RATIONAL POINTS ON VARIETIES AND THE BRAUER-MANIN OBSTRUCTION

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Problem Sessions

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The problems are listed under the lecture that is most relevant, but you should feel free to attempt the problems in whichever problem session you like (although you may have an easier time if you try the problem after that lecture). All of the problems are likely too much to complete in the given problem sessions. You should work on the parts that interest you the most, and use parts freely in later problems.

Included with the problems are some background that did not make it into the lectures. Some of the statements are given for your information and I do not expect you to prove them. The parts of the problems that I think are reasonable in scope explicitly say "Prove", "Show", "Determine", etc.

All of the computational parts of the problems are feasible to do by hand (although some may be involved). However, you may prefer to do them using computer algebra software, particularly if you want to practice some of what you learned in Drew Sutherland's lectures. Computing by hand and computing using software are both valuable skills and you should choose whichever appeals to you the most.

1. Lecture 1: Introduction to the Brauer-Manin obstruction

Recall that for any field k, $\mathbb{P}^n(k) := \{[a_0 : \cdots : a_n] : a_i \in k, \text{at least one } a_i \neq 0\} / \sim$ where $[a_0 : \cdots : a_n] \sim [\lambda a_0 : \cdots : \lambda a_n]$ for any $\lambda \in k^\times$.

(1) Find a rational point on
$$X := V(x^3 + 2y^3 + 10z^3) \subset \mathbb{P}^2.$$

(2) Show that
$$X := V(x^2 + y^2 + 7z^2) \subset \mathbb{P}^2$$
has no $\mathbb{Q}_7$-points. Conclude that $X(\mathbb{Q}) = \emptyset$.

(3) *Note:* For this problem, the following specific consequence of Hensel's Lemma will be useful. If p be a prime and $u \in \mathbb{Z}_p^\times$, then
$$u \in \mathbb{Z}_p^{\times 2} \Leftrightarrow \begin{cases} u \bmod p \in \mathbb{F}_p^{\times 2} & \text{if } p \neq 2 \\ u \equiv 1 \pmod 8 & \text{if } p = 2 \end{cases}$$

(a) Let p be an odd prime and let $a, b, c \in \mathbb{Z} - p\mathbb{Z}$. Show that $\{ax^2 : x \in \mathbb{F}_p\}$ and $\{c - by^2 : y \in \mathbb{F}_p\}$ both have cardinality $\frac{p+1}{2}$ and therefore that the sets contain a common value. Use this to show that

$$X := V(ax^2 + by^2 + cz^2) \subset \mathbb{P}^2$$

has a $\mathbb{Q}_p$-point.

(b) Determine whether $X := V(5x^2 + 7y^2 - 3z^2) \subset \mathbb{P}^2$ has $\mathbb{A}_\mathbb{Q}$-points.

(c) Let $a, b, c \in \mathbb{Z}$ be squarefree, pairwise relatively prime integers. Prove that $X := V(ax^2 + by^2 + cz^2)$ has $\mathbb{A}_\mathbb{Q}$-points if and only if a, b, c are not all the same sign and $ax^2 + by^2 + cz^2$ has solutions in $\mathbb{Z}/8abc\mathbb{Z}$ such that for every $p | 8abc$, at least two of the coordinates are nonzero modulo p.

Note: More generally, for any geometrically integral variety X over k, $X(k_v) \neq \emptyset$ for all but finitely many places v, and one can algorithmically determine a finite set of places S such that $X(k_v) \neq \emptyset$ for all $v \notin S$. (For X as in part (c), the exercise implies that $S = \{2, \infty\} \cup \{p : p | abc\}$.) See [Poo17, Section 7.7.2] for more details.

(4) Let F be a field of characteristic different from 2 and let $a, b \in F^\times$. Complete one of the following two problems.

(a) Let $\mathcal{A}_{a,b}$ be the 4-dimensional F-algebra with basis $1, i, j, ij$ defined by the following multiplication rules

$$i^2 = a, \quad j^2 = b, \quad ji = -ij.$$

Prove that $\mathcal{A}_{a,b} \cong \mathrm{M}_2(F)$ if and only if there is some $x, y, z \in F$, not all zero, such that $ax^2 + by^2 = z^2$.

