Canonical Forms in Geometry and Soliton Theory

Chuu-Lian Terng[1]

[1]Department of Mathematics
University of California, Irvine

Glimpses of Mathematics, Now and Then:
A Celebration of Karen Uhlenbeck 80th Birthday
Institute for Advanced Study
September 16-18, 2022

Photo from Karen's Abel Prize ceremony 2019

Bozeman and Irvine

Outline of my Talk

Use of canonical forms (slices) of group actions:

- Canonical forms in submanifold geometry
- Invariant solutions for partial differential equations
- Canonical forms and differential invariants for curves in homogeneous spaces
- Soliton equations and Integrable curve flows

I. Canonical Form in Submanifold Geometry

Definition. (Palais 1960)
A Lie group G acts on a manifold M, H a subgroup of G. A submanifold S is an *H-slice* if

- $G \cdot S = \{g \cdot p | g \in G, p \in S\}$ is open in M,
- $H \cdot S \subset S$,
- if $g \in G$ and $(g \cdot S) \cap S \neq \emptyset$, then $g \in H$.

Examples

1. $SO(n)$ acts on $\mathbb{R}^n$ by $g \cdot y = gy$, $S = \mathbb{R} \times 0$ is a Z_2-slice.
2. $SU(n)$ acts on $su(n)$ by Adjoint action, $g \cdot X = gXg^{-1}$. Then $\{X | X \in su(n)\, \mathrm{diagonal}\}$ is an S_n-slice.
3. $SO(n)$ acts on the space of $n \times n$ symmetric matrices by $g \cdot A = gAg^{-1}$, $S = \{\mathrm{diag}(c_1, \ldots, c_n) | c_i \in \mathbb{R}\}$ is a S_n-slice, where S_n is the permutation group of $\{1, \ldots, n\}$.

Polar Actions

Definition. (Palais-Terng 1985)
An isometric G-action on a Riemannian manifold M is polar if there is a closed submanifold S of M that meets all orbits and meets orthogonally. Such S is called a section.

Theorem (Dadok 1985)

(i) The Adjoint action of a compact Lie group G on its Lie algebra $\mathcal{G}$ is polar and a maximal abelian subalgebra is a W-slice, where $W = N(\mathcal{A})/Z(\mathcal{A})$.

(ii) The slice representation of a symmetric space G/K is polar, a maximal abelian subspace $\mathcal{A}$ in $\mathcal{P}$ is a W-slice, where $W = \frac{N(\mathcal{A})}{Z(\mathcal{A})}$ is the Weyl group associated to G/K and $\mathcal{P}$ is the orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$.

(iii) Orbit foliations of polar representations are the ones obtained in (i).

- Examples (i) and (iii) are the slice representations of $S^n = \frac{SO(n+1)}{SO(n)}$ and $\frac{U(n)}{O(n)}$ respectively.
- The polar representation on $\mathbb{R}^n$ gives a polar action on S^{n-1}.

Definition. (Terng 1985)
A submanifold M^n of S^{n+k-1} is isoparmetric if

1. the normal bundle $\nu(M)$ is flat,
2. principal curvatures along a parallel normal field are constant.

Remarks

- A principal orbit of a polar action on S^n is isoparametric.
- The definition of isoparametric hypersurfaces was given by É Cartan.

Isoparametric Theory

Theorem. (Terng 1985)
Let M^n be isoparametric in S^{n+k}, ξ a parallel normal field, and $M_\xi = \{\exp_p(\xi(p)) \mid p \in M\}$ a parallel to M. Then

- M_ξ is a smooth submanifold (may have lower dimension),
- $\{M_\xi | \xi \text{ parallel normal field}\}$ is a singular foliation of S^{n+k} and has similar properties as the orbit foliation of a polar action on S^{n+k}.
- Fix $p \in M$, there is a Weyl group W acts on the normal sphere $S_p := \exp_p(\nu_p(M))$ such that $M \cap S_p = W \cdot p$ and S_p meets all parallel submanifolds orthogonally.
- The map from the space of smooth functions on S^{n+k} that are constant on the leaves of the parallel foliation and the space of smooth W-invariant functions on S_p defined by restriction is a bijection. This is analogous to the Chevalley Restriction Theorem for symmetric spaces.

