Lower Bounds for Maximally Recoverable Tensor Codes and Higher Order MDS Codes

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Abstract

An (m, n, a, b)-tensor code consists of $m \times n$ matrices whose columns satisfy 'a' parity checks and rows satisfy 'b' parity checks (i.e., a tensor code is the tensor product of a column code and row code). Tensor codes are useful in distributed storage because a single erasure can be corrected quickly either by reading its row or column. Maximally Recoverable (MR) Tensor Codes, introduced by Gopalan et al. [GHK⁺17], are tensor codes which can correct every erasure pattern that is information theoretically possible to correct. The main questions about MR Tensor Codes are characterizing which erasure patterns are correctable and obtaining explicit constructions over small fields.

In this paper, we study the important special case when a = 1, i.e., the columns satisfy a single parity check equation. We introduce the notion of higher order MDS codes (MDS(ℓ) codes) which is an interesting generalization of the well-known MDS codes, where ℓ captures the order of genericity of points in a low-dimensional space. We then prove that a tensor code with a = 1 is MR iff the row code is an MDS(m) code. We then show that MDS(m) codes satisfy some weak duality. Using this characterization and duality, we prove that (m, n, a = 1, b)-MR tensor codes require fields of size $q = \Omega_{m,b}(n^{\min\{b,m\}-1})$. Our lower bound also extends to the setting of a > 1. We also give a deterministic polynomial time algorithm to check if a given erasure pattern is correctable by the MR tensor code (when a = 1).

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1 Introduction

In distributed storage, data is stored in individual servers each with a few terabytes of capacity. A large datacenter can have millions of such servers holding a few exabytes (millions terabytes) of data. Costs for building and running such datacenters run into billions of dollars. In such a large system, hard disks crash every minute. Servers can also become temporarily unavailable due to system updates or network bottlenecks. To avoid data loss and to serve user requests with low latency, some form or redundancy is necessary. Replicating data multiple times is too wasteful, doubling or tripling the costs. Erasure coding has been used to improve the storage efficiency while maintaining data reliability. For example a (k + h, k)-Reed-Solomon code can be used to add h redundant servers to every k data servers. This allows us to correct any h erasures (node failures). To improve storage efficiency, one is forced to choose large values of k. But this creates problems with latency, to recover a single erased node, one needs to read data from k other servers. When kis large, this is prohibitively slow. To balance this tension between storage efficiency and latency, erasure codes with *locality* were introduced in [GHSY12, SAP+13]. These codes allow fast recovery of an erased symbol by reading a small number of other unerased coordinates, this ability is referred to as locality. This requires that each coordinate of the code participates in a parity check equation involving a few other symbols. In addition to these local parity checks, the code also satisfies a small number of global parity checks which give it resilience to tolerate a large number of erasures in the worst case. Such codes with different architectures were deployed in practice to reduce the storage overhead while maintaining low latency and high durability [HSX⁺12, MLR⁺14].

The key notion of maximal recoverability for such local codes was introduced in [CHL07, GHSY12]. Maximal recoverability refers to the optimality of a code w.r.t to its ability to correct every erasure pattern that is information theoretically possible to correct given the code architecture (or topology). Therefore maximally recoverable (MR) codes have the best durability among all the codes with that particular architecture. In a seminal paper, Gopalan et al. [GHK⁺17] generalized and brought under a common framework various code topologies used in erasure coding, by introducing MR codes with grid-like topologies. Here the topology as specified by $T_{m\times n}(a, b, h)$. This means that the codewords are $m \times n$ matrices where each column satisfies a parity check equations and each row satisfies b parity check equations. In addition there are h global parity check equations that all the mn symbols satisfy. It is easy to show the existence of MR codes for any given topology over exponentially large fields by using randomization and Schwartz-Zippel lemma [GHK⁺17]. There are two main questions about these codes which are still wide open:

Question 1.1. What are the erasure patterns that are correctable by an MR code with topology $T_{m \times n}(a, b, h)$?

Question 1.2. What is the minimum field size required to construct an MR code with topology $T_{m \times n}(a, b, h)$? In particular, can we get explicit constructions over small fields?

Both questions are really important. Knowing which patterns are correctable allows to design the topology which gives the desired durability while minimizing the storage costs. And explicit constructions over small fields are important for them to be useful in practice. The amount of computation needed for encoding and recovering from erasures is very sensitive to the field size over which the code is defined. Typically, field sizes of 2^8 to 2^{16} are used in practice. Fields which are much bigger incur a large computational overhead and therefore infeasible to use in practice.

When h = 0, codes with topology $T_{m \times n}(a, b, 0)$ are pure tensor codes, i.e., the code is a tensor product of a column code and a row code. In this paper, we will denote a code with this topology

as an (m, n, a, b)-tensor code. Let us denote the row code by denoted by C_{row} , which is a (n, n-b)code. And denote the column code by C_{col} , which is a (m, m-a)-code. Then the tensor code is $C = C_{\text{col}} \otimes C_{\text{row}}$, the codewords are $m \times n$ matrices where each row belongs to C_{row} and each column belongs to C_{col} . The setting h = 0 is already very interesting for the following reason. In a recent work [HPYWZ21], it was shown that the set of erasure patterns correctable by an MR code with topology $T_{m \times n}(a, b, h)$ are precisely those obtained by adding h more erasures arbitrarily to erasure patterns correctable by an MR code with topology $T_{m \times n}(a, b, 0)$. Therefore answering Question 1.1 for h = 0 is enough to answering it for any h. Moreover, if one can construct an explicit MR code with topology $T_{m \times n}(a, b, 0)$ over \mathbb{F}_q , then one can get an explicit MR code with topology $T_{m \times n}(a, b, 0)$ over fields of size $q^{(m-a)(n-b)-h}$ [HPYWZ21], which also partially answers Question 1.2.

In this paper, we will focus on (m, n, a, b)-MR tensor codes in the special case of a = 1, i.e., there are no global parity checks and all the columns satisfy a single parity check equation. Firstly, this setting is very much practically relevant. The f4 storage architecture of Facebook [MLR⁺14] uses an (m = 3, n = 14, a = 1, b = 4)-tensor code, though they couldn't obtain an MR construction in their implementation. They simply use a tensor product of (14, 10)-Reed-Solomon code with a (3, 2)-parity check code, which need not be MR and therefore doesn't have the optimal durability. Moreover, as we will see, constructing MR tensor codes even in this special case of a = 1 is quite challenging and leads to some really interesting generalization of MDS (Maximum Distance Separable) codes.

1.1 Previous work

Correctable Patterns: In the paper where they introduce MR tensor codes, $[\text{GHK}^+17]$ characterize the set of correctable erasure patterns by an (m, n, a, b)-MR tensor code when a = 1 in terms of a combinatorial condition called 'regularity'. But they don't give a efficient procedure to check if an erasure pattern is regular, naively it would require checking an exponential number of constraints. They also conjectured that regularity characterizes correctable erasure patterns when a > 1. But this conjecture is false as shown in [HPYWZ21], we will later present a counterexample with an illuminating explanation of why regularity fails to capture correctability when a > 1. Currently, we do not have any characterization of correctable erasure patterns by an (m, n, a, b)-MR tensor codes when a, b > 1. A subset of correctable erasure patterns in MR tensor codes when a = 2 were obtained in [SRLS18].

Constructions: If we instantiate the row code and column code with random codes over fields of size $q \gg (an + bm - ab) \cdot {\binom{mn}{an+bm-ab}}$, by Schwartz-Zippel lemma and union bound, we can conclude that with high probability the tensor code will be MR. By doing a more careful union bound, [KMG21] show that there exists (m, n, a = 1, b)-MR tensor codes over fields of size $q = O_{m,b} \left(n^{b(m-1)}\right)$. In some special cases, (m = 4, n, a = 1, b = 2) and (m = 3, n, a = 1, b = 3), [KMG21] proved the existence of MR tensor codes over fields of size $q = O(n^5)$.

Lower bounds: Prior to our work there are no general lower bounds on the field size required for MR tensor codes. In the special case of (m = 4, n, a = 1, b = 2)-MR tensor code, [KMG21] prove a quadratic lower bound on the field size, i.e., $q = \Omega(n^2)$. For codes with topology given by $T_{n \times n}(a = 1, b = 1, h = 1)$, [GHK⁺17] prove a lower bound of exp ($\Omega(\log(n)^2)$). This was improved by [KLR19], where they proved the optimal field size is $q = \exp(O(n))$.

1.2 Our Contributions

We show an equivalent characterization of MR tensor codes with a = 1 in terms of their row codes. In particular, we introduce the notion of higher order MDS codes, which is a natural generalization of MDS (Maximum Distance Separable) codes and prove a tight relation to MR tensor codes.

Higher order MDS codes : A (n, k)-MDS code encodes k symbols into n symbols, such that one can correct any h erasures. Moreover, Reed-Solomon codes are explicit constructions of MDS codes over fields of size O(n). MDS codes play crucial role in coding theory and especially in erasure coding for distributed storage. Suppose the generator matrix of an (n, k)-code C over \mathbb{F} is given by a $k \times n$ matrix V, whose columns are denoted by $V_1, V_2, \ldots, V_n \in \mathbb{F}^k$. Let $V_A = \text{span} \{V_i : i \in A\}$. Then C is MDS iff $V_A = \mathbb{F}^k$ for all $A \subset [n]$ of size |A| = k. An equivalent way to state this is to say that for any subsets $A_1, A_2 \subset [n]$, dim $(V_{A_1} \cap V_{A_2}) = \dim(W_{A_1} \cap W_{A_2})$ for some generic matrix W.* One direction is obvious, this new condition clearly implies the usual MDS definition by taking $A_2 = [n]$. To prove the other direction note that

$$\dim(W_{A_1} \cap W_{A_2}) = \dim(W_{A_1}) + \dim(W_{A_2}) - \dim(W_{A_1} + W_{A_2})$$

$$= \dim(W_{A_1}) + \dim(W_{A_2}) - \dim(W_{A_1 \cup A_2})$$

$$= \min\{|A_1|, k\} + \min\{|A_2|, k\} - \min\{|A_1 \cup A_2|, k\} \qquad \text{(Genericity of } W\text{)}$$

$$= \dim(V_{A_1}) + \dim(V_{A_2}) - \dim(V_{A_1 \cup A_2}) \qquad (V \text{ is MDS})$$

$$= \dim(V_{A_1}) + \dim(V_{A_2}) - \dim(V_{A_1} + V_{A_2})$$

$$= \dim(V_{A_1} \cap V_{A_2}).$$

This leads to a natural generalization of MDS codes to higher order MDS codes.

Definition 1.3 (Higher order MDS code (MDS(ℓ))). Let C be an (n, k)-code with generator matrix $V_{k \times n}$. For $\ell \ge 2$, we say that C is an MDS(ℓ) code if for all $A_1, \ldots, A_\ell \subset [n]$,

$$\dim(V_{A_1} \cap V_{A_2} \cap \dots \cap V_{A_\ell}) = \dim(W_{A_1} \cap \dots \cap W_{A_\ell}),\tag{1}$$

where W is a $k \times n$ generic matrix.

Note that the definition of $MDS(\ell)$ is independent of the generator chosen to represent the code C and therefore purely a property of the code C. This is because of invariance of the condition (1) under basis change $V \to A \cdot V$ for any invertible $k \times k$ matrix A.

Example - MDS(3) : It is instructive to look at an example. Suppose C is an (n, 3)-MDS(3) code over \mathbb{F} . Let V be its generator matrix of size $3 \times n$ and let $V_1, V_2, \ldots, V_n \in \mathbb{F}^3$ be its columns. By abusing notation, we can think of V_i as points in the two dimensional projective space $\mathbb{P}^2(\mathbb{F})$, because scaling the vectors does affect the definition of MDS(3). Therefore the subspace span $\{V_i, V_j\}$ corresponds to a line passing through the points $V_i, V_j \in \mathbb{P}^2(\mathbb{F})$. Now the usual notion of MDS (which is equivalent to MDS(2)), corresponds to the condition that no three points among $V_1, V_2, \ldots, V_n \subset \mathbb{P}^2(\mathbb{F})$ are collinear. The MDS(3) condition corresponds to the condition that no three points are collinear and if we draw all the lines through pairs of points in V_1, V_2, \ldots, V_n , then no three lines

^{*}Genericity: A generic point X can be thought of either as a symbolic vector, or one can think of it as a point with entries in an infinite field \mathbb{F} which avoids any fixed low-dimensional algebraic variety. If $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , then one can think of a generic point as something which escapes any measure zero set. In particular, low-dimensional varieties are measure zero sets.

are concurrent (i.e., pass through the same point) other than the trivial concurrency which occurs when the three lines chosen pass through some V_i . Thus MDS(3) can be thought of a higher order genericity condition than MDS(2) and therefore strictly stronger condition than MDS(2). In particular an arbitrary MDS code may not be MDS(3). More generally, MDS(ℓ) require that ℓ -wise intersections behave generically. Thus ℓ can be thought of as the degree of genericity of the points V_1, V_2, \ldots, V_n .

