

Bézout's Caliper

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We discuss Bézout's identity, a fundamental theorem in number theory established by Étienne Bézout in 1779. We give a proof by a method that also serves as a basis for proving Fermat's little theorem and Euler's theorem. We then explore the mathematical underpinnings of Vernier calipers, elucidating their precision measurement capabilities and historical origins. We derive a formula relating the number of divisions on the main and Vernier scales of the caliper to the minimum length measurable by the Vernier, as determined by Bézout's identity, giving us a connection between these two concepts. As a conclusion, we ask ChatGPT if there is a connection between Bézout's identity and the Vernier caliper; the response prompts a reflection on the role of human imagination in scientific discovery and contrasts it with the limitations of current artificial intelligence systems.

I. PREAMBLE: SOME BASIC CONCEPTS IN NUMBER THEORY

Before plunging into our exposition, we review some basic number theory concepts and definitions that are used.

Every positive integer is either *prime*, that is not factorizable into integers, or *composite*, if it can be factored. A fundamental result is that every integer has a unique factorization into primes, for example $12 = 2 \times 2 \times 3$. Two positive integers j and k are said to be *relatively prime* if they have no prime factors in common. Thus their greatest common divisor is unity, abbreviated $\gcd\{k, j\} = 1$. For example, since $10 = 2 \times 5$ and $9 = 3 \times 3$, $\gcd\{10, 9\} = 1$.

Two integers m and n are said to be *congruent modulo j* if $m - n = Qj$, or equivalently, $m = n + Qj$, for some integer Q ; this is written as $m \equiv n \pmod{j}$. As an example, $22 \equiv 1 \pmod{7}$ because $22 = 1 + 3 \times 7$. If $mk \equiv 0 \pmod{j}$ and the integers k and j are relatively prime, then each prime factor of j must divide m , since it cannot divide k . But this means that m is divisible by j ,

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that is $m \equiv 0 \pmod{j}$. This will be applied with $m = n - l$ where n, l are both integers, with the conclusion that $nk \equiv lk \pmod{j}$, and k and j relatively prime, implies that $n \equiv l \pmod{j}$. This result is called the modular cancellation theorem.

Some further notation: We will use curly brackets to denote a set, for example $\{1, \dots, 3\}$ denotes the set with elements 1,2,3.

II. BÉZOUT'S IDENTITY

Best known for his work on the number of solutions of polynomial equations, Etienne Bézout (1730-1783) left the world of number theory with an essential lemma or subsidiary theorem, Bézout's identity. It was officially published in 1779 in his book, "Théorie Générale des Équations Algébrique" (General Theory of Algebraic Equations) [1], which explored systems of equations in many variables [2]. Buried in this book, Bézout's identity appears and relates to many basic results in number theory; the proof method we give below is also used in proving Fermat's little theorem and Euler's theorem. In our case, we will use a specific form of the identity: for some k and j that are relatively prime, there exists some m and n such that, for any positive integer d less than the larger of j and k , one has $mk - nj = d$, where m and n range over all positive integers [3].

Since k and j are relatively prime, they cannot be equal, so one must be larger than the other. Without loss of generality, we assume j is the larger one, and start from the following,

$$\gcd(k, j) = 1, j > k \tag{1}$$

We begin by showing that the set of integers $\{1, \dots, j - 1\}k$ is congruent, \pmod{j} , to the set of integers $\{1, \dots, j - 1\}$, that is, we want to show

$$\{1, \dots, j - 1\}k \pmod{j} \equiv \{1, \dots, j - 1\} \pmod{j} \tag{2}$$

The sets on the left and right of this equation contain the same number of elements as long as there are no duplications. There are clearly no duplications in the set on the right. Suppose there is a duplication in the set on the left, that is there are positive integers l and n such that

$$nk \equiv lk \pmod{j} \tag{3}$$

$$\tag{4}$$

This can be rewritten as

$$(n - l)k \equiv 0 \pmod{j} \quad (5)$$

Using the modular cancellation theorem, which applies since k and j are relatively prime (Eq. (1))

$$(n - l) \equiv 0 \pmod{j} \quad (6)$$

However, for n and l to be both positive and less than j , they must be equal. So the set $\{1, \dots, j - 1\}k$ contains no duplications since the multipliers of k in this set are all distinct.