Theorem (Thorbergsson 1991)
If M^n is irreducible isoparametric in S^{n+k} with $k \geq 2$, then the parallel foliation is the orbit foliation of the slice representation of some rank $k+1$ symmetric space restricted to unit sphere.

Remarks

- There are infinitely many inhomogeneous examples of isoparmetric hypersurfaces in S^{n+1}.
- Many works concerning singular foliations that behave like orbit foliations of polar actions: Terng-Thorbergsson, Heintze-Liu, Lytchak, Radeschi, ... etc.

II. Construction of invariant solutions of PDE

Reduction of Variables

If G acts on M with S as an H-slice, then the construction of G-invariant solutions of a PDE on M often reduces to the construction of H-invariant solutions on the slice S. In particular,

1. if the action is transitive, the reduced equation is algebraic,
2. if principal orbits are of co-dimension 1 then the reduced equation becomes an ODE.

Some Examples

- Schwarzchild (1915): $SO(3)$-invariant solution to the Einstein field equations.

- Hsiang-Lawson (1972) used the isotropy representation of rank two symmetric space to construct S^1-invariant minimal surfaces in spheres.

- Uhlenbeck (1982) showed that the ODE given by S^1 equivariant harmonic map from the 2-sphere to S^{2n} is a completely integrable Hamiltonian system and constructed a complete set of commuting conservation laws.

- Terng-Liu (2009) proved that the mean curvature flow (MCF) starting from an isoparametric submanifold M in S^n flows along parallel submanifolds of M, and collapses in finite time with type II singularity. We also proved in 2020 that these solutions are ancient solution to the MCF.

- Liu-Radeschi (2022) proved that the MCF starts from a principal orbit of a polar action on a symmetric space also collapses in finite time and is an ancient solution.

III. Differential invariants of curves in G/K

Let G act on M transitively, $p_0 \in M$, H subgroup of G_{p_0}, and V an affine subspace of $\mathcal{G}$. A smooth map $g : \mathbb{R} \to G$ is a (H,V)-moving frame along $\gamma : \mathbb{R} \to M$ if it satisfies

(1) $\gamma(t) = g(t) \cdot p_0$,

(2) $g^{-1} g_x \in V$,

(3) if g_1 also satisfies (1), (2), then there is a constant $c \in H$ such that $g_1 = gc$.

Remark

- $g : \mathbb{R} \to G$ is a (H, V)-moving frame along $\gamma : \mathbb{R} \to G/K$ iff $C^\infty(\mathbb{R}, V)$ is a H-slice of the gauge action of $C^\infty(\mathbb{R}, K)$ on $C^\infty(\mathbb{R}, \mathcal{G})$.
- The map $g^{-1} g_x$ gives a complete set of *differential invariants* for curves in M.

Frenet Frame

The rigid motion group acts on $\mathbb{R}^3$ transitively by $\begin{pmatrix} A & y \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + y \\ 1 \end{pmatrix}$.

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be parametrized by its arc-length, the Frenet frame $g = \begin{pmatrix} e_1 & n & b & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}$ satisfies

$$g^{-1} g_x = \begin{pmatrix} 0 & -k & 0 & 1 \\ k & 0 & -\tau & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

where k, τ are the curvature and torsion resp. Here $H = \{e\}$.

Parallel frame

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be parametrized by its arc-length. Let $e_1 = \gamma_x$, and (e_2, e_3) a parallel normal frame. Then $g = \begin{pmatrix} e_1 & e_2 & e_3 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an (H, V)-moving frame along γ with $H = 1 \times SO(2)$ and

$$g^{-1} g_x = \begin{pmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \end{pmatrix} \in V$$

k_1, k_2 are the principal curvatures w.r.t. e_2, e_3 resp.