We now state one of our main theorems relating MR tensor codes with higher order MDS codes.

Theorem 1.4. Let $C = C_{col} \otimes C_{row}$ be an (m, n, a = 1, b)-tensor code where C_{col} is a simple parity check code. Then C is MR iff C_{row} is an MDS(m) code.

Therefore constructing MR tensor codes when a = 1 is equivalent to constructing higher order MDS codes. Our next result is a lower bound on the field size required for higher order MDS codes. We will be stating it in the regime when the codimension of the code is a constant, which is the regime of interest in practice.

Theorem 1.5. Let C be an (n,k)-MDS (ℓ) code over \mathbb{F}_q . Then

$$q \ge \Omega_\ell \left(n^{\min\{\ell,k,n-k\}-1} \right).$$

This immediately implies the following corollary for MR tensor codes.

Corollary 1.6. Let C be an (m, n, a, b)-MR tensor code over \mathbb{F}_q . Then

$$q \ge \Omega_m \left(n^{\min\{m-a+1,b,n-b\}-1} \right).$$

This is the first general lower bound on the field size required for MR tensor codes. Prior to our work, only a quadratic lower bound was known in the special case of (m = 4, n, a = 1, b = 2) [KMG21], but unfortunately we cannot recover this bound from our general result.

We also show the following upper bound on field size for MR tensor codes, generalizing [KMG21] where they obtained such a result for a = 1.

Theorem 1.7. There exist (m, n, a, b)-MR tensor codes over fields of size

$$q = O_{m,b}\left(n^{b(m-a)}\right).$$

Finally, we also give an efficient polynomial time algorithm for checking if an erasure pattern is correctable by an MR tensor code when a = 1.

Theorem 1.8. There exists an efficient algorithm to check if an erasure pattern is correctable by an (m, n, a = 1, b)-MR tensor code in time $m(m + n)^3$.

1.3 **Proof Overview**

Higher order MDS - MR tensor code equivalence. The proof of the equivalence between (m, n, a = 1, b)-MR tensor codes and MDS(m) codes follows from some linear algebra and inductive arguments.

Field size lower bound. Our lower bound for MDS(m) is inspired by lower bounds for Maximally Recoverable Local Reconstruction Codes from [GGY20] and works as follows. We will first prove a lower bound when the code dimension is small. We then prove some weak duality for MDS(m) codes, which implies the lower bound when the codimension is small. By a reduction, we can assume that the dimension of the code is equal to m. Suppose V be the generator matrix of an (n, k)-MDS(k) code where k is a constant and suppose k divides n for simplicity. We will partition V_1, V_2, \ldots, V_n in to k parts of size n/k each, say the partition is given by $[n] = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_k$. Now consider arbitrary subsets $A_i \subset \mathcal{P}_i$ of size $|A_i| = k - 1$ for $i \in [k]$. For V to be MDS(k), it is necessary that $V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_k} = 0$. To see this, note that $x \in V_{A_i}$ imposes 1 generic linear equation on x. And so $x \in V_{A_i}$ for all $i \in [k]$ imposes k equations which should be linearly independent if V behaves generically. Fix any $i \in [k]$ and fix a subset $A_i \subset \mathcal{P}_i$. A random point X of \mathbb{F}_a^k lies in V_{A_i} for some $A_i \subset \mathcal{P}_i$ with high probability. This is because for a fixed subset $A_i \subset \mathcal{P}_i$, X lies in V_{A_i} with probability $\frac{1}{q}$. Since there are $\binom{n/k}{k-1}$ subsets $A_i \subset \mathcal{P}_i$, the expected number of $A_i \subset \mathcal{P}_i$ such that $X \in V_{A_i}$ is $\binom{n/k}{k-1} \cdot \frac{1}{q}$ which is $\gg 1$ if $q \ll_k n^{k-1}$. By using pairwise independence and carefully calculating second moments, we can conclude that if $q \ll_k n^{k-1}$, with high probability there exists some $A_i \subset \mathcal{P}_i$ such that $X \in V_{A_i}$. By union bound over $i \in [m], X \in V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_m}$ for some $A_i \subset \mathcal{P}_i$ and X is non-zero with high probability which violates the MDS(m) property. Therefore $q \gg_k n^{k-1}$. The lower bound for MR tensor codes when a > 1, is obtained by a simple reduction to the a = 1 case.

Efficient correctability checking when a = 1. The previous work of [GHK⁺17] showed that E is a correctable pattern for a (m, n, a = 1, b)-MR tensor code if and only if the erasure pattern satisifes a combinatorial condition called *regularity*. The regularity condition upper bounds the size of the intersection of E with any subrectangle of $[m] \times [n]$. We show that the inequalities in this regularity condition correspond to capacity constraints in a suitable max-flow problem. We call this property *excess-compatability*. We show that excess-compatibility is requivalent to regularity by applying a generalization of Hall's marriage theorem on the existence of matchings [BBJ12]. As a result, we show that checking regularity is equivalent to a polynomial-sized maximum flow problem.

1.4 Open Questions

The biggest open question is to obtain constructions of (n, n - b)-MDS(m) codes (or equivalently (m, n, a = 1, b)-MR tensor codes). Our lower bounds show that we need fields of size at least $\Omega_{m,b}\left(n^{\min\{b,m\}-1}\right)$ whereas the upper bounds (which are not explicit) are $q = O_{m,b}(n^{b(m-1)})$. Closing this gap and getting explicit constructions of MDS(m) codes over small fields is the main question we leave open. Concretely, our lower bound shows that (n, n - 3)-MDS(3) codes require fields of size $q \ge \Omega(n^2)$. We conjecture that this is tight. By MDS(3) duality (Corollary 3.12), this is equivalent to constructing (n, 3)-MDS(3) codes.

Conjecture 1.9. There exist (n, 3)-MDS(3) codes over fields of size $q = O(n^2)$.

We give evidence for this conjecture by giving constructions of codes over fields of size $q = O(n^2)$ which come very close to being MDS(3), see Appendix C for these constructions. In these constructions, we relax (1) by restricting which sets A_1, A_2, A_3 we consider. For the first construction, we split n into two halves, and require that each A_i is a two-element set using one element from each set.[†] For the second construction, we split n into three parts and require that A_i is a two-element subset of the *i*th part.

Another important open question is further characterizing correctable patterns for MR tensor codes.

Question 1.10. Can we efficiently detect which erasure patterns of an (m, n, a, b)-MR tensor code are correctable when a, b > 1?

Currently, there is no "simple" condition like that of regularity and no efficient deterministic algorithm which is known for testing correctability when a, b > 1. We shall investigate this question more deeply in a future work.

Organization

In Section 2, we formally define MR tensor codes. In Section 3, we develop the theory of MDS(m) codes, including studying its duality properties. In Section 4, we shows the field size lower bounds for MDS(m) and MR-tensor codes. In Section 5, we show how to efficiently testing whether an erasure pattern is correctable for (m, n, a = 1, b)-MR tensor codes.

In Appendix A, we prove Theorem 1.7 on randomized constructions of MR-tensor codes. In Appendix B, we provide a number of proofs omitted from the main exposition. In Appendix C, we give constructions of codes which partially satisfy the MDS(3) property.

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2 Preliminaries

Let \mathbb{F} be any field. Let $n > k \ge 1$ be integers. For any $A \subset [n]$ and a matrix V with columns v_1, v_2, \ldots, v_n , we use V_A to denote the submatrix of V formed by columns $\{v_i : i \in A\}$.

A (n, k)-code C is a k-dimensional subspace of \mathbb{F}^n . It can either be described using a generator matrix $G_{k \times n}$ such that $C = \{G^T x : x \in \mathbb{F}^k\}$ or using a parity check matrix $H_{(n-k) \times n}$ such that $C = \{y \in \mathbb{F}^n : Hy = 0\}$. Note that G and H have full row rank and $HG^T = 0$. The rows of Gform a basis for C and the rows of H form a basis for the dual code C^{\perp} . Note that G, H are not uniquely determined by C, but they are unique up to basis change.

Proposition 2.1. Let C be a (n,k)-code with generator matrix $G_{k\times n}$ and parity check matrix $H_{(n-k)\times n}$. Let $E \subset [n]$ be an erasure pattern and let $\overline{E} = [n] \setminus E$. The following conditions are equivalent.

- 1. E is correctable i.e. given $x_{\overline{E}}$ for some unknown $x \in C$, we can recover x.
- 2. $G_{\overline{E}}$ has rank k.
- 3. H_E has full column rank.

[†]This may seem very restrictive, but by Lemma 3.1, we assume that $|A_1| + |A_2| + |A_3| = 6$ and even each $|A_i| = 2$ (as long as the code is MDS).

Therefore a maximal correctable erasure pattern has size n - k. And codes which correct all erasure patterns of size n - k are called MDS codes.

Definition 2.2. (MDS Code) A (n,k)-code is called an MDS code if it can correct every erasure pattern of size n - k.

Reed-Solomon codes are explicit MDS codes and they can be constructed for all k, n over fields of size O(n) which is tight. We now present several equivalent properties of MDS codes.

Proposition 2.3. Let C be a (n,k)-code with generator matrix $G_{k\times n}$ and parity check matrix $H_{(n-k)\times n}$. The following conditions are equivalent.

- 1. C has distance n k + 1.
- 2. Every erasure pattern of size at most n k is correctable.
- 3. Every $k \times k$ minor of G is non-zero.
- 4. Every $(n-k) \times (n-k)$ minor of H is non-zero.

3 Higher order MDS codes

3.1 Basic properties of higher-order MDS codes

In this section, we will prove some properties of higher-oder MDS codes that we will need. The proofs are given in Appendix B. The following proposition gives an equivalent definition of $MDS(\ell)$ codes.

Lemma 3.1. Let $V \in \mathbb{F}^{k \times n}$ be an (n, k)-MDS code and let $\ell \ge 2$. Let $W \in \mathbb{R}^{k \times n}$ be a generic real matrix. Then V is $MDS(\ell)$ if and only if for all $A_1, \ldots, A_\ell \subseteq [n]$ such that $|A_i| \le k$, $|A_1| + \cdots + |A_\ell| = (\ell - 1)k$ and $A_1 \cap \cdots \cap A_k = \emptyset$, we have that

$$V_{A_1} \cap V_{A_2} \cap \dots \cap V_{A_{\ell}} = 0 \iff W_{A_1} \cap \dots \cap W_{A_{\ell}} = 0,$$

The following proposition shows that $MDS(\ell)$ property is preserverd under puncturing and shortening of codes. If C is any (n, k) code, the punctured code at position i is an (n - 1, k) code obtained given by projecting all the codewords of C onto the subset $[n] \setminus \{i\}$. The shortened code at position i is an (n - 1, k - 1) code obtained by projecting only the codewords of C which are supported on $[n] \setminus \{i\}$ onto $[n] \setminus \{i\}$.

Proposition 3.2. Let C be an (n, k)-MDS (ℓ) code. Then

1. C is also an $MDS(\ell - 1)$ code.

- 2. The code C_0 obtained by puncturing C at any position is an (n-1,k)-MDS (ℓ) code.
- 3. The code C_1 obtained by shortening C at any position is an (n-1, k-1)-MDS (ℓ) code.

3.2 Equivalence between MR tensor codes with a = 1 and $MDS(\ell)$

The following lemma shows that $MDS(\ell)$ codes are intimately connected to MR Tensor codes with a = 1 or b = 1.