Hence by the definition of congruence, we have now

$$\{1, \dots, j - 1\}k = \{m_1, \dots, m_{j-1}\}j + \{1, \dots, j - 1\} \quad (7)$$

However, all the m_i in this equation must be nonnegative. Suppose m_I were negative, $m_I = -|m_I|$ with $|m_I| > 0$, then transposing we have

$$Ik + |m_I|j = I, \quad (8)$$

but this is impossible because the left hand side is bigger than j , and the right hand side is smaller than j .

So we conclude,

$$\{1, \dots, j - 1\}k - \{m_1, \dots, m_{j-1}\}j = \{1, \dots, j - 1\} \quad (9)$$

for m positive, or in the Bézout form, the set of integers $nk - mj$ for n and m positive and $\gcd(k, j) = 1$, always contains the set of integers $\{1, \dots, j - 1\}$.

A. Exercises and Examples

Here, we give some concrete examples of Eq. (2), some of which you may prove on your own.

Example 1:

Given that $j = 10$, $k = 9$

$$\begin{aligned}
 1 \times 9 &\equiv 9 \pmod{10} \\
 2 \times 9 &= 18 \equiv 8 \pmod{10} \\
 3 \times 9 &= 27 \equiv 7 \pmod{10} \\
 &\dots \\
 9 \times 9 &= 81 \equiv 1 \pmod{10}.
 \end{aligned}
 \tag{10}$$

That is, in set notation,

$$\{1, 2, 3, \dots, 9\}9 \equiv \pmod{10} \{9, 8, 7, \dots, 1\} = \{1, 2, 3, \dots, 9\} \quad .
 \tag{11}$$

,

Exercise: Prove that when $k = j - 1$, a reverse ordering is always produced

We will encounter the case $k = j - 1$ below, where it is used in the simplest realization of the Vernier caliper.

But this is not true in general, for example when k and j are not separated by 1, we get a scrambling in the ordering.

Example 2:

Given that $j = 7$, $k = 5$

$$\begin{aligned}
 1 \times 5 &= 5 \equiv 5 \pmod{7} \\
 2 \times 5 &= 10 \equiv 3 \pmod{7} \\
 3 \times 5 &= 15 \equiv 1 \pmod{7} \\
 4 \times 5 &= 20 \equiv 6 \pmod{7} \\
 5 \times 5 &= 25 \equiv 4 \pmod{7} \\
 6 \times 5 &= 30 \equiv 2 \pmod{7}
 \end{aligned}
 \tag{12}$$

So on the right we get 1,...,6, but in scrambled order: 5,3,1,6,4,2.

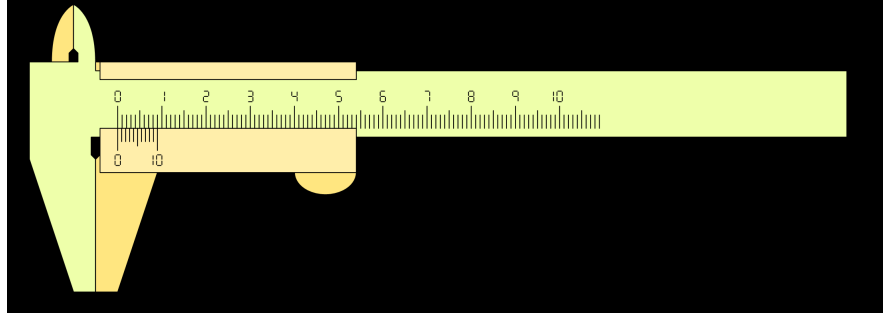


FIG. 1: Drawing of a Vernier caliper [4]. The bottom jaws are for measuring outer diameter, for which we give formulas below.

Exercise: Make up your example and work out the scrambling order.

Exercise: Search for Euler's theorem, and prove it using the set congruence method used above to prove Bézout's identity.