The Adjoint frame

Terng-Uhlenbeck 2006

Let $a = i\mathrm{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k}) \in u(n)$, $M = \{g^{-1}ag \in g \in U(n)\}$ adjoint orbit at $a, \mathcal{U}_a = u(k) \times u(n-k)$, and

$$\mathcal{P} = \mathcal{U}_a^{\perp} = \left\{ \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \right\}.$$

Then there exists $(U_a, \mathcal{P})$ moving frame g along $\gamma : \mathbb{R} \to M$, i.e.,

(i) $\gamma = gag^{-1}$,

(ii) $g^{-1}g_x \in \mathcal{P}$,

(iii) if g_1 satisfies (i)-(ii) then there is a constant $c \in U_a$ such that $g_1 = gc$.

Central affine curve frame

(Pinkall 1995 for $n - 2$, Callini, Ivey, Mari Beffa 2013 for $n \geq 4$)

$SL(n, \mathbb{R})$ acts on $\mathbb{R}^n \setminus 0$ transitively by $A \cdot y = Ay$.
If $\gamma : \mathbb{R} \to \mathbb{R}^n \setminus 0$ satisfying $\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1})$ never vanishes, then we can change parameter such that

- $\det(g) \equiv 1$, where $g = t(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \in SL(n, \mathbb{R})$,
-

$$g^{-1} g_x = \begin{pmatrix} 0 & 0 & & & & u_1 \\ 1 & 0 & 0 & & & u_2 \\ 0 & 1 & 0 & & & \cdot \\ & & & & & \\ & & & 1 & 0 & u_{n-1} \\ & & & & 1 & 0 \end{pmatrix}.$$

Such g is called the central affine moving frame and $u_1, \ldots, u_{n-1}$ the central affine curvatures of γ.

Lagrangian frame

Let $w(X, Y) = X^t SY$ be the symplectic form on $\mathbb{R}^{2n}$, and $Sp(2n) = \{g \in GL(2n, \mathbb{R}) \mid g^t Sg = S\}$ the group of linear isomorphisms that preserves w, where $S = \sum_{i=1}^{2n} (-1)^{i+1} e_{i,2n+1-i}$.

(Terng- Wu 2021) Given a smooth $\gamma : \mathbb{R} \to \mathbb{R}^{2n}$ satisfying $\det(\gamma, \ldots, \gamma_x^{(2n-1)})$ never vanishes and $\gamma, \gamma_x, \ldots, \gamma_x^{n-1}$ span a Lagrangian subspace, then

- we can change parameter so that $w(\gamma_x^{(n)}, \gamma_x^{(n-1)}) = (-1)^n$,
- there exists $g = (\gamma_x, \ldots, \gamma_x^{(n)}, g_{n+2}, \ldots, g_{2n}) : \mathbb{R} \to Sp(2n)$ such that

$$g^{-1} g_x = b + u = \sum_{i=1}^{2n-1} e_{i+1,i} + \sum_{i=1}^{n} u_i e_{n+1-i,n+i}.$$

g Lagrangian frame and u Lagrangian curvature along γ.

(IV) Soliton eqs and Integrable curve flows

Schrödinger flow on Grassmanian

The Adjoint orbit M of $U(n)$ in $u(n)$ at $a = i\,\mathrm{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})$ equipped with the metric induced from the Killing form of $u(n)$ is isometric to $\mathrm{Gr}(k, \mathbb{C}^n)$. The Schrödinger flow, $\gamma_t = J_\gamma(\nabla_{\gamma_x}\gamma_x)$ is

$$\gamma_t = [\gamma, \gamma_{xx}]. \tag{1}$$

Theorem (Terng-Uhlenbeck 2006)

If $\gamma : \mathbb{R}^2 \to M$ is a solution of (1), then there is $g : \mathbb{R}^2 \to U(n)$ such that

- $g(\cdot, t)$ is an Adjoint frame along $\gamma(\cdot, t)$ for each t,
- $$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \\ g^{-1}g_t = \begin{pmatrix} -iqq^* & iq_x \\ iq_x^* & iqq^* \end{pmatrix}, \end{cases} \tag{2}$$

  where $q(\cdot, t)$ is the differential invariant of $\gamma(\cdot, t)$.