Lemma 3.3. Let $C = C_{col} \otimes C_{row}$ be an (m, n, a = 1, b)-Tensor Code where C_{col} is a parity check code and C_{row} is an (n - b, n)-MDS code. Let $E \subset [m] \times [n]$ be a maximal erasure pattern i.e. |E| = mn - (m-1)(n-b) and suppose that each row has at least b erasures. Let $\overline{E} = \bigcup_{i=1}^{m} \{i\} \times A_i$ be the partition of \overline{E} into subsets of each of the m rows (here $A_i \subset [n]$). Then E is correctable iff $\dim(V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_m}) = 0.$

Proof. The conditions on E translate to the following conditions on A_1, A_2, \ldots, A_m .

- 1. $|A_i| \leq n-b$
- 2. $\sum_{i} |A_i| = (m-1)(n-b).$

Since C_{col} is a simple parity check code, each column of C sum to zero. Let $V_{\bar{b}\times n}$ be a generator matrix for C_{row} where $\bar{b} = n - b$. The following statements are equivalent.

- 1. E is not correctable.
- 2. There exists a non-zero codeword of C which is supported on E.
- 3. There exist $r_1, r_2, \ldots, r_m \in C_{\text{row}}$, not all zero, such that
 - $\operatorname{supp}(r_i) \subset \overline{A}_i \text{ for } i \in [m],$
 - $\sum_{i=1}^{m} r_i = 0.$

(Since $r_i = y_i^T V$ for some $y_i \in \mathbb{F}^{\bar{b}}$, we have the following equivalent statement.)

- 4. There exist $y_1, y_2, \ldots, y_m \in \mathbb{F}^{\bar{b}}$, not all zero, such that
 - $y_i^T V_{A_i} = 0$ for $i \in [m]$,
 - $\sum_{i=1}^{m} y_i = 0.$

(Since $y_i \in V_{A_i}^{\perp}$ for each $i \in [m]$, we have the following equivalent statement.)

5. There exists $y_i \in V_{A_i}^{\perp}$, not all zero, such that $\sum_{i=1}^m y_i = 0$. (Since $|A_i| \leq n-b$, and V is a generator matrix of an MDS code, we have $\dim(V_{A_i}^{\perp}) = (n-b) - \dim(V_{A_i}) = (n-b) - |A_i|$. Therefore

$$\sum_{i=1}^{m} \dim(V_{A_i}^{\perp}) = \sum_{i=1}^{m} (n-b-|A_i|) = m(n-b) - (m-1)(n-b) = n-b.$$

So we have the following equivalent statement.)

6. $V_{A_1}^{\perp} + V_{A_2}^{\perp} + \dots + V_{A_m}^{\perp} \neq \mathbb{F}^{\overline{b}}.$ 7. $V_{A_1} \cap V_{A_2} \cap \dots \cap V_{A_m} \neq 0.$

This completes the proof.

We can now prove Theorem 1.4.

Corollary 3.4 ((Theorem 1.4)). Let $C = C_{col} \otimes C_{row}$ be an (m, n, a = 1, b) tensor code. Let C_{col} be the parity check code. Then C is an MR tensor code iff C_{row} is (n, n - b) - MDS(m).

Proof. If C_{row} is MDS(m), then by Lemma 3.3, C is MR. The other direction follows from Lemma 3.3 and Lemma 3.1.

In the style of Lemma 3.3, we prove a similar but more intricate lemma which captures the case $a \ge 2$. Recall we have $a \le m$ and $b \le n$.

Lemma 3.5. Let $C = C_{col} \otimes C_{row}$ be an (m, n, a, b) tensor code and let $\bar{a} = m - a, \bar{b} = n - b$. Let E be a maximal erasure pattern of size $|E| = mn - \bar{a}\bar{b}$. and let $\bar{E} = \bigcup_{i \in [m]} \{i\} \times A_i = \bigcup_{j \in [n]} B_j \times \{j\}$. If U, V are generator matrices of C_{row} and C_{col} and P, Q are their respective parity check matrices. Then correctability of E is equivalent to each of the following conditions:

$$\sum_{i=1}^{m} U_i \otimes V_{A_i} = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{\bar{b}}$$
⁽²⁾

$$\sum_{j=1}^{n} U_{B_j} \otimes V_j = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{\bar{b}}$$

$$\tag{3}$$

$$\sum_{i=1}^{m} P_i \otimes V_{A_i}^{\perp} = \mathbb{F}^a \otimes \mathbb{F}^{\bar{b}}$$

$$\tag{4}$$

$$\sum_{j=1}^{n} U_{B_j}^{\perp} \otimes Q_j = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{b}.$$
(5)

Notice that if a = 1 and C_{col} is a parity check code, then $P_i = 1$ for all *i*. So the expression becomes $\sum_{i=1}^{m} V_{A_i}^{\perp} = \mathbb{F}^{\bar{b}}$. Taking duals, we get $\bigcap_{i=1}^{m} V_{A_i} = 0$ which is equivalent to Lemma 3.3.

3.3 Weak duality of $MDS(\ell)$

A natural conjecture is that for all $\ell \ge 2$, a code C is (n, k)-MDS (ℓ) iff the dual code C^{\perp} is (n, k)-MDS (ℓ) . This is true for $\ell = 2$ because MDS(2) is equivalent to the usual MDS, and MDS codes satisfy duality. We will later show that duality also holds for $\ell = 3$. But suprisingly, this fails for $\ell \ge 4$. We exhibit a counterexample in Appendix B.

As duality of higher-order MDS codes is false in general, we instead prove a weaker form of duality-the dual of an $MDS(\ell)$ code satisfies what we call "cycle- $MDS(\ell)$ " property which is a weaker form of $MDS(\ell)$. This result will also imply that the dual of any MDS(3) code is indeed MDS(3).

3.3.1 Cycle MDS

Call a collection of subsets $S_1, S_2, \ldots, S_\ell \subset [n]$ a cycle family if for all $j \in [n]$, the set $T_j := \{i : j \in S_i\}$ is an interval modulo ℓ , i.e., $T_j = \{c_j, c_j + 1, \ldots, d_j\} \mod \ell$ for some c_j and d_j . If you visualize S_1, S_2, \ldots, S_ℓ as subsets of rows of an $\ell \times n$ matrix, then T_1, T_2, \ldots, T_n are the subsets of columns corresponding to S_1, S_2, \ldots, S_ℓ .

Definition 3.6. Say that an $k \times n$ matrix V is (n, k)-cycle-MDS (ℓ) if for any cycle family $S_1, \ldots, S_\ell \subset [n]$ such that $|S_i| \leq k$ for all $i, |S_1| + \cdots + |S_\ell| \leq (\ell - 1)k$, and $S_1 \cap \cdots \cap S_\ell = \emptyset$, we have that $V_{A_1} \cap \cdots \cap V_{A_\ell} = 0$ if and only if it generically holds.

Lemma 3.7. Let C be an (n,k)-MDS(m) code. Then the dual code C^{\perp} is a cycle (n, n-k)-cycle-MDS(m) code.

Proof. In the proof, we crucially use the fact that the generator matrix of any (m, m - 1)-MDS code is equivalent to a generic (m, m - 1) matrix upto symmetries.[‡] In particular any (m, m - 1)-MDS code is also MDS (ℓ) for all $\ell \ge 2$. This is because there is a unique (m, 1)-MDS code upto symmetries which is the parity check code (whose generator matrix has a single row of all ones).

Let $V_{k\times n}$ be the generator matrix of C and let $Q_{(n-k)\times n}$ be its parity check matrix. Note that Q is also the generator matrix for the dual code C^{\perp} . Let a = 1, and b = n - k. Let $U_{(m-1)\times m}$ be a generator matrix of an (m, m-1)-MDS code (and thus also MDS(m)).

Let $S_1, S_2, \ldots, S_m \subseteq [n]$ be a cycle family, each of size at most b = n - k and of total size (m-1)band $\bigcap_{i=1}^m S_i = \emptyset$. To show that Q is (n, b)-cycle-MDS(m), it suffices to show that $Q_{S_1} \cap \cdots \cap Q_{S_m} = 0$ whenever it holds generically.

For all $j \in [n]$, let $T_j := \{i \in [m] : j \in S_i\}$. Recall that since S_1, S_2, \ldots, S_m is a cycle MDS family, we have $T_j = \{c_j, \ldots, d_j\} \mod m$ for some c_j, d_j or $T_j = \emptyset$. Note that $|T_j| \leq m - 1$ since $\bigcap_{i=1}^m S_i = \emptyset$.

Construct $B_1, B_2, \ldots, B_n \subset [m]$ as follows (again indices are considere modulo m):

$$B_i = \begin{cases} [m] \setminus \{c_j, \dots, d_j + 1\} & \text{if } T_j \neq \emptyset \\ [m-1] & \text{otherwise} \end{cases}$$

Note that each $|B_i| \leq m-1$ and

$$\sum_{i=1}^{n} |B_i| = (m-1)n - \sum_{j=1}^{m} |S_j| = (m-1)n - (m-1)b = (m-1)k.$$

For all $i \in [m]$, let $W_i := U_{[m] \setminus \{i, i+1\}}^{\perp}$, where indices are taken modulo m. Claim 3.8. For any MDS U, the family W_1, \ldots, W_m is MDS.

Proof. By symmetry, it suffices to show that W_1, \ldots, W_{m-1} are linearly independent. Consider the following matrix product.

$$\begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_{m-1}^T \end{bmatrix} \cdot \begin{bmatrix} W_1 & W_2 & \cdots & W_{m-1} \end{bmatrix} = \begin{bmatrix} * & & & & \\ * & * & & & \\ & * & * & & \\ & & \ddots & \ddots & \\ & & & & & * & * \end{bmatrix}$$

where * corresponds to a non-zero entry and all the unmarked entries are 0. Here we used that fact that $\langle W_i, U_i \rangle \neq 0$ and $\langle W_i, U_{i+1} \rangle \neq 0$, but $\langle W_i, U_j \rangle = 0$ for all $j \notin \{i, i+1\}$, which follows from the MDS property of U. Since the RHS matrix is clearly full rank, the matrices on the LHS product are both full rank. Therefore, W_1, \ldots, W_m is indeed MDS.

Claim 3.9. $U_{B_i}^{\perp} = W_{T_j} = \text{span}\{W_{c_j}, \dots, W_{d_j}\}.$

[‡]If U is the generator matrix of an (m, m-1)-MDS code and $X_{(m-1)\times m}$ is any generic matrix, we can scale the columns of X and change basis (left multiply with an invertible $(m-1)\times(m-1)$ matrix) to make it equal to U.

Proof. Clearly $W_{c_j}, \ldots, W_{d_j} \in U_{B_i}^{\perp}$. Since they are part of an MDS code of dimension m-1 and $|T_j| \leq m-1, W_{c_j}, \ldots, W_{d_j}$ are linearly independent. Finally by the MDS property of U and since $|B_i| \leq m-1$,

$$\dim(U_{B_i}^{\perp}) = (m-1) - |B_i| = (m-1) - (m - (|T_i| + 1)) = |T_i|.$$

Therefore we get an equality by dimension counting.

By Lemma 3.5,

$$\bigcap_{i=1}^{m} Q_{S_{i}} = 0$$

$$\iff \sum_{i=1}^{m} W_{i} \otimes Q_{S_{m}} = \mathbb{F}^{m-1} \otimes \mathbb{F}^{b}$$

$$\iff \sum_{j=1}^{n} W_{T_{j}} \otimes Q_{i} = \mathbb{F}^{m-1} \otimes \mathbb{F}^{b}$$

$$\iff \sum_{j=1}^{n} U_{B_{j}}^{\perp} \otimes Q_{i} = \mathbb{F}^{m-1} \otimes \mathbb{F}^{b}$$

$$\iff \sum_{j=1}^{n} U_{B_{j}} \otimes V_{i} = \mathbb{F}^{m-1} \otimes \mathbb{F}^{n-b}$$

$$\iff \sum_{i=1}^{m} U_{i} \otimes V_{A_{i}} = \mathbb{F}^{m-1} \otimes \mathbb{F}^{n-b}$$

$$(A_{i} = \{j : i \in B_{j}\})$$

$$\iff \bigcap_{i=1}^{m} V_{A_{i}} = 0.$$

Since V is MDS(m), it is enough to show that $\bigcap_{i=1}^{m} \widetilde{Q}_{S_i} = 0$ for a generic $\widetilde{Q}_{b \times n}$ implies $\bigcap_{i=1}^{m} \widetilde{V}_{A_i} = 0$ for a generic $\widetilde{V}_{(n-b)\times n}$. This follows by Lemma 3.5 by the same chain of equivalances as above and noting that the dual of a generic code is also generic, i.e., the parity check matrix corresponding to a generic generator matrix is also generic.