III. THE CALIPER SYSTEM

The Vernier caliper was first created in 1631 by Pierre Vernier, a French Mathematician known for inventing various measurement devices. The goal of the Vernier caliper was to gain more accurate measurements using two separate graduated scales, the fixed main scale and the movable Vernier scale, by exploiting the fact that the human eye excels at aligning fine lines with high precision but lacks accuracy when interpolating within an open interval. The main scale measures the desired unit's integer and first decimal value (e.g. 1.1), and the Vernier scale measures values in the second decimal place. (e.g. 0.01). To be specific we assume that the Vernier scale consists of ten divisions, each of which are $0.9 \text{ mm} = .09 \text{ cm}$ [5], whereas the main scale (for a 10 cm caliper capacity) is split into 100 divisions of $1 \text{ mm} = .1 \text{ cm}$. The discrepancy in division size lets the scale align at certain interval values, yielding a measurement to the hundredth decimal place. The difference between the main scale division and the Vernier scale division, in this example, $(.1 - .09) \text{ cm} = .01 \text{ cm}$ is called the Vernier constant, and it determines the caliper's precision. The smaller the Vernier constant is, the higher the accuracy of the caliper. In this example, using the caliper to measure an object $.01 \text{ cm}$ wide, since $.01 + .09 = .1$, the first tick on the Vernier scale would align with the first tick on the main scale. To measure an object $.03 \text{ cm}$ wide, since $.03 + 3 \times .09 = .03 + .27 = .3$, the third tick on the Vernier scale would align with the third tick on the main scale, and so forth.

IV. THE BÉZOUT VERNIER

Let k be the number of divisions per centimeter on the main scale of the caliper, and let j be the number of divisions per centimeter on the Vernier scale of the caliper, where $j > k$ and both

j and k are relatively prime. Let L be length in centimeters of the Vernier scale, up to the zero point when the jaws are closed, and let D be the distance between the inner sides of the jaws, one of which moves the Vernier scale.

Length of each division on the main scale: L/k

Length of each division on the Vernier scale: L/j

There are R divisions of size L/k on the main scale and S divisions of size L/j on the Vernier scale at the place where the markings on the main and Vernier scales exactly align.

The total length on the main scale where markings align, $R(L/k)$, is equal to the distance between the caliper's jaws plus the length on the Vernier scale, $D + S(L/j)$. So as an equation, we get

$$R(L/k) = D + S(L/j) \quad (13)$$

Transposing the length of the Vernier scale to the other side we may write it in terms of the distance between the jaws of the caliper,

$$D = R(L/k) - S(L/j) = \frac{RL}{k} - \frac{SL}{j} \quad (14)$$

Combining both terms on the right side, we get

$$D = \frac{RLj - SLk}{kj} = L\left(\frac{Rj - Sk}{kj}\right) . \quad (15)$$

The minimum value of D reduces the numerator, $Rj - Sk$, to 1, which is always attainable by Bézout's identity,

$$D_{min} = L\left(\frac{1}{kj}\right) = \frac{L}{k} \frac{1}{j} \quad (16)$$

This gives the standard formula for the "Vernier constant", which is the minimum length (or increment of length) measurable by use of the Vernier. An alternative formula is also given in an

online engineering text [5] as the difference in widths of the main scale and Vernier scale divisions, which is

$$\frac{L}{k} - \frac{L}{j} = L \frac{j-k}{jk} \quad , \quad (17)$$

which agrees with Eq. (16) when $j - k = 1$. This is the customary choice in caliper systems, and what is illustrated in Fig. 1 where $k = 9$ and $j = 10$. A specific example of both formulas for the Vernier constant is given next.

A. Convenient Versus Inconvenient Verniers

To give a specific example, we use millimeters as our unit of measurement. Thus, the main scale will be divided into ten one-millimeter intervals, while the Vernier scale will be divided into ten intervals of 0.9 millimeters.

Vernier: 0.9 mm width

Main: 1 mm width

According to online literature, one can calculate the Vernier constant in two equivalent ways [5]:

$$[1] \quad 1 \text{ mm} - 0.9 \text{ mm} = 0.1 \text{ mm} = \text{Vernier Constant}$$

$$[2] \quad \frac{\text{Main Scale Width}}{\text{Number of Vernier Scale Intervals}} = \frac{1 \text{ mm}}{10} = 0.1 \text{ mm} = \text{Vernier Constant}$$

As noted in our general discussion, these formulas agree because $j - k = 1$.

Bézout's identity gives more general calipers, but the alignment is not convenient because of the scrambled order of tick labels, as seen already in Example 2 above, where we got a scrambled order when $k \neq j - 1$. To create easily read alignments, we must use the case where $k = j - 1$, as it produces a simple caliper, where the ticks align at consecutive interval values.

