Generalized Extremal Numbers In Various Ambient Graphs

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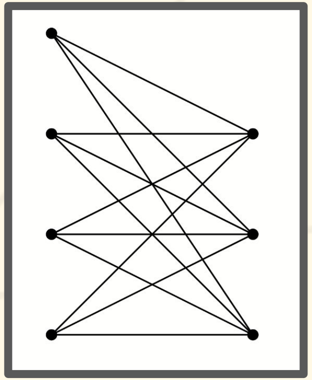
What is the largest bipartite graph not containing H as a subgraph?

Define $\mathrm{ex}_B(n, H)$ to be the largest number of edges in a bipartite H-free graph with n vertices.

If H is not bipartite, then $\mathrm{ex}_B(n, H) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

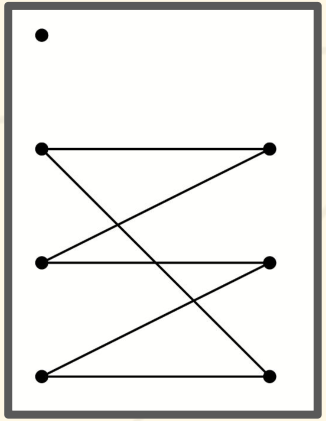


- Always H-free
- Max num. of edges

Trees

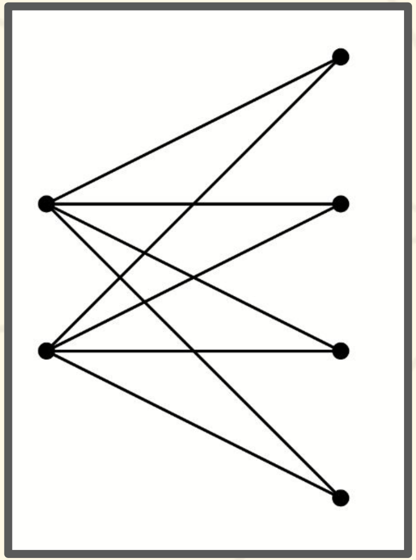
$$\mathrm{ex}_B(n, K_{1,t}) = \left\lfloor \frac{(t-1)n}{2} \right\rfloor - (n \bmod 2)$$

- Degree argument.
- If n odd, bipartite G can't be perfectly regular, so we lose an edge.



Trees

$$\mathrm{ex}_B(n, P_{t+1}) \approx \frac{(t-1)n}{2}$$



$$\mathrm{ex}_B(n, P_{t+1}) \leq \mathrm{ex}(n, P_{t+1}) = \left\lceil \frac{(t-1)n}{2} \right\rceil.$$

$$\mathrm{ex}_B(n, P_{t+1}) \geq e(K_{\lfloor \frac{t}{2} \rfloor - 1, n - \lfloor \frac{t}{2} \rfloor + 1}) \approx \frac{(t-1)n}{2} \text{ for large n.}$$

Key fact:

$$\frac{1}{2}\mathrm{ex}(n, H) \leq \mathrm{ex}_B(n, H) \leq \mathrm{ex}(n, H)$$

Any graph G, has a bipartite subgraph with at least $\frac{e(G)}{2}$ edges.

Apply this to $\mathrm{ext}(n, H)$.

- Cor: $\mathrm{ex}_B(n, H) = \Theta(\mathrm{ex}(n, H))$
- Extends easily to k-partite ambient graph case

Q: What is the largest bipartite graph with part sizes n,m not containing H as a subgraph?

Call this number $ex(n,m,H)$.

$\boxed{K_{s,t}}$ $\mathrm{ex}(n, m, K_{2,2}) \leq \lfloor \frac{n}{2}(1 + \sqrt{4n-3}) \rfloor$

By the same argument to show:

$$\mathrm{ex}(n, K_{2,2}) \leq \lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \rfloor$$

Except that each vertex can have at most n/2-1 2-paths, instead of n-1.

This implies $ex_B(n, K_{2,2}) = \Theta(n^{3/2})$ (point-line incidence is bipartite)

The KST theorem's method can be carried over to give better constants, but doesn't give a better upper bound on the exponent.

A probabilistic result on extremal numbers of this type:

Color the vertices of ext(n,H) red with probability p, blue with probability 1-p.

$$\mathbb{P}(\#\ \text{non-monochromatic edges in ext(n,H)} \geq (\text{ex}(n,H)+1)2p(1-p)-1) > \frac{1}{2}$$

$$\mathbb{P}(\#\ \text{red vertices} \in [np-o(n), np+o(n)]) \geq \frac{1}{2}$$

For all p there is a bipartite subgraph of ext(n,H) with an arbitrarily close proportion of vertices in A to p for large enough n, |B|=n-|A|, and (ex(n,H)+1)2p(1-p)-1 edges.

(You can also get a graph with $|A| \geq pn$ or $|A| \leq pn$, since

$$\mathbb{P}(\#\ \text{red vertices} \geq np) \approx \frac{1}{2} \text{ for large n.)}$$

Alteration method: take this graph and move vertices to have |A|=pn. Proportion of vertices needed to move is negligible for large n, so still have ~ex(n,H)2p(1-p) edges.

