Canonical Forms in Geometry and Soliton Theory

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Glimpses of Mathematics, Now and Then:
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Outline of my Talk

Use of canonical forms (slices) of group actions:
- Canonical forms in submanifold geometry
- Invariant solutions for partial differential equations
- Canonical forms and differential invariants for curves in homogeneous spaces
- Soliton equations and Integrable curve flows
I. Canonical Form in Submanifold Geometry

Definition. (Palais 1960)
A Lie group $G$ acts on a manifold $M$, $H$ a subgroup of $G$. A submanifold $S$ is an $H$-slice if

- $G \cdot S = \{ g \cdot p | g \in G, p \in S \}$ is open in $M$,
- $H \cdot S \subset S$,
- if $g \in G$ and $(g \cdot S) \cap S \neq \emptyset$, then $g \in H$.

Examples

1. $SO(n)$ acts on $\mathbb{R}^n$ by $g \cdot y = gy$, $S = \mathbb{R} \times 0$ is a $\mathbb{Z}_2$-slice.

2. $SU(n)$ acts on $su(n)$ by Adjoint action, $g \cdot X = gXg^{-1}$. Then \{X | X \in su(n) diagonal\} is an $S_n$-slice.

3. $SO(n)$ acts on the space of $n \times n$ symmetric matrices by $g \cdot A = gAg^{-1}$, $S = \{ \text{diag}(c_1, \ldots, c_n) | c_i \in \mathbb{R} \}$ is a $S_n$-slice, where $S_n$ is the permutation group of $\{1, \ldots, n\}$. 
Polar Actions

Definition. (Palais-Terng 1985)
An isometric $G$-action on a Riemannian manifold $M$ is polar if there is a closed submanifold $S$ of $M$ that meets all orbits and meets orthogonally. Such $S$ is called a section.

Theorem (Dadok 1985)

(i) The Adjoint action of a compact Lie group $G$ on its Lie algebra $\mathcal{G}$ is polar and a maximal abelian subalgebra is a $W$-slice, where $W = N(A)/Z(A)$.

(ii) The slice representation of a symmetric space $G/K$ is polar, a maximal abelian subspace $\mathcal{A}$ in $\mathcal{P}$ is a $W$-slice, where $W = \frac{N(A)}{Z(A)}$ is the Weyl group associated to $G/K$ and $\mathcal{P}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$.

(iii) Orbit foliations of polar representations are the ones obtained in (i).
• Examples (i) and (iii) are the slice representations of $S^n = \frac{SO(n+1)}{SO(n)}$ and $\frac{U(n)}{O(n)}$ respectively.

• The polar representation on $\mathbb{R}^n$ gives a polar action on $S^{n-1}$.

**Definition. (Terng 1985)**

A submanifold $M^n$ of $S^{n+k-1}$ is isoparametric if

1. the normal bundle $\nu(M)$ is flat,
2. principal curvatures along a parallel normal field are constant.

**Remarks**

• A principal orbit of a polar action on $S^n$ is isoparametric.
• The definition of isoparametric hypersurfaces was given by É Cartan.
Isoparametric Theory

Theorem. (Terng 1985)

Let $M^n$ be isoparametric in $S^{n+k}$, $\xi$ a parallel normal field, and $M_\xi = \{\exp_p(\xi(p)) | p \in M\}$ a parallel to $M$. Then

- $M_\xi$ is a smooth submanifold (may have lower dimension),
- $\{M_\xi|\xi \text{ parallel normal field}\}$ is a singular foliation of $S^{n+k}$ and has similar properties as the orbit foliation of a polar action on $S^{n+k}$.
- Fix $p \in M$, there is a Weyl group $W$ acts on the normal sphere $S_p := \exp_p(\nu_p(M))$ such that $M \cap S_p = W \cdot p$ and $S_p$ meets all parallel submanifolds orthogonally.
- The map from the space of smooth functions on $S^{n+k}$ that are constant on the leaves of the parallel foliation and the space of smooth $W$-invariant functions on $S_p$ defined by restriction is a bijection. This is analogous to the Chevalley Restriction Theorem for symmetric spaces.
Theorem (Thorbergsson 1991)
If $M^n$ is irreducible isoparametric in $S^{n+k}$ with $k \geq 2$, then the parallel foliation is the orbit foliation of the slice representation of some rank $k + 1$ symmetric space restricted to unit sphere.