Example:

Lets give an example of this phenomena, working in integers, as we did in previous sections.

Given that $j = 10$, $k = 9$,

$$1 \times 10 - 1 \times 9 = 1$$

$$2 \times 10 - 2 \times 9 = 2$$

$$3 \times 10 - 3 \times 9 = 3$$

.....

$$9 \times 10 - 9 \times 9 = 9$$

(18)

The correspondence between the numbers 1..9 on left and right leads to an easily readable Vernier: alignment of ticks at an integer n out of the set $\{1, 2, \dots, 9\}$ means the extra fraction of diameter is $.n$ mm.

Let's try it now with k and j differing by 2:

Given that $j = 7$, $k = 5$,

$$3 \times 7 - 4 \times 5 = 1$$

$$1 \times 7 - 1 \times 5 = 2$$

$$4 \times 7 - 5 \times 5 = 3$$

.....

$$7 \times 7 - 8 \times 5 = 9$$

(19)

In order to maintain an increasing increment of 1, the tick values on the left had to be scrambled. This would correspond to a harder to read Vernier.

In conclusion, we find that the difference between the number of divisions per unit value of the Vernier scale and main scale should be one, as it produces a serially ordered tick alignment, allowing the caliper to be conveniently read. That is why this specific configuration is deliberately adopted in practical engineering calipers. Further examples of more general Vernier systems, from a number theoretic point of view, have been given by Evans [6], with specific application of the Chinese Remainder Theorem for simultaneous congruences rather than the Bézout identity.

V. CHATGPT'S OPINION

Clearly, there is a connection between Bézout's identity and the theory of the Vernier caliper, but as far as we can tell it is not discussed anywhere on the internet. Moreover, the association with Bézout's identity is not immediately apparent, so when we asked ChatGPT "What is the connection between Bézout's identity and the Vernier caliper?", we got this response:

"Bézout's identity and the Vernier caliper are unrelated concepts from different mathematical and practical domains.

Bézout's Identity:

Bézout's identity is a theorem in number theory named after the French mathematician Étienne Bézout. It states that for any two integers a and b , there exist integers x and y such that $ax + by = \gcd(a, b)$, where $\gcd(a, b)$ is the greatest common divisor of a and b . In other words, Bézout's identity provides a way to express the greatest common divisor of two integers as a linear combination of those integers.

Vernier Caliper:

The Vernier caliper is a precision measuring instrument used in various fields, such as physics and engineering, to measure the dimensions of objects with high accuracy. It consists of a main scale and a sliding Vernier scale. The Vernier scale allows for measurements with a higher degree of precision than what is possible with just the main scale. The difference between one main scale division and one Vernier scale division is known as the least count of the Vernier caliper.

In summary, there is no direct connection between Bézout's identity and the Vernier caliper. Bézout's identity is a mathematical theorem related to integers and their greatest common divisors, while the Vernier caliper is a physical measuring tool used for accurate length measurements. They belong to different areas of study and application [7]."

VI. CONCLUSION

So, there is a moral in this: many discoveries in mathematics and physics require a scientist’s imagination, an irreplaceable tool which separates humans from computers. At this stage, artificial intelligence (AI) is still reliant on manmade information databases, trained and biased by human intellect, so self-taught software is not yet prevalent. The ability to think abstractly and make intuitive leaps is a unique feature of human intelligence. Mathematical and physical theories often emerge from a combination of rigorous logical reasoning and imaginative insight, whereas computer artificial intelligence systems operate based on patterns and information present in their training data. While computer based AI can generate novel combinations and responses, these are essentially rearrangements of existing knowledge rather than true imaginative leaps.

Research in the development of explainable artificial intelligence, a system which allows for humans to retain intellectual oversight, is gaining traction, aiming to make AI systems more transparent and interpretable. This transparency is essential for human researchers to trust and collaborate effectively with artificial intelligence, especially in domains as complex as mathematics and theoretical physics, but still does not go far enough. Eventually, if a self-learning software is created, analogous to the human brain, computers may be able to construct creative connections between physical and theoretical concepts, potentially unlocking a new pathway of scientific development.

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