For all p and large enough n there is a bipartite subgraph of ext(n,H) with $|A| = \lfloor np \rfloor$, $|B| = n - |A|$, and ~$(ex(n, H) + 1)2p(1 - p)$ edges.

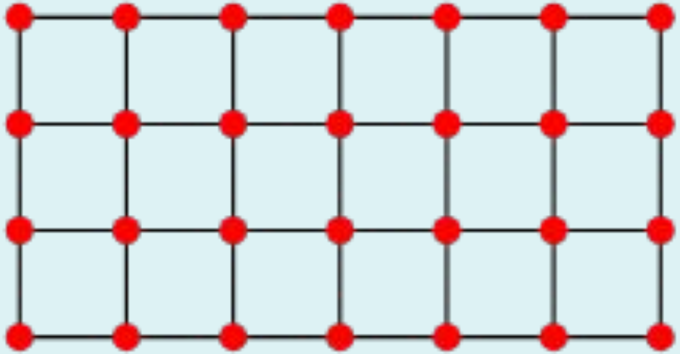
As some corollaries with varied choices of p,

$$\text{For large n, } 2\text{ex}(n, n, H) \geq \text{ex}(2n, H)$$

$$\text{ex}(n, kn, H) = \Theta(\text{ex}(n + kn, H))$$

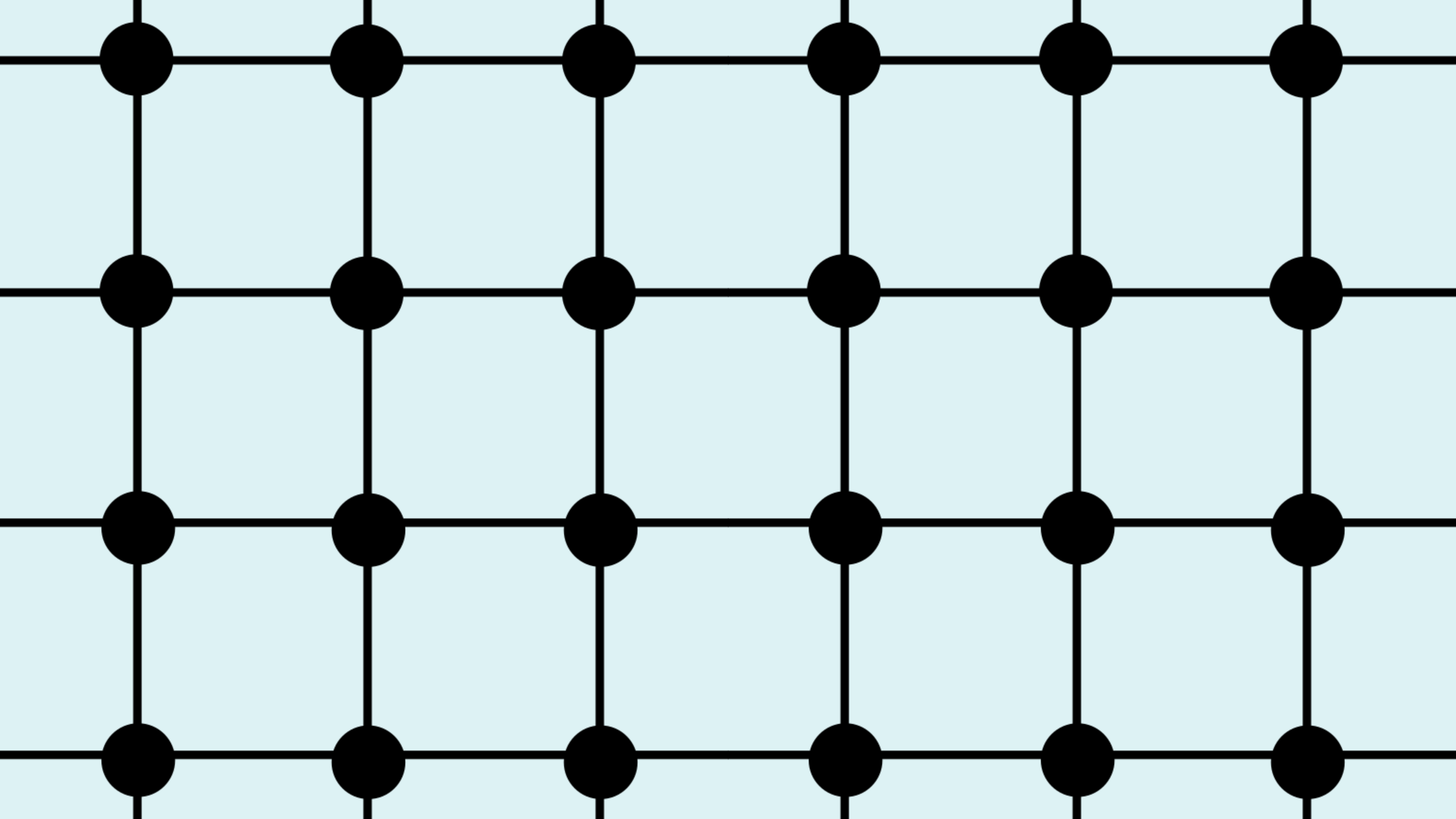
$$\text{ex}(n, n^2, H) = \Omega\left(\frac{\text{ex}(n + n^2, H)}{\sqrt{n + n^2}}\right)$$