(b) Let $C_{a,b}$ be the conic $ax^2 + by^2 = z^2$. Prove that $C_{a,b} \cong \mathbb{P}^1_F$ if and only if $C_{a,b}(F) \neq \emptyset$.

Conclude that $(a, b) := [\mathcal{A}_{a,b}] = [C_{a,b}] \in \operatorname{Br} F$ is trivial if and only if $C_{a,b}(F) \neq \emptyset$, and that $C_{a,b}(F) \neq \emptyset$ if and only if $a \in \mathrm{N}(k(\sqrt{b})^\times)$. (By symmetry this is equivalent to $b \in \mathrm{N}(k(\sqrt{a})^\times)$)

Note: One can also show that $\mathcal{A}_{a,b} \oplus_k \mathcal{A}_{a,c} \cong \mathrm{M}_2(\mathcal{A}_{a,bc})$, (see [GS17, Lemma 1.5.2]) which implies that in $\operatorname{Br} F$, we have $(a, b)(a, c) = (a, bc)$.

(5) Let X be a smooth projective geometrically integral variety over a number field k and let π denote the structure morphism $\pi \colon X \to \operatorname{Spec} k$.

(a) Let $\alpha_0 \in \operatorname{Br} k$. Show that $X(\mathbb{A}_k)^{\pi^*\alpha_0} = X(\mathbb{A}_k)$.

(b) Let $\alpha, \beta \in \operatorname{Br} X$. Show that

$$X(\mathbb{A}_k)^\alpha \cap X(\mathbb{A}_k)^\beta = \bigcap_{\gamma \in \langle \alpha, \beta \rangle} X(\mathbb{A}_k)^\gamma$$

(6) Let $X \subset \mathbb{P}^4$ be given by the vanishing of the following two quadrics

$$st - x^2 + 5y^2, \quad (s+t)(s+2t) - x^2 + 5z^2.$$

This variety was first studied by Birch and Swinnerton-Dyer [BSD75].

(a) Note that an intersection of quadrics in $\mathbb{P}^3$ is a genus 1 curve, and any smooth genus 1 curve over a finite field $\mathbb{F}$ has an $\mathbb{F}$-point. Prove that $X \cap V(z)$ is smooth modulo p for all $p \neq 2, 5$. Use this to prove that $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 2, 5$.

(b) Building on the previous part, show that $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$.

(c) Using (4), show that $(5, \frac{s}{t})$ and $(5, \frac{s+t}{s+2t})$ are trivial in $\operatorname{Br} \mathbf{k}(X)$.

(d) Using the previous part and the remark at the end of (4) show that, in $\operatorname{Br} \mathbf{k}(X)$

$$\mathcal{A} := \left(5, \frac{s+t}{s}\right) = \left(5, \frac{s+2t}{s}\right) = \left(5, \frac{s+t}{t}\right) = \left(5, \frac{s+2t}{t}\right).$$

Additionally show that for every point $P \in X - V(s,t)$, there is an open set $P \in U_P \subset X - V(s,t)$ such that at least one of $\frac{s+t}{s}, \frac{s+t}{t}, \frac{s+2t}{s}, \frac{s+2t}{t}$ is regular and invertible on U.

Note: Since $V(s,t) \subset X$ is codimension 2 in X, this together with the purity theorem (see [Poo17, Thm. 6.8.3]) allows us to conclude that $\mathcal{A} \in \operatorname{Br} X$.

(e) Show that $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$ and hence $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}} = \emptyset$. (*Sketch*: First show that for all $p \neq 5$ and $P_p \in X(\mathbb{Q}_p)$, at least one of $\frac{s+t}{s}, \frac{s+t}{t}, \frac{s+2t}{s}, \frac{s+2t}{t}$ is a p-adic unit at P_p. Then, noting that $\mathbb{Q}_p(\sqrt{5})/\mathbb{Q}_p$ is unramified for $p \neq 5$, use Problem (4) to deduce that $\mathcal{A}(P_p) = 0 \in \operatorname{Br} \mathbb{Q}_p$. Lastly, show that $\mathcal{A}(P_5) \neq 0 \in \operatorname{Br} \mathbb{Q}_5$ for all $P_5 \in X(\mathbb{Q}_5)$.)