- q satisfies the matrix NLS,

$$q_t = i(q_{xx} + 2qq^*q).$$

- Conversely, if q is a solution of Matrix NLS, then (2) is solvable; moreover, if g is a solution of (2), then $\gamma = gag^{-1}$ is a solution of (1).
- Use techniques from soliton theory, we solve the Cauchy problem, construct infinitely many families of explicit soliton solutions, and write down commuting higher flows, ... etc.

Theorem (Liu-Terng-Wu 2021)

The above results also hold for Schrödinger flows on compact Hermitian symmetric spaces.

Lax equation

1. The following is equivalent:
   - $\begin{cases} g^{-1}g_x = A, \\ g^{-1}g_t = B \end{cases}$ is solvable,
   - $A_t = B_x + [A, B]$,
   - $A_t = (\partial_x + A)_t = [\partial_x + A, B]$.
2. The last equation is a Lax equation, i.e., evolution equation for maps $y : \mathbb{R} \to \mathcal{B}$ of the form $y_t = [y, B(y)]$, where $\mathcal{B}$ is an associated algebra of operators and $B : \mathcal{B} \to \mathcal{B}$.
3. Completely integrable Hamiltonian systems and soliton equations often can be written as Lax equations.
4. If $x : \mathbb{R} \to gl(n)$ satisfies $x_t = [x, B(x)]$ then $\mathrm{tr}(x^k(t))$ and eigenvaluies of $x(t)$ are constants.
   Proof Compute directly: $(x^k)_t = [x^k, B(x)]$. So we have $(\mathrm{tr}(x^k))_t = \mathrm{tr}([x^k, B(x)]) = 0$.

Adler-Kostant-Symes' construction of completely integrable systems

L_+, L_- subgroups of L such that $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as Linear subspaces ($(\mathcal{L}_+, \mathcal{L}_-)$ is a splitting of $\mathcal{L}$).

Let (,) be an ad-invariant non-degenerate bilinear form on $\mathcal{L}$. Then:

- $\mathcal{L}_+^{\perp}$ can be identified as $\mathcal{L}_-^*$.
- If $V_i : \mathcal{L} \to \mathcal{L}$ satisfies $[\xi, V_i(\xi)] = 0$ and $V_i(g\xi g^{-1}) = gV_i(\xi)g^{-1}$, then

$$\xi_t = [\xi, (V_i(\xi))_+]$$

is Hamiltonian w.r.t. the natural Poisson structure on the dual $\mathcal{L}_-^*$ and they all commute, where η_+ is the projection of η onto $\mathcal{L}_+$ along $\mathcal{L}_-$.

Toda Lattice

Let $\mathcal{L} = sl(n, \mathbb{R})$, $\mathcal{L}_+ = so(n)$, and $\mathcal{L}_- = \mathcal{B}_n$ upper triangular. Then $(\mathcal{L}_+, \mathcal{L}_-)$ is a splitting of $\mathcal{L}$. Let $(\xi, \eta) = \mathrm{tr}(\xi\eta)$. Then $\mathcal{L}_+^\perp =$ the space of trace zero symmetric $n \times n$ matrices, which is $\mathcal{L}_-^*$.

The Toda lattice (in Flaschka variables) is the flow on the co-adjoint orbit M of tri-diagonal trace 0 symmetric $n \times n$ matrices constructed from $V(\xi) = \xi$, i.e.,

$$\xi_t = [\xi, \xi_+],$$

and flows $\xi_t = [\xi, (\xi^k)_+]$ all commute for $1 \leq k \leq n-1$. So this is a completely Hamiltonian system.

A standard splitting of Loop algebra

Let $\mathcal{G}$ be a finite dimensional simple Lie algebra,

$$\mathcal{L}(\mathcal{G}) = \{\sum_{i \le i_0} \xi_i \lambda^i \,|\, \xi_i \in \mathcal{G}\},$$

$$\mathcal{L}_+(\mathcal{G}) = \{\sum_{i \ge 0} \xi_i \lambda^i \in \mathcal{L}(\mathcal{G})\},$$

$$\mathcal{L}_-(\mathcal{G}) = \{\sum_{i < 0} \xi_i \lambda^i \in \mathcal{L}(\mathcal{G})\}.$$