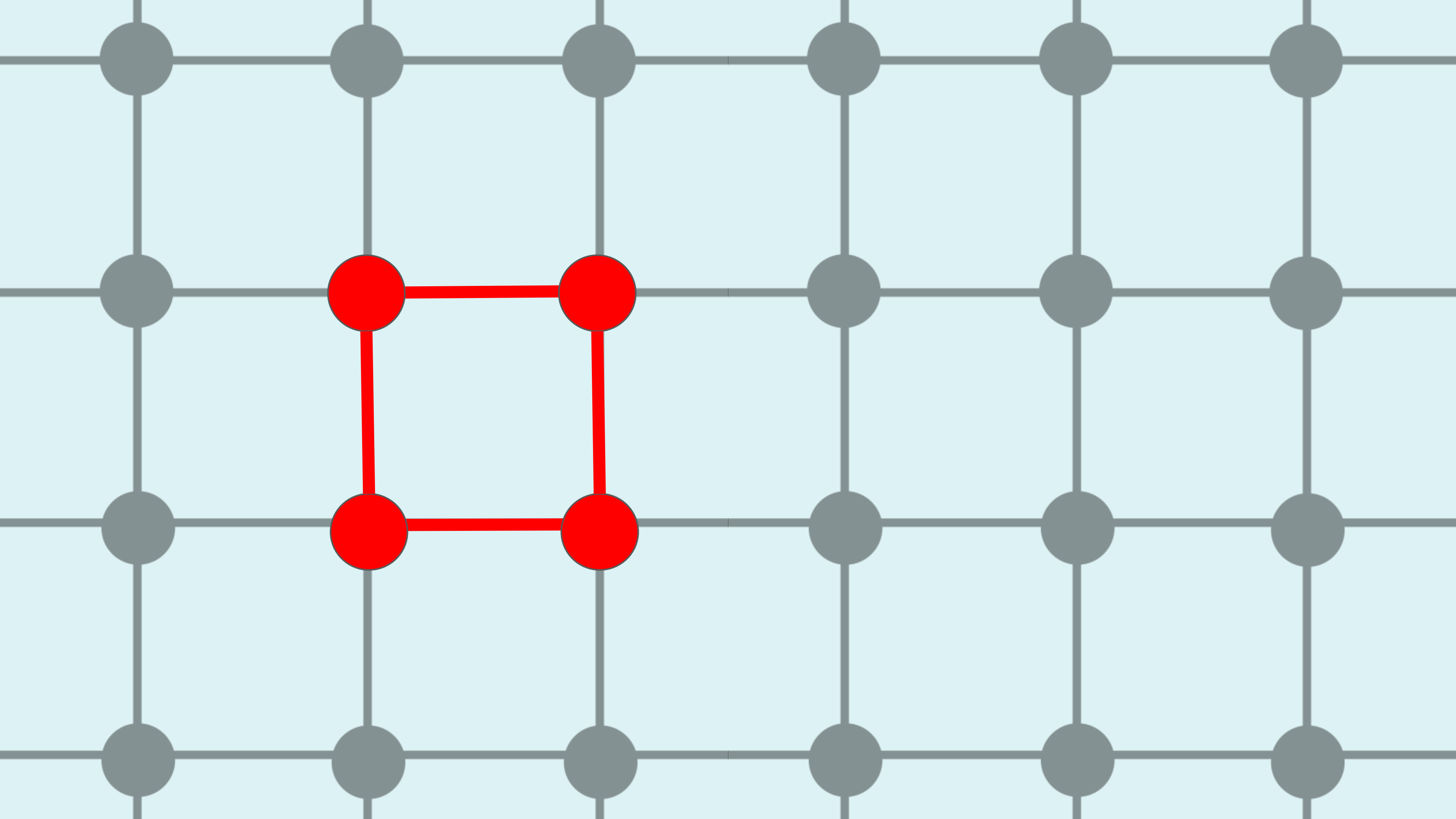
Avoiding C_4 in Grid Graphs

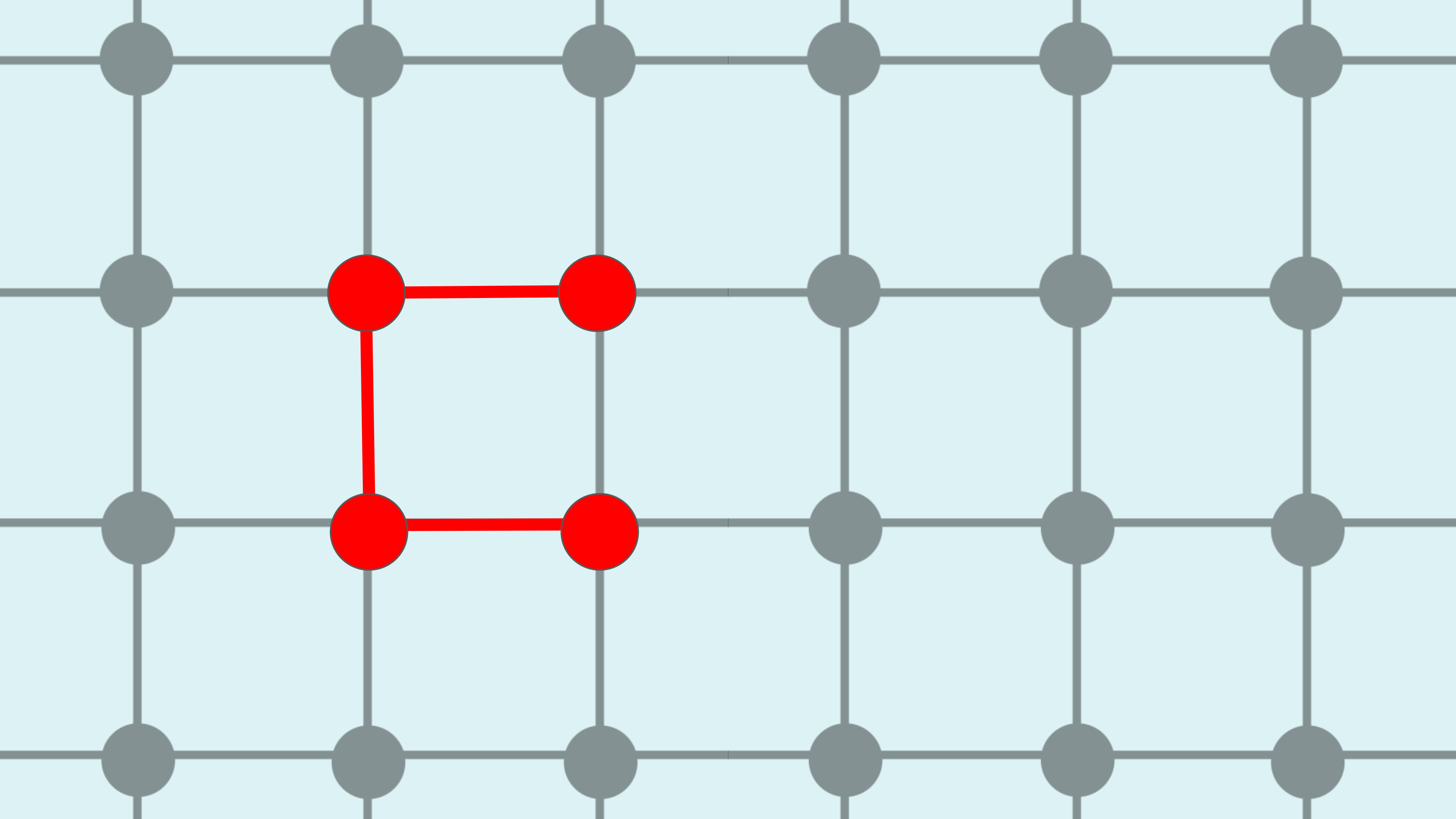