A special case of a cycle family is when the sets S_1, \ldots, S_ℓ are all disjoint.

Definition 3.10. Say that an $k \times n$ matrix V is (n, k)-weak-MDS (ℓ) if for any ℓ disjoint subsets $S_1, \ldots, S_\ell \subset [n]$ such that $|S_i| \leq k$ for all i and $|S_1| + \cdots + |S_\ell| \leq (\ell-1)k$, we have that $V_{S_1} \cap \cdots \cap V_{S_\ell} = 0$.

Thus, Lemma 3.7 implies that the dual of any (n, k)-MDS(m) code is a (n, n-k)-weak-MDS(m) code. This observation is useful in proving our field size lower bounds.

3.3.2 MDS(3) duality

It turns out that the cycle MDS(m) condition is equivalent to the cycle-MDS(m) condition when m = 3.

Proposition 3.11. Let V be a (n, k)-cycle-MDS(3) code. Then, V is an (n, k)-cycle-MDS(3) code.

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Proof. By Lemma 3.1, to prove that V is (n,k) - MDS(3) it suffices to check that for all sets A_1, A_2, A_3 with $|A_i| \leq k, |A_1| + |A_2| + |A_3| = 2k$, and $A_1 \cap A_2 \cap A_3 = \emptyset$, we have that $V_{A_1} \cap V_{A_2} \cap V_{A_3} = 0$ if and only if it should happen generically. Since every subset of $\{1, 2, 3\}$ is either an emptyset or an "interval" modulo 3, we have that A_1, A_2, A_3 is a cycle family. Thus, the required condition holds because V is (n, k)-cycle-MDS(3).

Corollary 3.12. The dual of an (n, k)-MDS(3) code is a (n, n - k)-MDS(3) code.

As mentioned previously, the dual of an MDS(4) code is not necessarily an MDS(4) code (but it must be an MDS(3) code).

4 Lower bounds on field size for $MDS(\ell)$

Lemma 4.1. Assume $n \gg k \ge 3$. Let V be an (n,k)-weak-MDS(k) code over field \mathbb{F}_q . Then $q \ge \Omega_k(n^{k-1})$.

Proof. Let $s = \lfloor n/k \rfloor$. Let for all $i \in [k]$, let $I_i = \{(i-1)s + 1, \ldots, is\}$. Let k' = k - 1. Let S_i be all subsets of I_i of size k'.

Since V is (n, k)-weak-MDS(k) we have that

for all
$$A_1 \in \mathcal{S}_1, \dots, A_k \in \mathcal{S}_k$$
, $\dim(V_{A_1} \cap \dots \cap V_{A_k}) = 0.$ (6)

We seek to show that if $q \ll_k n^{k-1}$, then the above condition is violated. Consider the following random process. Sample $x \in \mathbb{F}^k$ uniformly at random, and for all $i \in [k]$, let X_i be the number of $A_i \in S_i$ such that $x \in V_{A_i}$. If we can show that with nonzero probability all the X_i 's are simultaneously nonzero and x is nonzero, then we know that (6) is violated.

Observe that for all $i \in [k]$ by linearity of expectation

$$\mathbb{E}[X_i] = \sum_{A_i \in \mathcal{S}_i} \Pr_{x \sim \mathbb{F}^k} [x \in V_{A_i}] = \binom{s}{k'} \frac{1}{q^{k-k'}} = \binom{s}{k-1} \frac{1}{q}.$$
(7)

Note that $\mathbb{E}[X_i] \gg 1$ if $q \ll_k n^{k-1}$. To conclude that $\Pr[X_i > 0] \approx 1$, we will show that second moment $\mathbb{E}[X_i^2] \approx \mathbb{E}[X_i]^2$ and use $\Pr[X_i > 0] \ge \mathbb{E}[X_i]^2 / \mathbb{E}[X_i^2]$. We will use the fact that for $A_i, A'_i \in S_i$,

$$\dim(V_{A_i} \cap V_{A'_i}) = \dim(V_{A_i}) + \dim(V_{A'_i}) - \dim(V_{A_i \cup A_i})$$
$$= 2k' - \min\{k, 2k' - |A_i \cap A'_i|\}$$
$$= \max\{2k' - k, |A_i \cap A'_i|\}$$
$$= \max\{k - 2, |A_i \cap A'_i|\}$$

$$\begin{split} \mathbb{E}[X_i^2] &= \sum_{A_i, A_i' \in \mathcal{S}_i} \Pr_{x \sim \mathbb{F}^k} [x \in V_{A_i} \cap V_{A_i'}] \\ &= \sum_{j=0}^{k'} \sum_{\substack{A_i, A_i' \in \mathcal{S}_i \\ |A_i \cap A_i'| = j}} \frac{q^{\max(j,k-2)}}{q^k} \\ &= \sum_{j=0}^{k'} \binom{s}{k'} \binom{k'}{j} \binom{s-k'}{k'-j} \frac{q^{\max(j,k-2)}}{q^k} \\ &= \mathbb{E}[X_i]^2 \sum_{j=0}^{k'} \frac{\binom{s}{k'} \binom{s'}{j} \binom{s-k'}{k'-j}}{\binom{s}{k'}^2} \cdot \frac{q^{\max(j,k-2)} \cdot q^2}{q^k} \\ &= \mathbb{E}[X_i]^2 \sum_{j=0}^{k'} \frac{\binom{k'}{j} \binom{s'}{k'-j}}{\binom{s}{k'}} \cdot q^{\max(j-(k-2),0)} \\ &\leqslant \mathbb{E}[X_i]^2 \left(1 + O_k(1) \sum_{j=1}^{k-1} \frac{q^{\max(j-(k-2),0)}}{n^j}\right) \\ &= \mathbb{E}[X_i]^2 \left(1 + O_k(1) \left(\frac{1}{n} + \frac{1}{n^2} + \dots + \frac{1}{n^{k-2}} + \frac{q}{n^{k-1}}\right)\right) \\ &\leqslant \mathbb{E}[X_i]^2 \left(1 + \frac{1}{k}\right). \end{split}$$
 (If $q \ll_k n^{k-1}$ and $n \gg_k 1$)

Thus, for each $i \in [k]$,

$$\Pr[X_i > 0] \ge \frac{\mathbb{E}[X_i]^2}{\mathbb{E}[X_i^2]} \ge \frac{1}{1+1/k} \ge \frac{k}{k+1}.$$

Therefore, by the union bound, all the X_i 's are at least 1 simultaneously and the sampled $x \in \mathbb{F}^k$ is nonzero with probability at least $1 - k \cdot \frac{1}{k+1} - \frac{1}{q^k} > 0$, a contradiction.

Combining Lemma 4.1, Proposition 3.2, and Lemma 3.7 allows us to prove Theorem 1.5 and Corollary 1.6.

Corollary 4.2 (Theorem 1.5). Let C be an (n,k)-MDS (ℓ) code over \mathbb{F}_q . Then

$$q \ge \Omega_\ell \left(n^{\min\{k, n-k, \ell\} - 1} \right).$$

Proof. Case 1: k > n/2. Let $b = \min(n - k, \ell)$. By Proposition 3.2, we can puncture the code C at n - k - b locations to get a code C_0 which is (k + b, k)-MDS(b) code. By Lemma 3.7, C_0^{\perp} is (k, b)-cycle-MDS(b). By Lemma 4.1, we have that the field size of C_0^{\perp} (and thus C) is

$$q \ge \Omega_b(k^{b-1}) = \Omega_\ell\left(n^{\min\{\ell, n-k\}-1}\right).$$

Case 2: $k \leq n/2$. Let $b = \min(k, \ell)$. By Proposition 3.2, we can shorten the code C at k - b locations to get a code C_0 which is (n - k + b, b)-MDS(b) code (also trivially cycle-MDS(b)). By Lemma 4.1, we have that the field size of C_0^{\perp} (and thus C) is

$$q \ge \Omega_b(n^{b-1}) = \Omega_\ell\left(n^{\min\{\ell,k\}-1}\right).$$

Corollary 4.3 (Corollary 1.6). Let $C = C_{\text{col}} \otimes C_{\text{row}}$ be an (m, n, a, b)-MR tensor code. The minimum field size of C is at least $\Omega_m(n^{\min\{b-1,n-b-1,m-a\}})$.

Proof. Like in Proposition 3.2, let C'_{row} be an (m-a+1,m-a)-MDS code formed by puncturing C_{row} at any a-1 positions. Observe that $C_{\text{col}} \otimes C'_{\text{row}}$ must be an (m-a+1,n,1,b)-tensor code code. This is because the generator matrix of $C_{\text{col}} \otimes C'_{row}$ is $U \otimes V_{[n]\setminus A}$ where A is the set of punctured positions which is a part of $U \otimes V$, the generator matrix for $C_{\text{col}} \otimes C'_{\text{row}}$. Moreover, there is a canonical injective map from correctable erasure patterns of $C_{\text{col}} \otimes C'_{\text{row}}$ and those of $C_{\text{col}} \otimes C_{\text{row}}$. By Corollary 3.4, C_{col} is an (n, n-b)-MDS(m-a+1) code. Thus, by Theorem 1.5 the field size of C_{col} is at least $\Omega_m(n^{\min\{b,n-b,m-a+1\}-1})$, as desired.

5 Efficient regularity testing for MR Tensor Codes when a = 1

In this section, we will assume that a = 1. Therefore WLOG, we can assume that C_{col} is a parity check code. In this case, we have a neat characterization of the generically correctable patterns in terms of *regularity*.

5.1 Characterizing correctable patterns: Regularity

Definition 5.1 (Regular pattern). An erasure pattern $E \subset [m] \times [n]$ is regular for an (m, n, a, b)-Tensor Code if for every $S \subset [m]$ of size at least a and $T \subset [n]$ of size at least b, we have

$$|E \cap (S \times T)| \leqslant sb + ta - ab \tag{8}$$

where s = |S| and t = |T|.

Remark 5.2. We can rewrite the regularity condition (8) as:

$$\left|\overline{E} \cap (S \times T)\right| \ge (s-a)(t-b) \tag{9}$$

We will first prove that correctability implies regularity for any value of a, b. We will need the following key lemma.

Lemma 5.3. If U_1, U_2 are subspaces of U and V_1, V_2 are subspaces of V, then

 $\dim(U_1 \otimes V_1 + U_2 \otimes V_2) = \dim(U_1) \cdot \dim(V_1) + \dim(U_2) \cdot \dim(V_2) - \dim(U_1 \cap U_2) \cdot \dim(V_1 \cap V_2).$

Proof. It is enough to show that $(U_1 \otimes V_1) \cap (U_2 \otimes V_2) = (U_1 \cap U_2) \otimes (V_1 \cap V_2)$. The rest follows from the fact that for any two spaces $A, B, \dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B)$ and

 $\dim(A \otimes B) = \dim(A) \cdot \dim(B)$. We will show that the duals of both sides are equal.

$$((U_1 \otimes V_1) \cap (U_2 \otimes V_2))^{\perp} = (U_1 \otimes V_1)^{\perp} + (U_2 \otimes V_2)^{\perp}$$
$$= (U_1^{\perp} \otimes V + U \otimes V_1^{\perp}) + (U_2^{\perp} \otimes V + U \otimes V_2^{\perp})$$
$$= (U_1^{\perp} \otimes V + U_2^{\perp} \otimes V) + (U \otimes V_1^{\perp} + U \otimes V_2^{\perp})$$
$$= (U_1^{\perp} + U_2^{\perp}) \otimes V + U \otimes (V_1^{\perp} + V_2^{\perp})$$
$$= (U_1 \cap U_2)^{\perp} \otimes V + U \otimes (V_1 \cap V_2)^{\perp}$$
$$= ((U_1 \cap U_2) \otimes (V_1 \cap V_2))^{\perp}.$$

Theorem 5.4 (Correctability \Rightarrow Regularity [GHK⁺17]). If an erasure pattern E is correctable by an (m, n, a, b)-tensor code, then it is regular.