Remarks

- There are infinitely many inhomogeneous examples of isoparmetric hypersurfaces in $S^{n+1}$.
- Many works concerning singular foliations that behave like orbit foliations of polar actions: Terng-Thorbergsson, Heintze-Liu, Lytchak, Radeschi, ... etc.
II. Construction of invariant solutions of PDE

Reduction of Variables
If $G$ acts on $M$ with $S$ as an $H$-slice, then the construction of $G$-invariant solutions of a PDE on $M$ often reduces to the construction of $H$-invariant solutions on the slice $S$. In particular,

1. if the action is transitive, the reduced equation is algebraic,
2. if principal orbits are of co-dimension 1 then the reduced equation becomes an ODE.

Some Examples

- Schwarzchild (1915): $SO(3)$-invariant solution to the Einstein field equations.
- Hsiang-Lawson (1972) used the isotropy representation of rank two symmetric space to construct $S^1$-invariant minimal surfaces in spheres.
• Uhlenbeck (1982) showed that the ODE given by $S^1$ equivariant harmonic map from the 2-sphere to $S^{2n}$ is a completely integrable Hamiltonian system and constructed a complete set of commuting conservation laws.

• Terng-Liu (2009) proved that the mean curvature flow (MCF) starting from an isoparametric submanifold $M$ in $S^n$ flows along parallel submanifolds of $M$, and collapses in finite time with type II singularity. We also proved in 2020 that these solutions are ancient solution to the MCF.

• Liu-Radeschi (2022) proved that the MCF starts from a principal orbit of a polar action on a symmetric space also collapses in finite time and is an ancient solution.
III. Differential invariants of curves in $G/K$

Let $G$ act on $M$ transitively, $p_0 \in M$, $H$ subgroup of $G_{p_0}$, and $V$ an affine subspace of $\mathcal{G}$. A smooth map $g : \mathbb{R} \to G$ is a (H,V)-moving frame along $\gamma : \mathbb{R} \to M$ if it satisfies

1. $\gamma(t) = g(t) \cdot p_0$,
2. $g^{-1}g_x \in V$,
3. if $g_1$ also satisfies (1), (2), then there is a constant $c \in H$ such that $g_1 = gc$.

Remark

- $g : \mathbb{R} \to G$ is a $(H, V)$-moving frame along $\gamma : \mathbb{R} \to G/K$ iff $C^\infty(\mathbb{R}, V)$ is a $H$-slice of the gauge action of $C^\infty(\mathbb{R}, K)$ on $C^\infty(\mathbb{R}, \mathcal{G})$.

- The map $g^{-1}g_x$ gives a complete set of differential invariants for curves in $M$. 

The rigid motion group acts on $\mathbb{R}^3$ transitively by
\[
\begin{pmatrix}
A & y \\
0 & 1
\end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + y \\ 1 \end{pmatrix}.
\]

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be parametrized by its arc-length, the Frenet frame $g = \begin{pmatrix} e_1 & n & b & \gamma \end{pmatrix}$ satisfies
\[
g^{-1}g_x = \begin{pmatrix}
0 & -k & 0 & 1 \\
k & 0 & -\tau & 0 \\
0 & \tau & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

where $k, \tau$ are the curvature and torsion resp. Here $H = \{e\}$. 
Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be parametrized by its arc-length. Let $e_1 = \gamma_x$, and $(e_2, e_3)$ a parallel normal frame. Then

$$g = \begin{pmatrix} e_1 & e_2 & e_3 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an $(H, V)$-moving frame along $\gamma$ with $H = 1 \times SO(2)$ and

$$g^{-1}g_x = \begin{pmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \end{pmatrix} \in V$$

$k_1, k_2$ are the principal curvatures w.r.t. $e_2, e_3$ resp.
The Adjoint frame