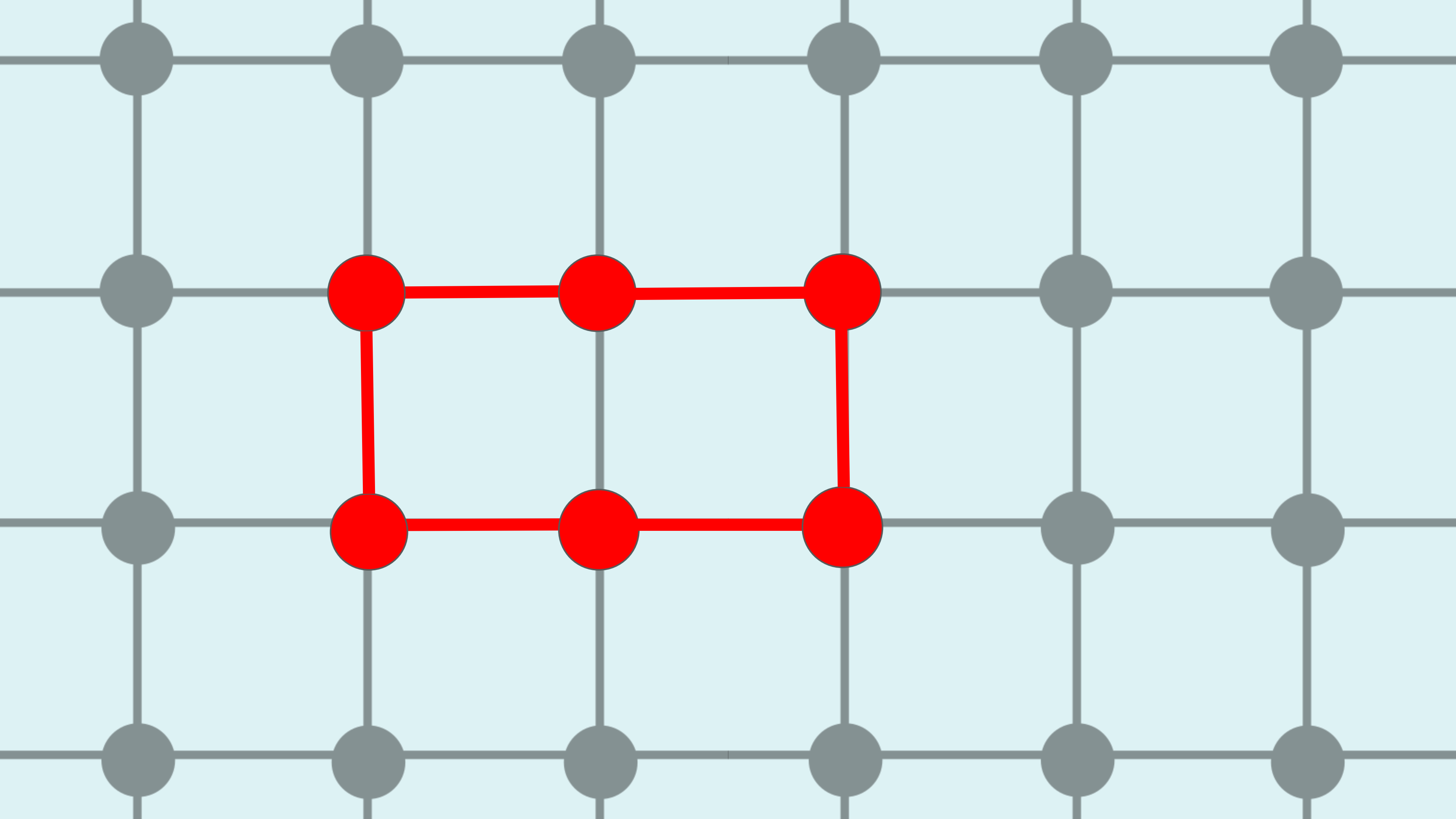
$$7 \times 4$$

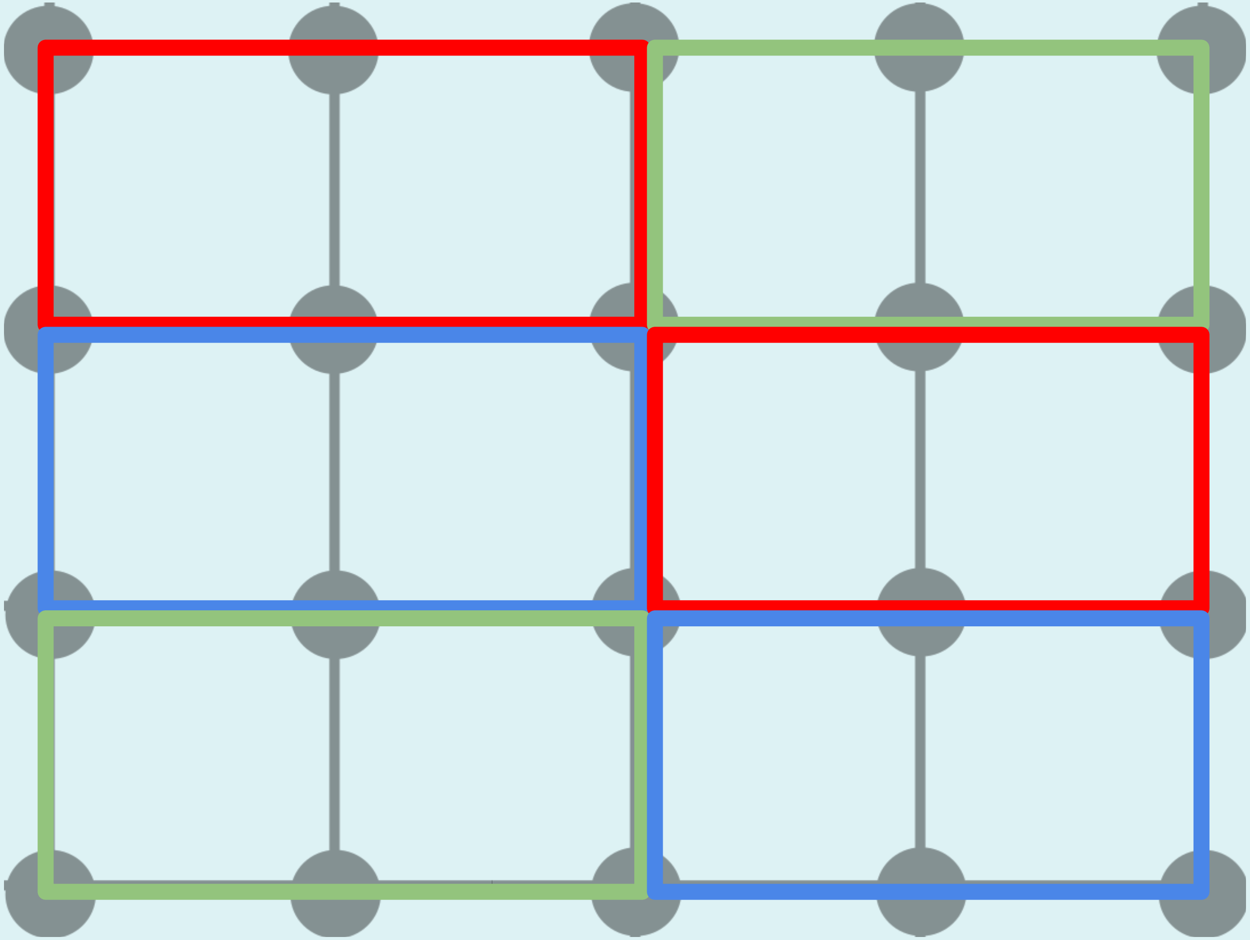
