbb{R}_+$. In fact, Sidorenko also required $e(G) \geq \max\{v_1(G), v_2(G)\}$ but this condition follows from (11) (see Remark 2.2).

It is obvious that any bigraph G in the class $\mathcal{F}$ is a strong Sidorenko bigraph in the sense of Definition 2.1 by restricting to probability spaces and setting F and G to be identically 1.

For the other direction, suppose G is a strong Sidorenko bigraph and $X, \mu, Y, \nu, F, f_v, G, g_w, W$ are as in (11). Replacing all functions h by $h + \epsilon$ and using the Dominated Convergence Theorem letting $\epsilon \to 0$, it is sufficient to show the case when all functions are strictly positive. Define then the probability measures μ' and ν' by

$$d\mu'(x) \stackrel{\text{def}}{=} \frac{F(x)}{M} \; d\mu(x), \qquad d\nu'(y) \stackrel{\text{def}}{=} \frac{G(x)}{N} \; d\nu(y),$$

where $M \stackrel{\text{def}}{=} \int_X F \; d\mu$ and $N \stackrel{\text{def}}{=} \int_Y G \; d\nu$ and let

$$f'_v \stackrel{\text{def}}{=} \frac{f_v}{F}, \qquad g'_w \stackrel{\text{def}}{=} \frac{g_w}{G}$$

so that interpreting W as a bigraphon over (X, μ') and (Y, ν'), the left-hand side of (11) is written as

$$t(G; f', g'; W) \cdot (M \cdot N)^{e(G)}$$

and the right-hand side of the same equation is written as

$$\left(t \left(\rho; \prod_{v \in V_1(G)} (f'_v)^{1/e(G)}, \prod_{w \in V_2(G)} (g'_w)^{1/e(G)}; W \right) \cdot M \cdot N \right)^{e(G)}$$

and thus (11) follows from the fact that G is a strong Sidorenko bigraph.

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