2. Lecture 2: Elements that capture a Brauer-Manin obstruction

Problems (2) and (3) require some familiarity with divisors in algebraic geometry. Feel free to read through the exercises and to try only the parts that you think you have the background for.

(1) Let X is a smooth projective geometrically integral variety over a number field k. Then $X(\mathbb{A}_k) = \prod_v X(k_v)$ and $X(\mathbb{A}_k)$ is a compact set (for the adelic topology, which in this case is the product topology), and, given any $\alpha \in \operatorname{Br} X$, $\operatorname{ev}_\alpha \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$ is locally constant. (See [Poo17, Sections 2.6 and 8.2.4] for more details).

(a) Show that $\operatorname{ev}_\alpha \colon X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is locally constant.

(b) Let $\alpha \in \operatorname{Br} X$. Show that $X(\mathbb{A}_k)^\alpha$ is open and closed in the adelic topology. Conclude that $X(\mathbb{A}_k)^{\operatorname{Br}}$ is closed.

(c) Assume that $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$. Show that there exists a finite set $B \subset \operatorname{Br} X$ such that $X(\mathbb{A}_k)^B = \emptyset$.

(2) Let $C = V(ax^2 + by^2 - c^2)$ be a smooth projective conic over a field F of characteristic 0 such that $C(F) = \emptyset$. The exact sequence of low degree terms from the Hochschild-Serre spectral sequence (c.f. [Poo17, Section 6.7]) begins

$$0 \to \operatorname{Pic} C \to (\operatorname{Pic} \overline{C})^{G_F} \to \operatorname{Br} F \to \operatorname{Br} C \to \mathrm{H}^1(G_F, \operatorname{Pic} \overline{C}) \tag{2.1}$$

(a) Show that $\operatorname{Pic} \overline{C} \cong \mathbb{Z}$ with the trivial Galois action, and that

$$\operatorname{Pic} C \subset (\operatorname{Pic} \overline{C})^{G_F} \simeq \mathbb{Z}$$

corresponds to the subgroup $2\mathbb{Z}$, i.e., $\operatorname{Pic} C$ is exactly the subgroup of $\operatorname{Pic} \overline{C}$ consisting of divisors of even degree.

(b) Use the previous exercise together with the exact sequence (2.1) to conclude that we have an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \operatorname{Br} F \to \operatorname{Br} C \to 0$$

(c) Recall from Problem (4) in Lecture 1 that C can be viewed as giving a class in $\operatorname{Br} F$. Use Problem (4) to show that $[C] = 0$ in $\operatorname{Br} \mathbf{k}(C)$ and $[C] \neq 0$ in $\operatorname{Br} F$. Conclude that C generates the kernel of the map $\operatorname{Br} F \to \operatorname{Br} C$, and thus

$$\frac{\operatorname{Br} F}{\langle [C] \rangle} \xrightarrow{\sim} \operatorname{Br} C.$$

(3) Let $p(t) \in k[t]$ be a separable degree 4 polynomial with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \overline{k}^\times$. Let $a \in k^\times$ be an element that is *not* a square in the splitting field of $p(t)$. Let $X \to \mathbb{P}^1$ be a relatively minimal smooth proper model of the affine surface $y^2 - az^2 = p(t)x^2 \in \mathbb{P}^2 \times \mathbb{A}^1$, where the map to $\mathbb{P}^1$ agrees with the projection onto the t-coordinate; X is a Châtelet surface. (One can check that under these assumptions, $\pi^{-1}(\infty) \cong V(y^2 - az^2 = cx^2$, where c is the leading coefficient of $p(t)$.)