Proof. By Proposition 2.1, E is correctable iff $\dim((U \otimes V)_{\overline{E}}) = (m-a)(n-b)$. Let $S \subset [m]$ be of size at least a and $T \subset [n]$ be of size at least b. We can upper bound $\dim((U \otimes V)_{\overline{E}})$ as:

$$\dim((U \otimes V)_{\overline{E}}) \leq \dim((U \otimes V)_{\overline{E} \cap (S \times T)}) + \dim((U \otimes V)_{\overline{E} \setminus (S \times T)})$$
$$\leq |\overline{E} \cap (S \times T)| + \dim((U \otimes V)_{([m] \times [n]) \setminus (S \times T)}).$$

We now use Lemma 5.3 to calculate $\dim((U \otimes V)_{([m] \times [n]) \setminus (S \times T)})$.

$$\dim((U \otimes V)_{([m] \times [n]) \setminus (S \times T)}) = \dim((U \otimes V)_{\overline{S} \times [n]} + (U \otimes V)_{[m] \times \overline{T}})$$

$$= \dim(U_{\overline{S}} \otimes V_{[n]} + U_{[m]} \otimes V_{\overline{T}})$$

$$= \dim(U_{\overline{S}}) \cdot \dim(V_{[n]}) + \dim(U_{[m]}) \cdot \dim(V_{\overline{T}}) - \dim(U_{\overline{S}}) \cdot \dim(V_{\overline{T}})$$

$$= (m - s)(n - b) + (m - a)(n - t) - (m - s)(n - t).$$

Combining the above, we get $|\overline{E} \cap (S \times T)| \ge (s-a)(t-b)$.

The following theorem shows that every regular pattern is correctable if a = 1 or b = 1.

Theorem 5.5 ([GHK⁺17]). A erasure pattern $E \subset [m] \times [n]$ is generically correctable for an (m, n, a = 1, b)-tensor code iff E is regular.

The proof is more involved and proved later by relating regularity to a notion called excesscompatibility.

Remark 5.6. [GHK⁺17] conjecture that regularity is equivalent to correctability for arbitrary a, b. This conjecture was recently disproved by [HPYWZ21]. For their counterexample, they consider m = n = 5, a = b = 2 and let the complement of the erasure pattern be

$$\overline{E} = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$$

E is a regular pattern which is not correctable. Our analysis gives a short proof of noncorrectability. By Proposition 2.1, *E* is correctable iff $\dim((U \otimes V)_{\overline{E}}) = (m-a)(n-b) = 9$. By Lemma 5.3, $U_{\{2,3\}} \otimes V_{\{2,3\}}$ and $U_{\{4,5\}} \otimes V_{\{4,5\}}$ will have to intersect non-trivially. This shows that there is a linear dependency among $(U \otimes V)_{\overline{E}}$.

5.2 Efficiently Checking Regularity

Let $E \subset [m] \times [n]$ be any pattern for a (m, n, a, b) tensor code. For all $i \in [m]$, let $\deg_E(i)$ be the number of $j \in [n]$ such that $(i, j) \in E$. Likewise, for all $j \in [n]$, define $\deg_E(j)$ to be the number of $i \in [m]$ such that $(i, j) \in E$. Let $f : [n] \to \mathbb{Z}_{\geq 0}$ be supply constraints and $g : [m] \to \mathbb{Z}_{\geq 0}$ be demand constraints. We define a $(g, f)_E$ -quasi-matching (c.f., Definition 2 of [BBJ12]) to be a subset of the edges $E' \subset E$ such that all supply/demand constraints are met:

- 1. For all $i \in [m]$, $\deg_{E'}(i) \ge g(i)$.
- 2. For all $j \in [n]$, $\deg_{E'}(j) \leq f(j)$.

We shall use the following Hall-like condition for testing if a quasi-matching exists:

Lemma 5.7 (Theorem 20 of [BBJ12]). There does not exist a $(g, f)_E$ -quasi-matching if and only if there exists a Hall-blocker, that is a $U \subseteq [m]$ with

$$\sum_{i \in U} g(i) > \sum_{j \in [n]} \min(f(j), \deg_{E \cap (U \times [n])}(j))$$

Definition 5.8. Fix $V \subset [n]$ and let $E_V := E \cap ([m] \times V)$. For all $i \in [m]$ define the excess $e(i) = \max(\deg_E(i) - b, 0)$. For all $j \in [n]$ define a(j) := a. Define a V-excess flow $E' \subseteq E$ to be an $(e, a)_{E_V}$ -quasi-matching.

We say that a E is excess-compatible if for all $V \subset [n]$ of size $\overline{b} = n - b$, there exists a V-excess flow $E'_V \subset E_V$.

Theorem 5.9. E is regular if and only if E is excess-compatible.

Proof. We proceed by showing both directions.

Regularity implies excess-compatibility. First, assume E is regular but not excess-compatible. Thus, there exists $V \subset [n]$ of size \overline{b} such that $E_V := E \cap ([m] \times V)$ lacks a V-excess flow. By Lemma 5.7, there exists a Hall-blocker $U_1 \subset [m]$ and neighborhood $V_1 := N(U_1) \cap V$ for which demand exceeds supply. That is, if we let $E_1 = E \cap (U_1 \times V_1)$ then

$$\sum_{i \in U_1} e(i) > \sum_{j \in V_1} \min(a, \deg_{E_1}(j)).$$
(10)

We may assume without loss of generality that $e(i) \ge 1$ for all $i \in U_1$ -deleting any exceptions would keep the LHS the same and perhaps decrease the RHS. In particular, $e(i) + b = \deg_E(i)$ for all $i \in U_1$.

Let $V_2 \subset V_1$ be the vertices $j \in V_1$ for which $\deg_{E_1}(j) < a$. Let $V_3 = V_1 \cup ([n] \setminus V)$. Thus, since

E is regular,

$$= \sum_{i \in U_1} \deg_E(i) - \sum_{j \in V_2} \deg_{E_1}(j)$$
(extra vertices on right side do not change right-degrees)

$$= \sum_{i \in U_1} (b + e(i)) - \sum_{j \in V_2} \deg_{E_1}(j)$$

> $|U_1|b + \sum_{j \in V_1} \min(a, \deg_{E_1}(j)) - \sum_{j \in V_2} \deg_{E_1}(j)$ (by (10))
= $|U_1|b + |V_1 \setminus V_2|a$ (by definition of V_2)

Therefore, $a(|V_3 \setminus V_1| - b) > 0$, which contradicts that n - |V| = b since $V_3 \setminus V_1 = [n] \setminus V$.

Excess-compatibility implies regularity. Second, assume *E* is excess-compatible. We show by induction on the size of $U \subseteq [m]$ that for all $S \subseteq [n]$, with $|S| \ge b$, we have that.

$$|E \cap (U \times S)| \leq |U|b + |S|a - ab.$$

The base case of $U = \emptyset$ is trivial as the LHS equals 0 for all S. For nontrivial U we may assume for all $i \in U$ that $\deg(i) \ge b + 1$. Otherwise, let $U' \subset U$ be the set of all i with $\deg(i) \ge b + 1$ and note that for all $S \subseteq [n]$ of size at least b, we have by the induction hypothesis that

$$|E \cap (U \times S) \leq |E \cap (U' \times S)| + b|U \setminus U'|$$

$$\leq |U'|b + |S|a - ab + b|U \setminus U'|$$

$$\leq |U|b + |S|a - ab,$$

as desired.

Thus, we may now assume that $\deg(i) \ge b + 1$ for all $i \in U$. To prove regularity, it suffices to consider the worst-case choice of S. That is, consider

$$S' := \{ j \in [n] : \deg_{E \cap U \times [n]}(j) \ge a \}.$$

If $|S'| \ge b$, we set S = S'. Otherwise, the worst-case S is the set of b vertices of largest degrees; however since the regularity inequalities are trivial when |S| = b, we do not need to consider this case any further. That is, we may assume that $|S| \ge b$.

To apply the excess compatibility condition, let $V \subset [n]$ be the n-b vertices with *lowest* degree with respect to $E \cap (U \times [n])$. Since $|S| \ge b$, we have that $|S \setminus V| = b$. By the contrapositive of Lemma 5.7, we have that

$$\sum_{i \in U} e(i) \leqslant \sum_{j \in V} \min(a, \deg_{E \cap (U \times V)}(j))$$
$$= \sum_{j \in V \setminus S} \deg_{E \cap (U \times V)}(j) + a|S \cap V|$$
$$= |E \cap (U \times (V \setminus S))| + a|S| - ab.$$

Also note that

$$\sum_{i \in U} e(i) = \sum_{i \in U} (\deg(i) - b)$$
$$= |E \cap (U \times [n])| - b|U|$$
$$= |E \cap (U \times S)| + |E \cap (U \times (V \setminus S))| - b|U|.$$

Combining the two equation blocks, we have that $|E \cap (U \times S)| \leq a|S| + b|U| - ab$, as desired.

By Theorem 5.5, regularity is equivalent to correctability of erasure patterns when a = 1. Since Theorem 5.9 shows that regularity is equivalent to excess-compatibility, we have the equivalence between all three notions.

Corollary 5.10. If a = 1, an erasure pattern E is excess-compatible iff E is generically correctable.

We will now prove that excess-compatibility of an erasure pattern can be reduced to a max flow problem which can be solved in polynomial time. This implies that correctability of an erasure pattern by an (m, n, a = 1, b)-MR tensor code can be checked in polynomial time. This proves Theorem 1.8.

Proposition 5.11. Excess-compatibility of an erasure pattern is testable in time

 $O(\min(\binom{m}{a}, \binom{n}{b}) \cdot (m+n)^3).$

Proof. We will prove that excess compatibility can be checked in the given time using a max flow algorithm.

Let $E \subset [m] \times [n]$ be the pattern we wish to test excess-compatibility. Fix a subset $V \subset [n]$ of size $\overline{b} = n - b$. Construct a directed graph on $m + \overline{b} + 2$ nodes: a source s, a sink t, c_i for $i \in [m]$ and d_j for $j \in V$. For each $i \in [m]$, have a (directed) edge from s to c_i with capacity e(i). For all $(i, j) \in E \cap ([m] \times V)$, have an edge from c_i to d_j with capacity 1. For all $j \in V$, have an edge from d_j to t with capacity a. Then, by definition of V-excess flow, a V-excess flow $E' \subset E \cap ([m] \times V)$ exists if and only if the maximum flow in our constructed directed graph saturates every edge from s to c_i . This maximum flow can be computed in $O(m + n)^3$ time. Since we need to perform this check for all $V \subset [n]$ of size \overline{b} , the total running time is $O(\binom{n}{b}(m + n)^3)$. Since we could have also performed this test by swapping m and n (and a and b), we can also test in $O(\binom{m}{a}(m + n)^3)$ time, as desired.

References

- [BBJ12] Drago Bokal, Boštjan Brešar, and Janja Jerebic. A generalization of hungarian method and hall's theorem with applications in wireless sensor networks. *Discrete Applied Mathematics*, 160(4-5):460–470, 2012.
- [CHL07] Minghua Chen, Cheng Huang, and Jin Li. On maximally recoverable property for multi-protection group codes. In *IEEE International Symposium on Information The*ory (ISIT), pages 486–490, 2007.
- [DL78] Richard A. DeMillo and Richard J. Lipton. A probabilistic remark on algebraic program testing. *Inf. Process. Lett.*, 7(4):193–195, 1978.

