Park City Experimental Mathematics Lab

Two (Unrelated) Problems: *H*-Intersecting Graphs and Polyominoes

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Our Two Problems

- Graph theory: maximum size of H-intersecting families of graphs on n vertices, where H is a bipartite graph
- Polyomino: asymptotic bounds for the behavior of the maximal number of bisections of a polyomino

H-intersecting graph families

Definition (H-intersecting graph family)

Given a graph H and arbitrary n, a family of graphs $\mathcal{G}$ living in K_n is H**-intersecting** if any two graphs $G_1, G_2 \in \mathcal{G}$ have intersection with H as a subgraph.

We can trivially bound $|\mathcal{G}|$ below, for $n \geq v(H)$, by $2^{\binom{n}{2}-e(H)}$: fix a copy of H and take all graphs G in which all edges of that copy of H are included. We can also bound it above by $\frac{1}{2}2^{\binom{n}{2}}$ when H is nonempty, as we cannot include any graph with its complement.

Can we improve this bound?

Improving the bound on $|\mathcal{G}|$

These bounds have been tightened in the following simple cases.

Conjecture (Conjecture of Simonovits-Sós)

If $\mathcal{G}$ is a K_3-intersecting family, then $|\mathcal{G}| \leq \frac{1}{8}2^{\binom{n}{2}}$.

Theorem (Ellis-Filmer-Friedgut 2010)

If H is a graph with chromatic number at least three, then $|\mathcal{G}| \leq \frac{1}{8}2^{\binom{n}{2}}$.

Theorem (Berger-Zhao 2023)

Any K_4-intersecting family $\mathcal{G}$ has size at most $\frac{1}{64}2^{\binom{n}{2}}$.

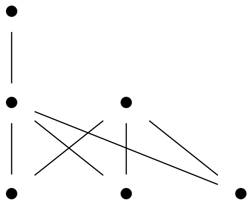
Another Simonovits-Sós conjecture

Simonovits and Sós additionally conjectured:

Conjecture (Second conjecture of Simonovits-Sós)

If $\mathcal{G}$ is a P_3-intersecting family, then $|\mathcal{G}| \leq \frac{1}{8} 2^{\binom{n}{2}}$.

Christofides provided the following construction for a counterexample, improving the bound to $\frac{17}{128} 2^{\binom{n}{2}}$:



Main problems

Problem (Stronger bounds for the P_3 case?)

Can we improve on Christofides' bound?

Problem (More general)

For other bipartite graphs, can we similarly improve on the trivial $2^{\binom{n}{2}-e(H)}$ bound?

What did we find?

Unfortunately, we did not make much progress in the P_3 case: it is a computationally hard problem.

Theorem (Trivial)

A graph has a $|\mathcal{G}|$ bound at most as big as its subgraph.

Theorem (Improvement on multipartite bound in special case)

Let k be a positive integer greater than 1. For $s_1, \ldots, s_{k-1} \in \mathbb{N}_{>0}$ and $s_1 + \cdots + s_{k-1} = m$. For $t \geq 2^m$, we can improve on the trivial bound. Let $H = K_{s_1,\ldots,s_{k-1},t}$ and $N = |E(H)|$. Then for $n \geq m + t + 2$, there exists an H-intersecting family of size at least

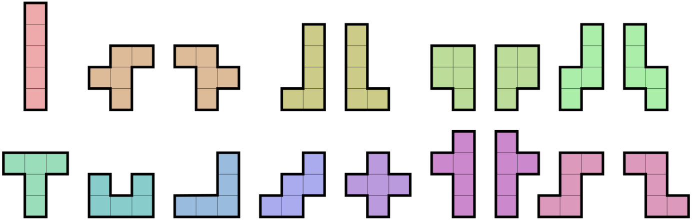
$$\frac{(t+2)(2^m - 1) + 1}{2^{|E(K_{s_1,\ldots,s_{k-1},t+2})|}} 2^{\binom{n}{2}} > \frac{1}{2^N} 2^{\binom{n}{2}}.$$

Introduction: The Problem

Definition (Polyomino)

A k-**polyomino** is a connected set of k unit squares in $\mathbb{R}^2$ (adjacency along edges).

- Equivalence up to translation and rotation.
- Holes are permitted.



Balanced Bisections

Definition (Balanced Bisection)

A partition of a $2k$-polyomino P into two sets, A and B, such that:

- Both A and B are k-polyominoes.

We denote the number of distinct bisections of P by $d(P)$.

Problem (The Extremal Question)

Let $Poly_{2k}$ be the set of all $2k$-polyominoes. We define the maximal number of bisections as

$$c_k := \max_{P \in Poly_{2k}} d(P).$$

- ***Goal:*** *Determine the asymptotic behavior of c_k as $k \to \infty$.*

Main Results

Theorem

The constant c_k exhibits exponential growth, $c_k \approx \beta^k \cdot k^\gamma \cdot (1 + o(1))$, and constrain the base of the exponent β to the interval

$$1.845 \lesssim \beta \leq 4.$$

Conjecture

$$\beta < \mu \approx 2.662.$$

- The rate of growth of self-avoiding walks on $\mathbb{Z}^2$ is μ, aka be the connective constant of the square lattice

Minimal Perimeter is "Best"

Conjecture

For any $k \geq 1$, if a $2k$-polyomino P satisfies $d(P) = c_k$, then P must have a minimal perimeter among all polyominoes of size $2k$.