Remark 2.1. The arithmetic of Châtelet surfaces is well-understood, due to landmark work of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [CTSSD87b, CTSSD87a]. They proved that for any Châtelet surface X over a number field k, $X(k)$ is dense (for the adelic topology) in the Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}}$. In addition, work of Coray and Colliot-Thélène implies that for any field F, $X(F) \neq \emptyset$ if and only if there is some extension L/F of odd degree such that $X(L) \neq \emptyset$ [CTC79, Thm. C].

In this problem you will compute $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})[2]$, under the assumption that $X(\mathbb{A}_k) \neq \emptyset$. For a Châtelet surface, $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})[2]$ is isomorphic to $\operatorname{Br} X / \operatorname{Br}_0 X$. With the approach I have outlined, it is enough to know that X is a smooth proper compactification of the surface $y^2 - az^2 = p(t)x^2 \in \mathbb{P}^2 \times \mathbb{A}^1$.

Note that restriction of divisors to the generic fiber of π, $\overline{X}_\eta$, gives a Galois-equivariant surjective map $\operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{X}_\eta$ whose kernel, which we denote N, is generated by fibral curves, i.e., curves on X that map to points under π. Additionally note that since $a \notin k^{\times 2}$, the conic X_η has no $\mathbf{k}(\eta)$-point. However, by Tsen's theorem, $\overline{X}_\eta$ does have a $\mathbf{k}(\eta_{\overline{k}})$-point.

(a) Taking cohomology of the short exact sequence $0 \to N \to \operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{X}_\eta \to 0$, prove that we have an exact sequence

$$(\operatorname{Pic} \overline{X})^{G_k} \to (\operatorname{Pic} \overline{X}_\eta)^{G_k} \to \mathrm{H}^1(G_k, N) \to \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}_\eta).$$

Since $X(\mathbb{A}_k) \neq \emptyset$, $(\operatorname{Pic} \overline{X})^{G_k} \simeq \operatorname{Pic} X$[Poo17, Ex. 6.10]. Using this and (2), conclude that we have a short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{H}^1(G_k, N) \to \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \to 0.$$

(b) Consider the fibral curves $F\colon V(t)$, and $F_i^\pm : V(y \pm \sqrt{a}z, t - \alpha_i)$, for $i = 1, \ldots, 4$. Show that

$$N \simeq \frac{\mathbb{Z}F \oplus \bigoplus_{i=1}^4 F_i^+ \oplus \bigoplus_{i=1^4} F_i^-}{\langle F_i^- - (F - F_i^+) : i = 1, \ldots, 4\rangle} \simeq \mathbb{Z}F \oplus \bigoplus_{i=1}^4 F_i^+$$

(c) By the assumption on a, observe that the absolute Galois group of k, denoted G_k, acts on the set of curves $\{F, F_i^\pm\}$ through a subgroup of $\operatorname{Gal}(k(\sqrt{a})/k) \times \operatorname{Aut}(\alpha_i)$. Determine how $\operatorname{Gal}(k(\sqrt{a})/k) \times \operatorname{Aut}(\alpha_i)$ acts on the set of curves $\{F, F_i^\pm\}$.

(d) Using the previous parts of the problem, compute that for every $f(t)|p(t)$ of *even* degree

$$\sum_{i, f(\alpha_i)=0} F_i^+ \in (N/2N)^{G_k}.$$

(e) Use the previous parts of the problem to compute that

$$\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X}) \simeq \begin{cases} \{0\} & \text{if } p(t) \text{ has an irreducible factor of degree at least 3} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p(t) \text{ has an irreducible quadratic factor} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } \alpha_i \in k^\times \text{ for all } i \end{cases}$$

Note: Using an argument similar to that in Problem 6(6d), one can show that if $f(x)$ is a quadratic factor of $p(x)$, then $(a, f(x)) = (a, p(x)/f(x))$ is an element of $\operatorname{Br} X$. Additionally, it is true that $(a, f(x))$ and $\sum_{i, f(\alpha_i)=0} F_i^+ \in (N/2N)^{G_k}$ have the same image in $\mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$.