Image Credit: Wolfram MathWorld











What if the number of squares is odd?

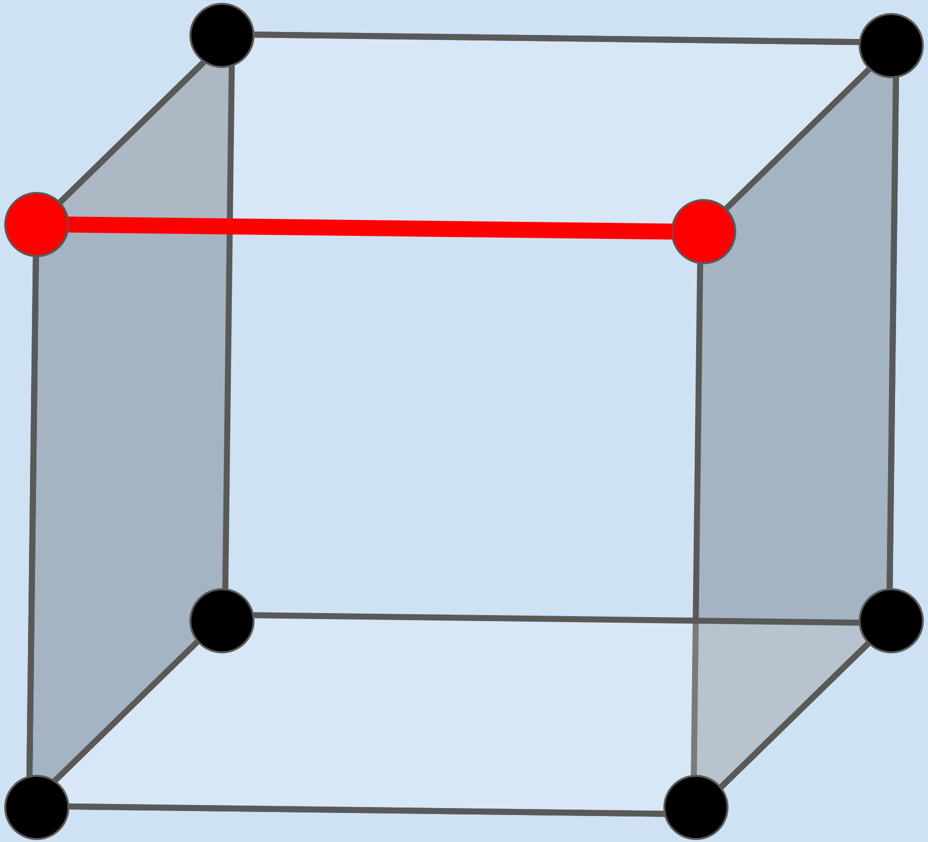
For the $m \times n$ grid graph, we have to remove