Terng-Uhlenbeck 2006

Let \( a = i \text{diag}(I_k, -I_{n-k}) \in u(n) \), \( M = \{ g^{-1}ag \in g \in U(n) \} \)
adjoint orbit at \( a \), \( U_a = u(k) \times u(n - k) \), and

\[
P = U_a^\perp = \left\{ \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \right\}.
\]

Then there exists \((U_a, P)\) moving frame \( g \) along \( \gamma : \mathbb{R} \to M \), i.e.,

(i) \( \gamma = gag^{-1} \),
(ii) \( g^{-1}g_x \in P \),
(iii) if \( g_1 \) satisfies (i)-(ii) then there is a constant \( c \in U_a \) such that \( g_1 = gc \).
Central affine curve frame

(Pinkall 1995 for $n - 2$, Callini, Ivey, Mari Beffa 2013 for $n \geq 4$)

$SL(n, \mathbb{R})$ acts on $\mathbb{R}^n \setminus 0$ transitively by $A \cdot y = Ay$. If $\gamma : \mathbb{R} \to \mathbb{R}^n \setminus 0$ satisfying $\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)})$ never vanishes, then we can change parameter such that

- $\det(g) \equiv 1$, where $g = t(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \in SL(n, \mathbb{R})$,
- $g^{-1}g_x = \begin{pmatrix} 0 & 0 & u_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ \cdot \\ \cdot \end{pmatrix}$.

Such $g$ is called the central affine moving frame and $u_1, \ldots, u_{n-1}$ the central affine curvatures of $\gamma$. 
Lagrangian frame

Let \( w(X, Y) = X^t SY \) be the symplectic form on \( \mathbb{R}^{2n} \), and \( Sp(2n) = \{ g \in GL(2n, \mathbb{R}) \mid g^t S g = S \} \) the group of linear isomorphisms that preserves \( w \), where
\[
S = \sum_{i=1}^{2n} (-1)^{i+1} e_{i,2n+1-i}.
\]

(Terng- Wu 2021) Given a smooth \( \gamma : \mathbb{R} \to \mathbb{R}^{2n} \) satisfying \( \det(\gamma, \ldots, \gamma_{(2n-1)x}) \) never vanishes and \( \gamma, \gamma_x, \ldots, \gamma_{x}^{n-1} \) span a Lagrangian subspace, then

- we can change parameter so that \( w(\gamma_x^{(n)}, \gamma_x^{(n-1)}) = (-1)^n \),
- there exists \( g = (\gamma_x, \ldots, \gamma_x^{(n)}, g_{n+2}, \ldots, g_{2n}) : \mathbb{R} \to Sp(2n) \) such that
\[
g^{-1} g_x = b + u = \sum_{i=1}^{2n-1} e_{i+1,i} + \sum_{i=1}^{n} u_i e_{n+1-i,n+i}.
\]

\( g \) Lagrangian frame and \( u \) Lagrangian curvature along \( \gamma \).
(IV) Soliton eqs and Integrable curve flows

Schrödinger flow on Grassmanian

The Adjoint orbit $M$ of $U(n)$ in $u(n)$ at $a = i \text{diag}(I_k, -I_{n-k})$ equipped with the metric induced from the Killing form of $u(n)$ is isometric to $\text{Gr}(k, \mathbb{C}^n)$. The Schrödinger flow, $\gamma_t = J_\gamma(\nabla_\gamma x \gamma_x)$ is

$$\gamma_t = [\gamma, \gamma_{xx}]. \quad (1)$$

Theorem (Terng-Uhlenbeck 2006)

If $\gamma : \mathbb{R}^2 \to M$ is a solution of (1), then there is $g : \mathbb{R}^2 \to U(n)$ such that

- $g(\cdot, t)$ is an Adjoint frame along $\gamma(\cdot, t)$ for each $t$,

  $$\begin{cases} 
  g^{-1} g_x = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \\
  g^{-1} g_t = \begin{pmatrix} -iqq^* & iq_x \\ iq_x^* & iqq^* \end{pmatrix},
  \end{cases} \quad (2)
  $$

  where $q(\cdot, t)$ is the differential invariant of $\gamma(\cdot, t)$. 
• $q$ satisfies the matrix NLS,

$$q_t = i(q_{xx} + 2qq^* q).$$

• Conversely, if $q$ is a solution of Matrix NLS, then (2) is solvable; moreover, if $g$ is a solution of (2), then $\gamma = gag^{-1}$ is a solution of (1).