- "Compactifying" flips
- How to deal with rectangles with an extra square?

Upper Bound

Conjecture

The quantity c_k is bounded above by

$$c_k \leq \beta^k$$

for a constant β that is strictly less than μ.

- Via bridges:

$$c_k \leq \sum_{L=O(\sqrt{k})}^{L_{\max}(k)} O(\sqrt{k})(\mu)^L.$$

- Unfortunately...

$$\lim_{k\to\infty} \frac{L_{\max}(k)}{k} = 2.$$

Percolation Theory

Conjecture

The number of bridges $b_L(A)$ of length L enclosing area A is approximately

$$b_L(A) \sim (\mu)^L exp\left(-L \cdot I\left(\frac{A}{L}\right)\right)$$

where I is a convex, non-negative rate function.

Theorem

If this conjecture is true, our upper bound conjecture is true.

Lower Bound

Theorem

The maximal number of bisections c_k satisfies the asymptotic lower bound:

$$c_k \geq \alpha^k \quad \textit{where} \quad \alpha \approx 1.845.$$

$4 \times n$ Case

1. The number of such bisections, $N_4(n)$, is given by

$$N_4(n) = \sum_{c=0}^{\lfloor n/3 \rfloor} 2^c \binom{n-2c}{c} \sum_{a=0}^{\lfloor (n-3c)/2 \rfloor} \binom{n-3c}{2a} \binom{2a}{a}.$$

2. $$N_4(n) \approx \frac{\sqrt{3} \cdot 3^n}{\sqrt{4\pi n}} \sum_{c=0}^{\lfloor n/3 \rfloor} \frac{1}{\sqrt{1 - 3c/n}} \binom{n-2c}{c} \left(\frac{2}{27}\right)^c.$$

3. $$\sqrt{e^{1.14346}} \approx 1.7714$$

Tool #1: Local Limit Theorem

Theorem (Local Limit Theorem)

Let $X_1, \ldots, X_N$ be i.i.d. integer-valued random variables with mean $\mathbb{E}[X_i] = \nu$ and variance $Var(X_i) = \sigma^2 < \infty$. Let $Z_N = \sum_{i=1}^{N} X_i$. Then the probability that Z_N takes a value m is given asymptotically for $N \to \infty$ by:

$$P(Z_N = m) \sim \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left(-\frac{(m - N\nu)^2}{2N\sigma^2}\right).$$

Lemma

For large N,

$$S(N) \sim \frac{3^N}{\sqrt{\frac{4\pi}{3}N}}.$$

Tool #2: Saddle-Point Methods

Theorem

The saddle-point approximation states that the sum $S_n(X_m)$ has the asymptotic behavior $S_n(X_m) \sim (C_m)^n$, where $C_m = \exp(g(\gamma_0))$ and $g(\gamma_0)$ is the value of the logarithmic exponent at the saddle point γ_0.

- Leads to a cubic:

$$162\gamma^3 - 162\gamma^2 + 45\gamma - 2 = 0.$$

- Unique real root in $(0, 1/3)$ at $\gamma_0 \approx 0.0545051$.
- Back substitute to get $\sqrt{e^{1.14346}} \approx 1.7714$

Step 1: The Setup

We tile the central $(m-2) \times n$ strip with:

1. **Simple Blocks:** $B_s = m - 1$ configurations of width 1.
2. **Complex Blocks:** $q_c(m)$ configurations of width 2, being tilings of an $(m-2) \times 2$ rectangle by two identical polyominoes.

Step 2: An Upper Bound

Theorem

Within the block-tiling model, the number of bisections, $N_m(n)$, is given by

$$N_m(n) = (m-1)^n \sum_{c=0}^{\lfloor n/2 \rfloor} \binom{n-c}{c} X_m^c,$$

where $X_m = q_c(m)/(m-1)^2$, $B_s = m-1$ is the number of simple block configurations, and $q_c(m)$ is the number of complex block configurations.

Step 3: Saddle Point Equations

Theorem

The dominant contribution to the sum $S_n(X_m)$ comes from terms where the fraction of complex blocks, $\gamma = c/n$, satisfies the saddle-point equation:

$$(1/X_m + 4)\gamma^2 - (1/X_m + 4)\gamma + 1 = 0.$$

Step 4: Conclusion

Theorem

The asymptotic growth of $N_m(n)$ as a function of $k = mn/2$ is of the form $(\alpha_m)^k$, where the base α_m is given by

$$\alpha_m = ((m-1) \cdot C_m)^{2/m},$$

and $C_m = \exp(g(\gamma_0))$ is the asymptotic correction factor derived from the saddle-point analysis.

Results

Rectangle Shape (m)	**Resulting Bound (α_m)**
4	1.7714
5	1.8170
6	**1.8445**
7	1.7688

Thank you!

Theorem (Problem 1)

Let k be a positive integer greater than 1. For $s_1, \ldots, s_{k-1} \in \mathbb{N}_{>0}$ and $s_1 + \cdots + s_{k-1} = m$. For $t \geq 2^m$, we can improve on the trivial bound. Let $H = K_{s_1,\ldots,s_{k-1},t}$ and $N = |E(H)|$. Then for $n \geq m + t + 2$, there exists an H-intersecting family of size at least

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Conjecture (Problem 2)

$$1.845 \lesssim \beta < \mu \approx 2.662.$$