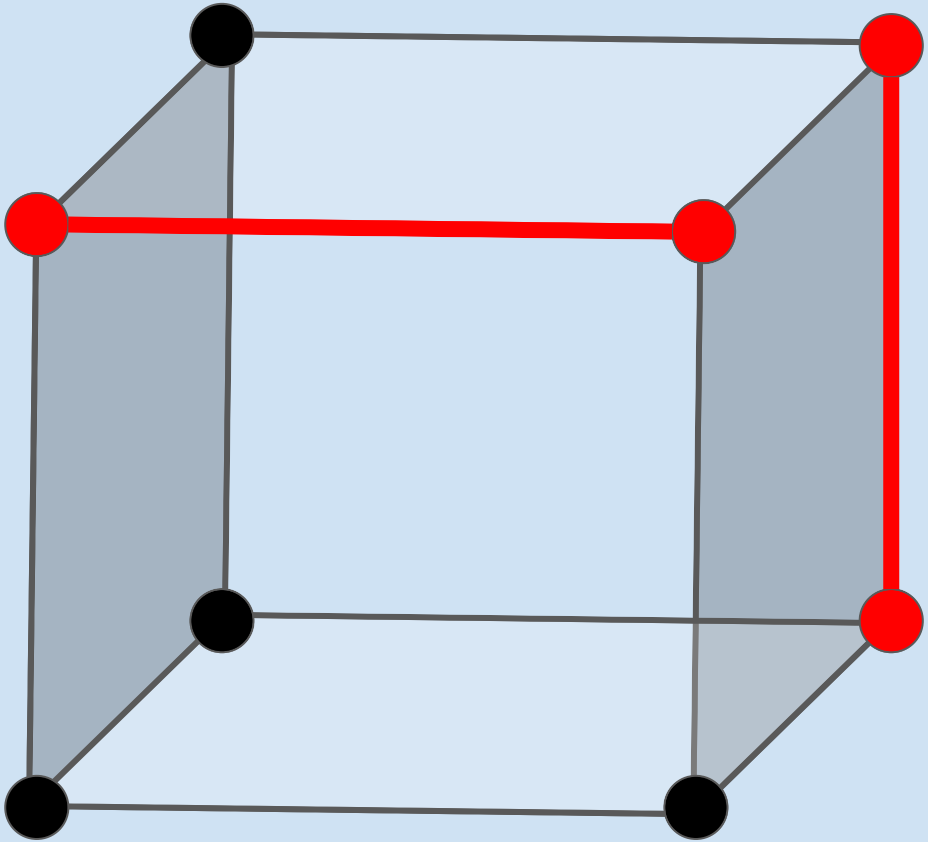
$$\left\lceil \frac{(m-1)(n-1)}{2} \right\rceil$$

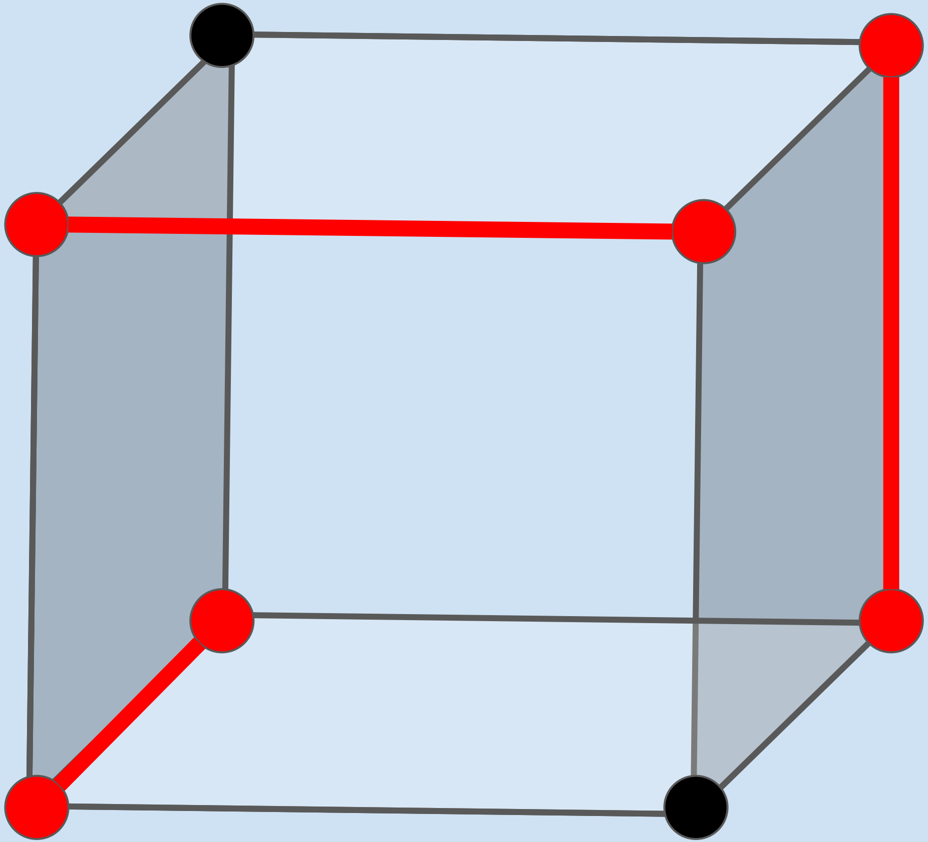
edges.

$$\mathrm{ex}(P_m \square P_n, C_4) = \boxed{2mn - m - n - \left\lceil \frac{(m-1)(n-1)}{2} \right\rceil}$$

What about <u>three</u> dimensions?







Is this optimal?