- [GGY20] Sivakanth Gopi, Venkatesan Guruswami, and Sergey Yekhanin. Maximally recoverable lrcs: A field size lower bound and constructions for few heavy parities. *IEEE Transactions on Information Theory*, 2020.
- [GHK⁺17] Parikshit Gopalan, Guangda Hu, Swastik Kopparty, Shubhangi Saraf, Carol Wang, and Sergey Yekhanin. Maximally recoverable codes for grid-like topologies. In 28th Annual Symposium on Discrete Algorithms (SODA), pages 2092–2108, 2017.
- [GHSY12] Parikshit Gopalan, Cheng Huang, Huseyin Simitci, and Sergey Yekhanin. On the locality of codeword symbols. *IEEE Transactions on Information Theory*, 58(11):6925 -6934, 2012.
- [HPYWZ21] Lukas Holzbaur, Sven Puchinger, Eitan Yaakobi, and Antonia Wachter-Zeh. Correctable erasure patterns in product topologies. *arXiv preprint arXiv:2101.10028*, 2021.
- [HSX⁺12] Cheng Huang, Huseyin Simitci, Yikang Xu, Aaron Ogus, Brad Calder, Parikshit Gopalan, Jin Li, and Sergey Yekhanin. Erasure coding in Windows Azure Storage. In USENIX Annual Technical Conference (ATC), pages 15–26, 2012.
- [KLR19] Daniel Kane, Shachar Lovett, and Sankeerth Rao. The independence number of the birkhoff polytope graph, and applications to maximally recoverable codes. *SIAM Journal on Computing*, 48(4):1425–1435, 2019.
- [KMG21] Xiangliang Kong, Jingxue Ma, and Gennian Ge. New bounds on the field size for maximally recoverable codes instantiating grid-like topologies. *Journal of Algebraic Combinatorics*, pages 1–29, 2021.
- [MLR⁺14] Subramanian Muralidhar, Wyatt Lloyd, Sabyasachi Roy, Cory Hill, Ernest Lin, Weiwen Liu, Satadru Pan, Shiva Shankar, Viswanath Sivakumar, Linpeng Tang, and Sanjeev Kumar. f4: Facebook's warm BLOB storage system. In 11th USENIX Symposium on Operating Systems Design and Implementation (OSDI), pages 383–398, 2014.
- [SAP⁺13] Maheswaran Sathiamoorthy, Megasthenis Asteris, Dimitris S. Papailiopoulos, Alexandros G. Dimakis, Ramkumar Vadali, Scott Chen, and Dhruba Borthakur. XORing elephants: novel erasure codes for big data. In *Proceedings of VLDB Endowment* (*PVLDB*), pages 325–336, 2013.
- [Sch80] Jacob T Schwartz. Fast probabilistic algorithms for verification of polynomial identities. Journal of the ACM (JACM), 27(4):701–717, 1980.
- [SRLS18] D. Shivakrishna, V. Arvind Rameshwar, V. Lalitha, and Birenjith Sasidharan. On maximally recoverable codes for product topologies. In 2018 Twenty Fourth National Conference on Communications (NCC), pages 1–6, 2018.
- [Zip79] Richard Zippel. Probabilistic algorithms for sparse polynomials. In *International symposium on symbolic and algebraic manipulation*, pages 216–226. Springer, 1979.

A MR Tensor Code: Upper bound on field size

Theorem A.1. There exists a (m, n, a, b)-MR Tensor Code with field size $O_{a,b,m}(n^{b(m-a)})$.

Proof. Let $C_{row} \otimes C_{col}$ be the code we seek to contrust. We shall exhibit a system of equations, the sum of whose degrees is $O_{a,b,m}(n^{b(m-a)})$. By the Schwartz-Zippel Lemma [DL78, Zip79, Sch80], this will imply the existence of a code over a field size of $O_{a,b,m}(n^{b(m-a)})$.

Let H_{row} be the parity check matrix of C_{row} of size $b \times n$, H_{col} be the parity check matrix of C_{col} of size $a \times m$. And let

$$H = \begin{bmatrix} I_m \otimes H_{row} \\ H_{col} \otimes D_{(n-b) \times n} \end{bmatrix}$$

be the parity check matrix of $C_{row} \otimes C_{col}$. Here I_m is the $m \times m$ identity matrix and D is the $(n-b) \times n$ matrix formed by the first n-b rows of I_n . Note that the number of columns of H is mn and the number of rows of H is mb + (n-b)a = mb + na - ab which is the codimension of the tensor code.

Call an erasure pattern $E \subset [m] \times [n]$ minimal if each nonempty row of E has size least b + 1and each nonempty column of E has size at least a + 1.

We impose the following three constraints on H,

- 1. H_{row} is an MDS code.
- 2. H_{col} is an MDS code.
- 3. For every $E \subset [m] \times [n]$ which is minimal and correctable, we impose that the minor $H|_E$ has rank |E|.

First, we show these conditions are sufficient to ensure that $C_{row} \otimes C_{col}$ is an MR Tensor Code. Let $E \subseteq [m] \times [n]$ be any correctable pattern. Since H_{row} is MDS, if any row of E has at most b entries, we can correct that row just by using H_{row} . Likewise, since H_{col} is MDS, if and row of E has at most a entries, we can correct that row just by using H_{col} . By removing any such rows and columns iteratively, it suffices to correct some $E' \subseteq E$ which is minimal and correctable (as E is correctable). The correctability of E' follows from condition 3.

To ensure the first condition, we sample the entries of H_{row} randomly. To check MDS, we need to ensure that each $a \times a$ minor of the [m, a] code has nonzero determinant. This is a system of equations of total degree $a\binom{m}{a} = O_{a,m}(1)$.

Likewise, for the second condition, we can ensure H_{col} is MDS with a system of equations of total degree $b\binom{n}{b} = O_b(n^b)$.

The last condition is a bit more tricky to analyze. First, we show that there are at most $O_{m,a,b}(n^{b(m-a)})$ minimal patterns, and each has $O_{m,a,b}(1)$ entries.

Assume that a minimal correctable pattern E has u nonempty rows and v nonempty columns. Because E is correctable, it is regular and so $|E| \leq ub + va - ab$. Also, since E is minimal, we have that $|E| \geq v(a + 1)$. Therefore,

$$ub + va - ab \ge |E| \ge v(a+1) \implies v \le b(u-a) \le b(m-a).$$

Thus, E spans at most b(m-a) columns and has at most $ub + va - ab \leq mb + b(m-a)a - ab = O_{a,b,m}(1)$ entries. Thus, the number of such E is at most

$$\binom{n}{\leqslant b(m-a)} \cdot \binom{m \cdot b(m-a)}{\leqslant mb + b(m-a)a - ab} = O_{a,b,m}(n^{b(m-a)})$$

Ensuring that $H|_E$ has full rank is equivalent to some $|E| \times |E|$ minor of $H|_E$ having nonzero determinant. If E is correctable, than one of these minors has at least one symbolically nonzero determinant. The constraint for this determinant has degree $|E| = O_{a,b,m}(1)$. Thus, we can specify all the necessary constraints with total degree $O_{a,b,m}(n^{b(m-a)})$.

B Omitted proofs

B.1 Proof of Lemma 3.1

To prove this lemma, we start with some foundational claims.

Claim B.1. For any $V \in \mathbb{F}^{k \times n}$ and for all $A_1, \ldots, A_\ell \subset [n]$,

$$\dim(V_{A_1} \cap \dots \cap V_{A_\ell}) = \sum_{i=1}^{\ell} \dim(V_{A_i}) - \operatorname{rank} \begin{bmatrix} V_{A_1} & V_{A_2} & & \\ V_{A_1} & V_{A_3} & & \\ \vdots & & \ddots & \\ V_{A_1} & & & V_{A_\ell} \end{bmatrix}.$$

Proof. We may assume that $\dim(V_{A_i}) = |A_i|$ for all *i*. Otherwise, some of the columns of V_{A_i} are linear combinations of other columns. Thus, we can remove elements of A_i corresponding to the redundant columns without changing the rank on either side of the main expression.

In particular, the RHS is equal to the dimension of the kernel of

$$X := \begin{bmatrix} V_{A_1} & V_{A_2} & & & \\ V_{A_1} & & V_{A_3} & & \\ \vdots & & \ddots & & \\ V_{A_1} & & & & V_{A_\ell} \end{bmatrix}.$$

It suffices to exhibit a linear bijection between ker X and $V_{A_1} \cap \cdots \cap V_{A_\ell}$. For any $x \in \ker W$, let x^i be the entries of x corresponding to A_i . In particular, we must have that

$$-V_{A_1}x^1 = V_{A_2}x^2 = V_{A_3}x^3 = \cdots V_{A_\ell}x^\ell.$$

Thus, $y := V_{A_2}x^2$ is in $V_{A_1} \cap \cdots \vee V_{A_\ell}$. This map has an inverse. For any $y \in V_{A_1} \cap \cdots \vee V_{A_\ell}$, there exists unique x^1, \ldots, x^ℓ (because dim $(V_{A_i}) = |A_i|$) such that $y = V_{A_1}x^1 = \cdots \vee V_{A_\ell}x^\ell$. In that case $x = (-x^1, x^2, \ldots, x^\ell)$ is in ker X. This establishes the bijection.

Claim B.2. Let $V \in \mathbb{F}^{k \times n}$ be MDS and $W \in \mathbb{R}^{k \times n}$ be generic. Then, for all $\ell \ge 2$ and $A_1, \ldots, A_\ell \subset [n]$,

$$\dim(V_{A_1} \cap \cdots \cap V_{A_\ell}) \ge \dim(W_{A_1} \cap \cdots \cap W_{A_\ell}).$$

Proof. Since V is MDS, $\dim(V_{A_i}) = \dim(W_{A_i})$ for all $i \in [\ell]$. Thus,

$$\dim(V_{A_1} \cap \dots \cap V_{A_{\ell}}) - \dim(W_{A_1} \cap \dots \cap W_{A_{\ell}})$$

$$= \operatorname{rank} \begin{bmatrix} W_{A_1} & W_{A_2} & & \\ W_{A_1} & W_{A_3} & & \\ \vdots & \ddots & & \\ W_{A_1} & & W_{A_{\ell}} \end{bmatrix} - \operatorname{rank} \begin{bmatrix} V_{A_1} & V_{A_2} & & & \\ V_{A_1} & V_{A_3} & & \\ \vdots & \ddots & & \\ V_{A_1} & & & V_{A_{\ell}} \end{bmatrix},$$

which is nonnegative because generic matrices maximize rank.

Claim B.3. Let $V \in \mathbb{F}^{k \times n}$ and $W \leq \mathbb{F}^k$ be a subspace. For any $A \subseteq B \subseteq [n]$,

$$0 \leq \dim(W \cap V_B) - \dim(W \cap V_A) \leq |B \setminus A|.$$

Proof. The left inequality is trivial. For the right, observe

$$\dim(W \cap V_B) - \dim(W \cap V_A)$$

$$= \dim(W) + \dim(V_B) - \dim(W + V_B) - \dim(W) - \dim(V_A) + \dim(W + V_A)$$

$$= (\dim(V_B) - \dim(V_A)) + (\dim(W + V_A) - \dim(W + V_B))$$

$$\leq (\dim(V_A + V_{B\setminus A}) - \dim(V_A)) + (0)$$

$$= \dim(V_{B\setminus A}) - \dim(V_A \cap V_{B\setminus A})$$

$$\leq |B \setminus A|.$$

Lemma B.4 (Padding Lemma). Let $V \in \mathbb{F}^{k \times n}$ be any MDS matrix. Let $\ell \ge 2$. Consider $A_1, \ldots, A_\ell \subset [n]$ of size at most k. Then the following statements are true:

- 1. If $\sum_{i=1}^{\ell} |A_i| > (\ell 1)k$, then $V_{A_1} \cap \cdots \cap V_{A_{\ell}} \neq 0$.
- 2. If $\sum_{i=1}^{\ell} |A_i| \leq (\ell-1)k$ and n is sufficiently large, then $V_{A_1} \cap \cdots \cap V_{A_{\ell}} = 0$ iff there exist $A'_1 \supseteq A_1, \ldots, A'_{\ell} \supseteq A_{\ell}$ such that
 - (a) $A'_1 \setminus A_1, A'_2 \setminus A_2, \ldots, A'_{\ell} \setminus A_{\ell}$ and $A_1 \cup A_2 \cup \cdots \cup A_{\ell}$ are mutually disjoint,
 - (b) $|A'_i| \leq k$ for all $i \in [\ell]$.
 - (c) $|A'_1| + \dots + |A'_{\ell}| = (\ell 1)k$,

(d)
$$V_{A'_1} \cap \cdots \cap V_{A'_e} = 0.$$

Proof. 1. By Claim B.1, we know there exists a matrix M of $(\ell - 1)k$ rows and $\sum_{i=1}^{\ell} |A_i|$ columns such that

$$\dim(V_{A_1} \cap \dots \cap V_{A_\ell}) = \sum_{i=1}^{\ell} \dim(V_{A_i}) - \operatorname{rank}(M)$$
$$\geqslant \sum_{i=1}^{\ell} |A_i| - (\ell - 1)k$$
$$> 0,$$

Thus, $V_{A_1} \cap \cdots \cap V_{A_\ell} \neq 0$.