Then

- $(\mathcal{L}_+(\mathcal{G}), \mathcal{L}_-(\mathcal{G}))$ is a splitting of $\mathcal{L}(\mathcal{G})$,
- $(\xi, \eta) = \mathrm{Res}(\mathrm{tr}(\xi\eta))$ is an ad-invariant non-degenerate bi-linear form,
- $\mathcal{L}_\pm(\mathcal{G})^* = \mathcal{L}_\pm(\mathcal{G})$,
- The coadjoint $L_-(G)$ orbit at $J = a\lambda$ or $J = a\lambda + b$ is $J + [a, \mathcal{G}]$.

- Given smooth $u : \mathbb{R} \to [a, \mathcal{G}]$, there exists a unique

$$P(u, \cdot) = a\lambda + \sum_{i \le 0} P_i(u)\lambda^i$$

in $\mathcal{L}(\mathcal{G})$ such that

$$\begin{cases} [\partial_x + J + u, P(u, \lambda)] = 0, \\ P(u, \lambda) \text{ is conjugate to } P(0, \lambda). \end{cases}$$

Then the AKS method gives commuting soliton flows

$$u_t = [\partial_x + b + u, P_i(u)].$$

Note these equations are Lax equations, hence invariants of $\partial_x + b + u$ are constants of these flows.

- The matrix NLS hierarchy is constructed with $\mathcal{G} = su(n)$ on the coadjoint orbit at $J = a\lambda$, where $a = i\mathrm{diag}(\mathrm{I}_k, -\mathrm{I}_{n-k})$.
- Let $\{\alpha_1, \ldots, \alpha_n\}$ be a system of simple negative roots of $\mathcal{G}$, $b = \sum_{i=1}^{n} \alpha_i$, α the highest root, and $J = \alpha\lambda + b$. The soliton hierarchy on $C^\infty(\mathbb{R}, [a, \mathcal{G}])$ constructed from J is invariant under the gauge action of $C^\infty(\mathbb{R}, N_+)$, where $\mathcal{N}_+$ is the positive nilpotent subalgebra. The Drinfeld-Sokolov's $\mathcal{G}$-KdV hierarchy is the quotient hierarchy induced on a cross section (canonical form) of the gauge action of $C^\infty(\mathbb{R}, N_+)$.
- The $sl(2, \mathbb{R})$-KdV hierarchy with $J = e_{21} + e_{12}\lambda$ gives the standard KdV hierarchy and the $sl(n, \mathbb{R})$-KdV gives the Gelfand-Dikki hierarchy on the space of n-th order linear differential operators on the line.

- The central affine moving frames and the Lagrangian moving frames give cross sections of the gauge actions in the constructions of the $sl(n, \mathbb{R})$-KdV and the $sp(2n)$-KdV flows resp.

- There is a hierarchy of central affine curve flows in $\mathbb{R}^n$ such that the induced flow for central affine curvature are the $sl(n, \mathbb{R})$-KdV flows ($n = 2$ by Pinkall (1995), $n = 3$ by Callini-Ivey-Mari Baffe (2013), and $n \geq 4$ by Terng-Wu (2019)).

- Terng-Wu (2019) also construct Darboux transforms, explicit soliton solutions, and gives the Hamiltonian properties of central affine curve flows on $\mathbb{R}^n$.

- (Terng-Wu 2020) Isotropic curve flow on $\mathbb{R}^{2n+1}$

  We constructed a hierarchy of isotropic curve flows in $\mathbb{R}^{2n+1}$ invariant under $O(n+1, n)$, whose isotropic curvature flows are the $o(n+1, n)$-KdV hierarchy, and also give explicit solitons solutions, and bi-Hamiltonian property for these curve flows.

- (Terng-Wu 2021) Lagrangian curve flow on $\mathbb{R}^{2n}$

  We construct a sequence of commuting Lagrangian curve flows on the symplectic space $\mathbb{R}^{2n}$ whose Lagrangian curvature are solutions of the $sp(2n)$-KdV flows. Darboux transforms, explicit soliton solutions, and the Hamiltonian properties of these Lagrangian curve flows are given.