• Use techniques from soliton theory, we solve the Cauchy problem, construct infinitely many families of explicit soliton solutions, and write down commuting higher flows, ... etc.

**Theorem (Liu-Terng-Wu 2021)**
The above results also hold for Schrödinger flows on compact Hermitian symmetric spaces.
Lax equation

1. The following is equivalent:
   \[
   \begin{cases}
   g^{-1}g_x = A, \\
   g^{-1}g_t = B
   \end{cases}
   \]
   is solvable,
   \[
   A_t = B_x + [A, B],
   \]
   \[
   A_t = (\partial_x + A)_t = [\partial_x + A, B].
   \]

2. The last equation is a Lax equation, i.e., evolution equation for maps \( y : \mathbb{R} \to B \) of the form \( y_t = [y, B(y)] \), where \( B \) is an associated algebra of operators and \( B : B \to B \).

3. Completely integrable Hamiltonian systems and soliton equations often can be written as Lax equations.

4. If \( x : \mathbb{R} \to gl(n) \) satisfies \( x_t = [x, B(x)] \) then \( \text{tr}(x^k(t)) \) and eigenvalues of \( x(t) \) are constants.
   
   **Proof** Compute directly: \( (x^k)_t = [x^k, B(x)] \). So we have \( (\text{tr}(x^k))_t = \text{tr}([x^k, B(x)]) = 0. \)
Adler-Kostant-Symes’ construction of completely integrable systems

$L_+, L_-$ subgroups of $L$ such that $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as Linear subspaces ( $(\mathcal{L}_+, \mathcal{L}_-)$ is a splitting of $\mathcal{L}$).

Let $(\ , \ )$ be an ad-invariant non-degenerate bilinear form on $\mathcal{L}$. Then:

- $\mathcal{L}^\perp_+$ can be identified as $\mathcal{L}^*_-$.
- If $V_i : \mathcal{L} \to \mathcal{L}$ satisfies $[\xi, V_i(\xi)] = 0$ and $V_i(g\xi g^{-1}) = gV_i(\xi)g^{-1}$, then
  \[ \xi_t = [\xi, (V_i(\xi))_+] \]
  is Hamiltonian w.r.t. the natural Poisson structure on the dual $\mathcal{L}^*_-$ and they all commute, where $\eta_+$ is the projection of $\eta$ onto $\mathcal{L}_+$ along $\mathcal{L}_-$. 
Let $\mathcal{L} = sl(n, \mathbb{R})$, $\mathcal{L}_+ = so(n)$, and $\mathcal{L}_- = B_n$ upper triangular. Then $(\mathcal{L}_+, \mathcal{L}_-)$ is a splitting of $\mathcal{L}$. Let $(\xi, \eta) = \text{tr}(\xi \eta)$. Then $\mathcal{L}_+ = \mathcal{L}_+^\perp$ the space of trace zero symmetric $n \times n$ matrices, which is $\mathcal{L}^\ast$.

The **Toda lattice (in Flaschka variables)** is the flow on the co-adjoint orbit $M$ of tri-diagonal trace 0 symmetric $n \times n$ matrices constructed from $V(\xi) = \xi$, i.e.,

$$\xi_t = [\xi, \xi_+]$$

and flows $\xi_t = [\xi, (\xi^k)_+]$ all commute for $1 \leq k \leq n - 1$. So this is a completely Hamiltonian system.
A standard splitting of Loop algebra