2. The 'if' direction follows from $V_{A_1} \cap \cdots \cap V_{A_\ell} \subseteq V_{A'_1} \cap \cdots \cap V_{A'_\ell} = 0$. For the 'only if' direction, assume $n \ge (\ell - 1)k$. By Claim B.1,

$$0 = \dim(V_{A_1} \cap \dots \cap V_{A_\ell}) = \sum_{i=1}^{\ell} \dim(V_{A_i}) - \operatorname{rank} \begin{bmatrix} V_{A_1} & V_{A_2} & & \\ V_{A_1} & V_{A_3} & & \\ \vdots & & \ddots & \\ V_{A_1} & & & V_{A_\ell} \end{bmatrix}.$$

Let $U(A_1, \ldots, A_\ell) \subset \mathbb{F}^{(\ell-1)k}$ be the column space of the block matrix in the above expression. We let let $U_i \subset U$ be the subspace of U which is supported on the *i*th block of k coordinates of $F^{(\ell-1)k}$. Note that $U_1 \oplus U_2 \oplus \cdots \oplus U_{\ell-1} \subseteq U$. Therefore, $U = \mathbb{F}^{(\ell-1)k}$ if and only if dim $U_i = k$ for all $i \in [\ell-1]$.

We now 'grow' A_1, \ldots, A_ℓ into the desired A'_1, \ldots, A'_ℓ through the following inductive process. Let $t = (\ell - 1)k - \sum_{i=1}^{\ell} |A_i|$. Let $A_1^{(0)}, \ldots, A_\ell^{(0)}$ be A_1, \ldots, A_ℓ .

- For $i \in \{0, \dots, t-1\}$.
- Since $\sum_{a=1}^{\ell} |A_a| < (\ell 1)k$, we have $U(A_1, \ldots, A_\ell) \neq \mathbb{F}^{(\ell-1)k}$. Thus, we can identify $j \in [\ell 1]$ such that $\dim(U_j) < k$.
- Add an element in $[n] \setminus \bigcup_{a=1}^{\ell} A_a^{(i)}$ to $A_{i+1}^{(i)}$. Call this new family $A_1^{(i+1)}, \ldots, A_{\ell}^{(i+1)}$.
- Repeat these steps.

We let $A'_i = A_i^{(t)}$ for all $i \in [\ell]$. Clearly A'_1, \ldots, A'_ℓ are supersets of A_1, \ldots, A_ℓ . We claim they also satisfy properties (a)-(d). Property (a) is satisfied because a new element of [n] is added at each step of the algorithm. Property (b) is satisfied because if no elements are added to A_1 and for all $j \ge 2$ an element is added to $A_j^{(i)}$ only if $\dim(U_j) < k$, which cannot happen if $|A_j^{(i)}| = k$; therefore no set will exceed k in size at any point. Property (c) is satisfied because the algorithm runs for $t = (\ell - 1)k - \sum_{i=1}^{\ell} |A_i|$ steps.

For property d, we claim by induction for all $i \in \{0, 1, ..., t\}$, $V_{A_1^{(i)}} \cap \cdots \cap V_{A_\ell^{(i)}} = 0$. The base case i = 0 follows by assumption. Note that at each stage, dim $U(A_1^{(i+1)}, \ldots, A_\ell^{(i+1)}) \ge \dim U(A_1^{(i)}, \ldots, A_\ell^{(i)}) + 1$ because one of the subspaces U_j increases in dimension by padding A_{j+1} . Thus,

$$\begin{split} \dim(V_{A_1^{(i+1)}} \cap \dots \cap V_{A_{\ell}^{(i+1)}}) &= \sum_{j=1}^{\ell} \dim(V_{A_j^{(i+1)}}) - \dim(U(A_1^{(i+1)}, \dots, A_{\ell}^{(i+1)})) \\ &\leqslant \left(1 + \sum_{j=1}^{\ell} \dim(V_{A_j^{(i)}})\right) - (\dim(U(A_1^{(i)}, \dots, A_{\ell}^{(i)})) + 1) \\ &= 0. \end{split}$$

Thus, $V_{A_1^{(i)}} \cap \cdots \cap V_{A_{\ell}^{(i)}} = 0$, completing the induction.

Now we prove Lemma 3.1.

Lemma 3.1. Let $V \in \mathbb{F}^{k \times n}$ be an (n, k)-MDS code and let $\ell \ge 2$. Let $W \in \mathbb{R}^{k \times n}$ be a generic real matrix. Then V is $MDS(\ell)$ if and only if for all $A_1, \ldots, A_\ell \subseteq [n]$ such that $|A_i| \le k$, $|A_1| + \cdots + |A_\ell| = (\ell - 1)k$ and $A_1 \cap \cdots \cap A_k = \emptyset$, we have that

$$V_{A_1} \cap V_{A_2} \cap \dots \cap V_{A_\ell} = 0 \iff W_{A_1} \cap \dots \cap W_{A_\ell} = 0,$$

Proof. Observe that "only if" direction follows immediately. Now we seek to show that "if" direction. We do this by showing the contrapositive.

Fix A_1, \ldots, A_ℓ such that $\dim(V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_\ell}) \neq \dim(W_{A_1} \cap \cdots \cap W_{A_\ell})$, Because $\dim(W_{A_1} \cap \cdots \cap W_{A_\ell})$ is the rank of a generic matrix, we must have that

$$\dim(V_{A_1} \cap V_{A_2} \cap \cdots \cap V_{A_\ell}) > \dim(W_{A_1} \cap \cdots \cap W_{A_\ell}) =: d,$$

Note that $|A_i| \ge d$ for all $i \in [\ell]$. We claim there exists subsets $A'_1 \subset A_1, A'_2 \subseteq A_2, \ldots, A'_\ell \subseteq A_\ell$ with $|A'_1| + |A'_2| + \cdots + |A'_\ell| = d$ such that

$$W_{A_1 \setminus A_1'} \cap W_{A_2 \setminus A_2'} \cap \dots \cap W_{A_\ell \setminus A_\ell'} = 0.$$

This follows from Claim B.1, as the block matrix has rank d less than the number of columns, so d columns can be removed without changing the rank, which decreases the dimension of the intersection by d. Observe that we must have

$$\dim(V_{A_1\setminus A_1'}\cap V_{A_2\setminus A_2'}\cap\cdots\cap V_{A_\ell\setminus A_\ell'})>0$$

because the dimension can decrease by at most d.

Let $B_i = A_i \setminus A'_i$ for all $i \in [\ell]$. If any $|B_i| > k$, then $V_{B_i} = \mathbb{F}^k$ and $W_{B_i} = \mathbb{R}^k$. We can replace both with an arbitrary subset of size k without changing anything. Thus, $\sum_i |B_i| \leq \ell k$. In fact since

$$0 = \dim(W_{B_1} \cap \dots \cap W_{B_\ell}) \leq k - \sum_i (k - |B_i|).$$

we have that $\sum_i |B_i| \leq (\ell - 1)k$. Now, if the inequality is strict, we can add an element to one of the B_i s without changing that the generic intersection of the Ws is nonempty (this is by looking at the matrix view and noting that some row must be less than full rank). this can only increase the intersection of the Vs. This finishes the argument that $\neg 2 \Rightarrow \neg 3$.

B.2 Proof of Lemma 3.5

Lemma 3.5. Let $C = C_{col} \otimes C_{row}$ be an (m, n, a, b) tensor code and let $\bar{a} = m - a, \bar{b} = n - b$. Let E be a maximal erasure pattern of size $|E| = mn - \bar{a}\bar{b}$. and let $\bar{E} = \bigcup_{i \in [m]} \{i\} \times A_i = \bigcup_{j \in [n]} B_j \times \{j\}$. If U, V are generator matrices of C_{row} and C_{col} and P, Q are their respective parity check matrices. Then correctability of E is equivalent to each of the following conditions:

$$\sum_{i=1}^{m} U_i \otimes V_{A_i} = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{\bar{b}}$$
⁽²⁾

$$\sum_{j=1}^{n} U_{B_j} \otimes V_j = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{\bar{b}}$$
(3)

$$\sum_{i=1}^{m} P_i \otimes V_{A_i}^{\perp} = \mathbb{F}^a \otimes \mathbb{F}^{\bar{b}}$$

$$\tag{4}$$

$$\sum_{j=1}^{n} U_{B_j}^{\perp} \otimes Q_j = \mathbb{F}^{\bar{a}} \otimes \mathbb{F}^{b}.$$
(5)

Proof. Note that by Proposition 2.1, correctability is equivalent to (2). Observe that (2) \iff (3) because the addition of vector spaces is commutative.

We now show that (2) \iff (4). By symmetric argument, we may show that (3) \iff (5). The conditions on E translate to the following conditions on A_1, A_2, \ldots, A_m .

1.
$$|A_i| \le n - b$$

2. $\sum_i |A_i| = (m - a)(n - b).$

The following statements are equivalent.

- 1. E is not correctable.
- 2. There exists a non-zero codeword of C which is supported on E.
- 3. There exist $r_1, r_2, \ldots, r_m \in C_{\text{row}}$, not all zero, such that
 - $\operatorname{supp}(r_i) \subset \overline{A}_i \text{ for } i \in [m],$

$$P \cdot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = \sum_{i=1}^m P_i \otimes r_i^T = 0.$$

(Since $r_i = y_i^T V$ for some $y_i \in \mathbb{F}^{\bar{b}}$ and $I \otimes V^T$ is an injective linear map, we have the following equivalent statement.)

- 4. There exist $y_1, y_2, \ldots, y_m \in \mathbb{F}^{\bar{b}}$, not all zero, such that
 - $y_i^T V_{A_i} = 0$ for $i \in [m]$,
 - $\sum_{i=1}^{m} P_i \otimes y_i = 0.$

(Since $y_i \in V_{A_i}^{\perp}$ for each $i \in [m]$, we have the following equivalent statement.)

5. There exists $y_i \in V_{A_i}^{\perp}$, not all zero, such that $\sum_{i=1}^{m} P_i \otimes y_i = 0$. (Since $|A_i| \leq n-b$, and V is a generator matrix of an MDS code, we have $\dim(V_{A_i}^{\perp}) = (n-b) - \dim(V_{A_i}) = (n-b) - |A_i|$. Therefore

$$\sum_{i=1}^{m} \dim(V_{A_i}^{\perp}) = \sum_{i=1}^{m} (n-b-|A_i|) = m(n-b) - (m-a)(n-b) = a(n-b) = a\bar{b}.$$

Since P_i is one-dimensional, we also have $\sum_{i=1}^{m} \dim(P_i \otimes V_{A_i}^{\perp}) = a\bar{b}$. So we have the following equivalent statement.)

6. $P_1 \otimes V_{A_1}^{\perp} + P_2 \otimes V_{A_2}^{\perp} + \dots + P_m \otimes V_{A_m}^{\perp} \neq \mathbb{F}^a \otimes \mathbb{F}^{\bar{b}}.$

This completes the proof.

B.3 Counterexample to $MDS(\ell)$ duality for $\ell \ge 4$

Consider the following matrices over \mathbb{F}_{13} .

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$
$$V^{\perp} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 6 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 7 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 8 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 9 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 10 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 11 & 1 \end{pmatrix}$$

One can check that V is MDS(4) (this follows from V being MDS) but V^{\perp} is not MDS. For example, let $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{1, 2, 3, 6, 7\}$, $A_3 = \{1, 2, 4, 6, 8\}$, and $A_4 = \{5, 7, 8\}$. If V^{\perp} were a generic matrix, then $V_{A_1}^{\perp} \cap V_{A_2}^{\perp} \cap V_{A_3}^{\perp} \cap V_{A_4}^{\perp} = 0$, but one can verify that $(5, 4, 3, 2, 8, 0) \in V_{A_1}^{\perp} \cap V_{A_2}^{\perp} \cap V_{A_3}^{\perp} \cap V_{A_4}^{\perp}$.