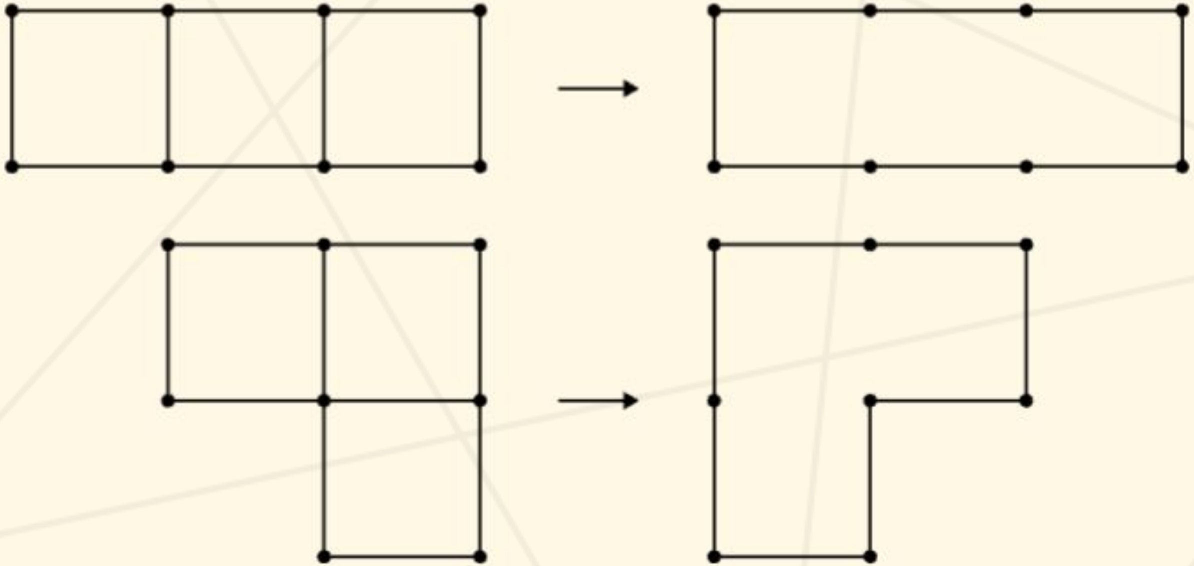
Avoiding C_6 in Grid Graphs

Case 1:

Case 2:

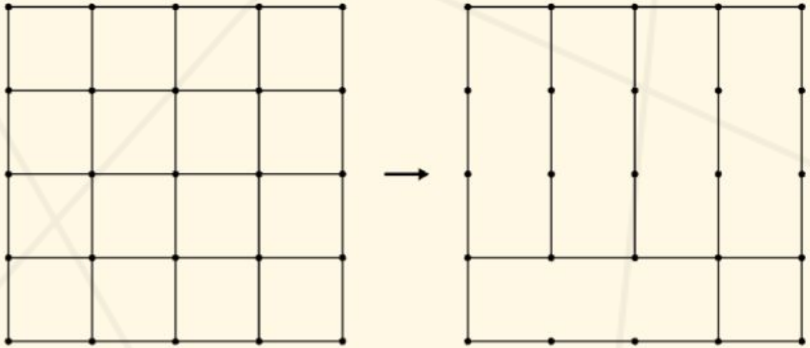
Extending to the Entire Grid

Standard Grid:

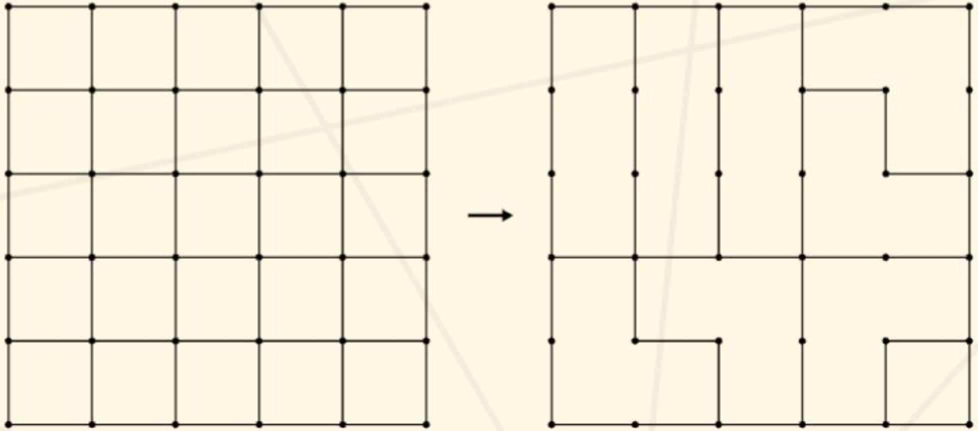


Avoiding C_6 in Grid Graphs: Strategy

n ≡ 2 mod 3:



n ≡ 0 mod 3:



Extremal Numbers for C_6 (Lower Bound)

$$\text{If n} \equiv 1 \text{ mod } 3\text{: } \mathrm{ex}(P_n \square P_n, C_6) = 2n^2 - 2n - 2(n-1)(n-1)/3$$

$$\text{If n} \equiv 0 \text{ or } 2 \text{ mod } 3\text{: } \mathrm{ex}(P_n \square P_n, C_6) = 2n^2 - 2n - 2[(n-1)(n-1)-1]/3$$

$$\text{Overall: } \mathrm{ex}(P_n \square P_n, C_6) = \boxed{2n^2 - 2n - 2\left\lfloor (n-1)(n-1)/3 \right\rfloor}$$

Thank you!

Next Steps:

- More variants of avoiding C_l within d dimensions
- Upper bounds
- $2ex(n,H) \leq ex(n,n,H)$ for all H?
- General lower bound on order of ex(n,n^c,H)