3. Lecture 3: The Brauer-Manin obstruction over extensions

(1) Let X be a smooth projective variety over a number field k that is everywhere locally soluble, and let $\alpha \in \operatorname{Br} X$. For any extension F/k, let α_F denote the image of α under the map $\operatorname{Br} X \to \operatorname{Br} X_F$.

For any $(P_v) \in X(\mathbb{A}_k)$ and any finite extension L/k, exhibit a point $(Q_w) \in X(\mathbb{A}_L)$ such that $\varphi_{\alpha_L}((Q_w)) = [L : k]\varphi_\alpha((P_v))$. (Recall that φ_α is what we used to denote the Brauer-Manin pairing.) This gives a proof of [CV, Lemma 2.1(2)] mentioned in today's lecture.

(2) In this problem we will show that the Châtelet surface given by

$$y^2 - 5z^2 = \frac{3}{5}\left(5t^4 + 7t^2 + 1\right)$$

fails to have points exactly over the extensions

$$\{L/\mathbb{Q} \text{ finite} : \exists w|3 \text{ such that } [L_w : \mathbb{Q}_3] \equiv 1 \bmod 2\} \cup \{L/\mathbb{Q}(\sqrt{29}) \text{ odd degree}\}.$$

This example was constructed by Sam Roven; he proved that X obtains a Brauer-Manin obstruction to the Hasse principle over $\mathbb{Q}(\sqrt{29})$ [Rov, Thm. 1.2].

(a) Show that $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 3$. (*Hint*: Try looking at fibers of π and then applying Problem 3 from Lecture 1)

(b) Show that $X(\mathbb{Q}_3) = \emptyset$ and that $X(F) \neq \emptyset$ for all extensions $F/\mathbb{Q}_3$ of even degree. Conclude (using the previous part as well) that

$$\{L/\mathbb{Q} \text{ finite} : X(\mathbb{A}_L) = \emptyset\} = \{L/\mathbb{Q} \text{ finite} : \exists w|3 \text{ such that } [L_w : \mathbb{Q}_3] \equiv 1 \bmod 2\}.$$

(c) Prove that $5t^4 + 7t^2 + 1$ is irreducible over $\mathbb{Q}$. Then, using Problem 3 from Lecture 2, compute that $\operatorname{Br} X_L = \operatorname{Br}_0 X_L$ for all L that are linearly disjoint from the splitting field of $5t^4 + 7t^2 + 1$. Conclude, using Remark 2.1, that for such L, $X(\mathbb{A}_L) \neq \emptyset \Leftrightarrow X(L) \neq \emptyset$.

(d) One can compute that the splitting field of $5t^4 + 7t^2 + 1$, which we will denote K, is a D_4 extension. (*Bonus:* Verify this using a computer algebra software.) Use this to prove that for every subfield $F \subset K$, exactly one of the following things occurs:
- $5t^4 + 7t^2 + 1$ remains irreducible over F,
- $5t^4 + 7t^2 + 1$ obtains a root over F,
- $F = \mathbb{Q}(\sqrt{29})$ and $5t^4 + 7t^2 + 1$ factors as two irreducible quadratic polynomials.

(e) Show that if $5t^4 + 7t^2 + 1$ has a root over F, then $X(F) \neq \emptyset$. (*Hint*: Consider the fiber of π above the root.)

(f) By [Rov, Theorem 1.2], $X(\mathbb{A}_{\mathbb{Q}(\sqrt{29})}) \neq \emptyset$ and $X(\mathbb{A}_{\mathbb{Q}(\sqrt{29})})^{\mathrm{Br}} = \emptyset$. Using this and Remark 2.1, prove that if $L/\mathbb{Q}(\sqrt{29})$ is an extension of odd degree $X(L) = \emptyset$. Additionally, using Problem (1) from Lecture 3, prove that if $L/\mathbb{Q}(\sqrt{29})$ is an extension of even degree then $X(\mathbb{A}_L)^{\mathrm{Br}} \neq \emptyset$ and hence $X(L) \neq \emptyset$.

(g) Assemble all of the previous parts to prove the desired statement.

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