A computer search can also find a number of other counterexamples, such as MDS(8, 4, 4) codes whose duals are not MDS(4). Another way to see the failure of duality of MDS(4) is by the lower bound of [KMG21] where they proved that (m = 4, n, a = 1, b = 2)-MR tensor codes require fields of size $\Omega(n^2)$. Constructing (m = 4, n, a = 1, b = 2)-MR tensor codes is equivalent to constructing (n, n-2)-MDS(4) code by Theorem 1.4. If MDS(4) duality is true, this is equivalent to constructing (n, 2) - MDS(4) codes. It is easy to see any (n, 2)-MDS code is also an MDS(4) code. Since there are MDS codes over linear size fields (Reed-Solomon codes), this would violate the lower bound of [KMG21].

B.4 Proof of Proposition 3.2

Proposition 3.2. Let C be an (n, k)-MDS (ℓ) code. Then

- 1. C is also an $MDS(\ell 1)$ code.
- 2. The code C_0 obtained by puncturing C at any position is an (n-1,k)-MDS (ℓ) code.
- 3. The code C_1 obtained by shortening C at any position is an (n-1, k-1)-MDS (ℓ) code.

Proof. (1) Follows trivially by taking A_{ℓ} in (1) to be the entire set.

(2) Let $V_{k \times n}$ be a generator matrix for C. Then dropping the i^{th} column of V, we get the generator matrix for C_1 , the puncturing of C at i. Therefore the condition (1) still holds.

(3) WLOG, let's assume that the code is shortened at position n. Let $V_{k\times n} = [V_1 V_2 \cdots V_n]$ be the generator matrix of C. By a basis change, we can assume that $V_n = e_k$, the k^{th} coordinate vector in \mathbb{F}^k . Let $\widetilde{V}_i \in \mathbb{F}^k$ be the vector formed by dropping the last coordinate of V_i . It is easy to see that $\widetilde{V} = [\widetilde{V}_1 \widetilde{V}_2 \cdots \widetilde{V}_{n-1}]$ is the generator matrix for the shortened code C_1 . We now want to prove that C_1 is $\text{MDS}(\ell)$. Let $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_\ell \subset [n-1]$ with $\sum_{i=1}^{\ell} |\widetilde{A}_i| = (k-1)(\ell-1)$ such that $\bigcap_{i=1}^{\ell} \widetilde{W}_{\widetilde{A}_i} = 0$ for a generic $(k-1) \times n$ matrix \widetilde{W} . By Lemma 3.1, it is enough to show that $\bigcap_{i=1}^{\ell} \widetilde{V}_{\widetilde{A}_i} = 0$. Define $A_1 = \widetilde{A}_1$ and $A_i = \widetilde{A}_i \cup \{n\}$ for $i \ge 2$.

Claim B.5. For $W_{k \times n}$ and $W_{(k-1) \times n}$ are generic matrices, then

$$\bigcap_{i=1}^{\ell} \widetilde{W}_{\widetilde{A}_i} = 0 \iff \bigcap_{i=1}^{\ell} W_{A_i} = 0.$$

Proof. WLOG, by a basis change we can assume $W_n = e_k$, the k^{th} coordinate vector. Let \widetilde{W}_i to be the vector formed by dropping the last coordinate of W_i , clearly \widetilde{W} is also generic. Now $\bigcap_{i=1}^{\ell} W_{A_i} = 0$ iff

$$X = \begin{bmatrix} W_{A_1} & W_{A_2} & & & \\ W_{A_1} & & W_{A_3} & & \\ \vdots & & \ddots & & \\ W_{A_1} & & & & W_{A_\ell} \end{bmatrix}$$

is full rank. Doing column operations, we can conclude that X is full rank iff

$$\widetilde{X} = \begin{bmatrix} \widetilde{W}_{\widetilde{A}_1} & W_{\widetilde{A}_2} & & & \\ W_{\widetilde{A}_1} & & W_{\widetilde{A}_3} & & \\ \vdots & & \ddots & \\ W_{\widetilde{A}_1} & & & & W_{\widetilde{A}_\ell} \end{bmatrix}$$

is full rank. This is equivalent to $\bigcap_{i=1}^{\ell} \widetilde{W}_{\widetilde{A}_i} = 0.$

Since V is the generator for an $MDS(\ell)$ code, we have $\bigcap_{i=1}^{\ell} W_{A_i} = 0 \Rightarrow \bigcap_{i=1}^{\ell} V_{A_i} = 0$. So therefore we will be by proving that $\bigcap_{i=1}^{\ell} V_{A_i} = 0 \Rightarrow \bigcap_{i=1}^{\ell} \widetilde{V}_{\widetilde{A}_i} = 0$, the proof of which is essentially same as that of Claim B.5.

C Near constructions of (n, 3)-MDS(3) codes

In this appendix, we exhibit some partial progress toward constructing (n, 3) - MDS(3) codes. By Theorem 1.5, we know such a construction needs $\Omega(n^2)$ field size. We construct a couple of different codes with field size $O(n^2)$ which which have some of the properties of MDS(3) codes.

The constructions are inspired by the fact that Reed-Solomon codes produce the (nearly) optimal field size for MDS codes.

C.1 A weak bipartite MDS(3) construction

Let p be a prime, and $q = p^2$. Assume that $\mathbb{F}_q = \mathbb{F}_p[X]/\langle p(X) \rangle$, where p(X) is a degree-2 irreducible in $\mathbb{F}_p[X]$.

Lemma C.1. There are explicit $u_0, u_1, \ldots, u_{p-1} \in \mathbb{F}_q^3$ and $v_0, v_1, \ldots, v_{p-1} \in \mathbb{F}_q^3$ with the following property. Let $W_{\alpha,\beta} = \operatorname{span}(u_{\alpha}v_{\beta})$. Then $W_{\alpha_1,\beta_1} \cap W_{\alpha_2,\beta_2} \cap W_{\alpha_3,\beta_3} \neq 0$ if they generically should.

We call it "weak bipartite" as the spaces we are considering the intersectin of form a bipartite graph.

Remark C.2. Note that the space $W_{1,1}$ has $q^2 - 1$ nonzero points, and the q - 1 nonzero points of $W_{i,j} \cap W_{1,1}$ must be unique for all $i, j \in [2, n]$. Therefore, any bipartite MDS(3) construction must have $O(\sqrt{q})$ points. Therefore, this construction is essentially optimal.

Remark C.3. Warning! In the construction the u_i 's and v_i 's are not MDS. In fact, they are collinear. Even so, this seems to be one of the few known algebraic constructions which gives a generic 3-wise intersection condition.

Proof. For all $\alpha, \beta \in \mathbb{F}_p$, let $u_{\alpha} = (\alpha, -1, 0)$ and $v_{\beta} = (\beta + \beta^2 X, 0, -1)$.

Now assume for arbitrary $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{F}_p$ that $W_{\alpha_1,\beta_1} \cap W_{\alpha_2,\beta_2} \cap W_{\alpha_3,\beta_3} \neq 0$. Let $w_{\alpha,\beta} = u_{\alpha} \times v_{\beta} = (1, \alpha, \beta + \beta^2 X)$. Then, we must have that $w_{\alpha_1,\beta_1}, w_{\alpha_2,\beta_2}, w_{\alpha_3,\beta_3}$ are coplanar. In other words

$$\det \begin{bmatrix} 1 & 1 & 1\\ \alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 + \beta_1^2 X & \beta_2 + \beta_2^2 X & \beta_3 + \beta_3^2 X \end{bmatrix} = 0.$$

Subtracting the first column from the second and third columns and then expanding we have that

$$0 = \det \begin{bmatrix} \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 \\ (\beta_2 - \beta_1)(1 + (\beta_1 + \beta_2)X) & (\beta_3 - \beta_1)(1 + (\beta_1 + \beta_3)X) \end{bmatrix}.$$

Expanding, we get that

$$(\alpha_2 - \alpha_1)(\beta_3 - \beta_1)(1 + (\beta_1 + \beta_2)X) = (\alpha_3 - \alpha_1)(\beta_2 - \beta_1)(1 + (\beta_1 + \beta_3)X).$$

Comparing coefficients of powers of X, we get that $\eta := (\alpha_2 - \alpha_1)(\beta_3 - \beta_1) = (\alpha_3 - \alpha_1)(\beta_2 - \beta_1)$ and $\eta\beta_2 = \eta\beta_3$.

If $\eta = 0$, then either $\alpha_1 = \alpha_2 = \alpha_3$, $\beta_1 = \beta_2 = \beta_3$, or $(\alpha_1, \beta_1) = (\alpha_i, \beta_i)$ for i = 2 or 3. In each of these cases, the spaces $W_{\alpha_1,\beta_1}, W_{\alpha_2,\beta_2}, W_{\alpha_3,\beta_3}$ generically intersect.

If $\eta \neq 0$ then $\beta_2 = \beta_3$ which then either $\beta_1 = \beta_2 = \beta_3$ or $\alpha_2 = \alpha_3$. Again, in each of these cases, the spaces $W_{\alpha_1,\beta_1}, W_{\alpha_2,\beta_2}, W_{\alpha_3,\beta_3}$ generically intersect.

Thus, our construction is bipartite MDS(3).

C.2 A very weak MDS(3) Construction

Note that the lower bound for the field size of $MDS(\ell)$, only needed the following structure: that there is *one* parition of the vectors into ℓ groups, such that $k - k/\ell$ -dimensional subspaces, one drawn from each group, has trivial intersection. Let's call this property *very weak* $MDS(\ell)$. We now show that the lower bound is essentially tight if $k = \ell = 3$.

We use a coset trick of [GGY20]. Let p be a prime such that $p \equiv 1 \mod 3$. Thus, there exists $\zeta \in \mathbb{F}_p^*$ which is a nontrivial cube root of 1. Let $S \subset \mathbb{F}_p^*$ be a subgroup of size (p-1)/3 not containing ζ . Let $c \in \mathbb{F}_p$ be a non-quadratic residue, and consider the field extension $\mathbb{F}_q := \mathbb{F}_p[X]/\langle X^2 - c \rangle$, where $q = p^2$. Now define three sets of vectors

$$U = \{ (1, \alpha, \alpha^2) : \alpha \in S \}$$

$$V = \{ (1, \zeta\beta, \zeta^2\beta^2) : \beta \in S \}$$

$$W = \{ (1, X\gamma, X^2\gamma^2) : \gamma \in \{1, 2, \dots, (p-1)/2\} \}$$

These three sets are disjoint, and their union is a subset of the Reed-Solomon code over \mathbb{F}_q^3 , so $U \cup V \cup W$ is MDS(2).

Now we seek to show that for any $u_1, u_2 \in U$, $v_1, v_2 \in V$, $w_1, w_2 \in W$ that $\operatorname{span}(u_1, u_2) \cap \operatorname{span}(v_1, v_2) \cap \operatorname{span}(w_1, w_2) = 0$. Have the notation $u_i := (1, \alpha_i, \alpha_i^2)$, etc. We know that $\operatorname{span}(u_1, u_2)^{\perp} = \operatorname{span}((\alpha_1\alpha_2, -(\alpha_1 + \alpha_2), 1))$, etc. Thus, $\operatorname{span}(u_1, u_2) \cap \operatorname{span}(v_1, v_2) \cap \operatorname{span}(w_1, w_2) \neq 0$ if and only if

$$\det \begin{bmatrix} \alpha_1 \alpha_2 & \zeta^2 \beta_1 \beta_2 & c \gamma_1 \gamma_2 \\ \alpha_1 + \alpha_2 & \zeta (\beta_1 + \beta_2) & X(\gamma_1 + \gamma_2) \\ 1 & 1 & 1 \end{bmatrix} = 0.$$

In particular, this implies that the coefficient of X in the expansion of the determinant is equal to 0. That is,

$$(\gamma_1 + \gamma_2) \left[\alpha_1 \alpha_2 - \zeta^2 \beta_1 \beta_2 \right] = 0.$$

But, $\gamma_1 + \gamma_2 \in \{1, \ldots, p-1\}$ and $\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \in S \not\ni \zeta^2$. Thus, the above expression cannot be 0, contradiction. Therefore, for all $u_1, u_2 \in U, v_1, v_2 \in V, w_1, w_2 \in W$ that $\operatorname{span}(u_1, u_2) \cap \operatorname{span}(v_1, v_2) \cap \operatorname{span}(w_1, w_2) = 0$, as desired.

Observe that $|U|, |V|, |W| \ge (p-1)/3 = \Omega(\sqrt{q})$, which is tight up to constant factors by the proof of Lemma 4.1.