Let $G$ be a finite dimensional simple Lie algebra,

$$L(G) = \left\{ \sum_{i \leq i_0} \xi_i \lambda^i \mid \xi_i \in G \right\},$$

$$L_+(G) = \left\{ \sum_{i \geq 0} \xi_i \lambda^i \in L(G) \right\},$$

$$L_-(G) = \left\{ \sum_{i < 0} \xi_i \lambda^i \in L(G) \right\}.$$

Then

- $(L_+(G), L_-(G))$ is a splitting of $L(G)$,
- $(\xi, \eta) = \text{Res}(\text{tr}(\xi \eta))$ is an ad-invariant non-degenerate bi-linear form,
- $L_\pm(G)^* = L_\pm(G)$,
- The coadjoint $L_-(G)$ orbit at $J = a \lambda$ or $J = a \lambda + b$ is $J + [a, G]$. 
• Given smooth $u : \mathbb{R} \to [a, G]$, there exists a unique

$$P(u, \cdot) = a\lambda + \sum_{i \leq 0} P_i(u)\lambda^i$$

in $\mathcal{L}(G)$ such that

$$\begin{cases}
[\partial_x + J + u, P(u, \lambda)] = 0, \\
P(u, \lambda) \text{ is conjugate to } P(0, \lambda).
\end{cases}$$

Then the AKS method gives commuting soliton flows

$$u_t = [\partial_x + b + u, P_i(u)].$$

Note these equations are Lax equations, hence invariants of $\partial_x + b + u$ are constants of these flows.
• The matrix NLS hierarchy is constructed with $\mathcal{G} = su(n)$ on the coadjoint orbit at $J = a\lambda$, where $a = i\text{diag}(I_k, -I_{n-k})$.

• Let $\{\alpha_1, \ldots, \alpha_n\}$ be a system of simple negative roots of $\mathcal{G}$, $b = \sum_{i=1}^{n} \alpha_i$, $\alpha$ the highest root, and $J = \alpha\lambda + b$. The soliton hierarchy on $C^\infty(\mathbb{R}, [a, \mathcal{G}])$ constructed from $J$ is invariant under the gauge action of $C^\infty(\mathbb{R}, \mathcal{N}_+)$, where $\mathcal{N}_+$ is the positive nilpotent subalgebra. The Drinfeld-Sokolov’s $\mathcal{G}$-KdV hierarchy is the quotient hierarchy induced on a cross section (canonical form) of the gauge action of $C^\infty(\mathbb{R}, \mathcal{N}_+)$.

• The $sl(2, \mathbb{R})$-KdV hierarchy with $J = e_{21} + e_{12}\lambda$ gives the standard KdV hierarchy and the $sl(n, \mathbb{R})$-KdV gives the Gelfand-Dikki hierarchy on the space of $n$-th order linear differential operators on the line.
• The central affine moving frames and the Lagrangian moving frames give cross sections of the gauge actions in the constructions of the $sl(n, \mathbb{R})$-KdV and the $sp(2n)$-KdV flows resp.

• There is a hierarchy of central affine curve flows in $\mathbb{R}^n$ such that the induced flow for central affine curvature are the $sl(n, \mathbb{R})$-KdV flows ($n = 2$ by Pinkall (1995), $n = 3$ by Callini-Ivey-Mari Baffe (2013), and $n \geq 4$ by Terng-Wu (2019)).

• Terng-Wu (2019) also construct Darboux transforms, explicit soliton solutions, and gives the Hamiltonian properties of central affine curve flows on $\mathbb{R}^n$. 
• (Terng-Wu 2020) Isotropic curve flow on $\mathbb{R}^{2n+1}$

We constructed a hierarchy of isotropic curve flows in $\mathbb{R}^{2n+1}$ invariant under $O(n + 1, n)$, whose isotropic curvature flows are the $o(n + 1, n)$-KdV hierarchy, and also give explicit solitons solutions, and bi-Hamiltonian property for these curve flows.

• (Terng-Wu 2021) Lagrangian curve flow on $\mathbb{R}^{2n}$

We construct a sequence of commuting Lagrangian curve flows on the symplectic space $\mathbb{R}^{2n}$ whose Lagrangian curvature are solutions of the $sp(2n)$-KdV flows. Darboux transforms, explicit soliton solutions, and the Hamiltonian properties of these Lagrangian curve flows are given.