IAS 2023 SUMMER COLLABORATORS RESEARCH REPORT: NEGATIVITY PRESERVERS IN FIXED AND ALL DIMENSIONS

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Background. The fundamental questions of metric geometry, particularly with regard to metric transforms, have evolved from purely theoretical bases to impact upon an array of applied fields; here we only mention prediction of stationary processes, approximation by splines, and the statistics of big data. Over Euclidean space, works by Fréchet and Schoenberg in [*Ann. of Math.* 1935] provided characterizations of such metric transforms in terms of matrix positivity, following Menger. A variant was also studied by Schoenberg in [*loc. cit.*], wherein he characterized metric embeddings into Euclidean spheres, and in [*Duke Math. J.* 1942] he classified the positive definite functions on them. This was recast as a positivity preserver problem and pursued by Rudin [*Duke Math. J.* 1959] in the dimension-free case, and by Loewner in fixed dimension (see Horn [*Trans. AMS* 1969]).

In recent works, we have refined and strengthened these results on both fronts: dimension-free (*à la* Schoenberg and Rudin) and in fixed dimension (following Loewner and Horn); an account of some of this theory, old and new, has appeared in book form.

In our Summer Collaboration from 10th to 21st July 2023 at the Institute for Advanced Study, we have made further progress along both of these fronts, as is now described.

1. Dimension-free setting: preservers of negativity. Schur's Theorem in [*Crelle* 1911] asserts that the cone of $N \times N$ positive semidefinite matrices is closed under the entrywise product for any N. Schoenberg and Rudin proved in the work cited above that a function $f : (-1,1) \to \mathbb{R}$ applied entrywise preserves the collection of such matrices of all sizes and entries in $(-1,1)$ if and only if f is represented by a power series with non-negative coefficients.

Recently, we have obtained enhanced versions of these results. In [*J. Eur. Math. Soc.* 2022], we significantly reduced the test set of matrices used in Schoenberg's characterization of dimension-free positivity preservers, in one and several variables:

Theorem 1. *Let $I := (-\rho,\rho)$, $(0,\rho)$, or $[0,\rho)$, where $0 < \rho \leqslant \infty$, and let m be a positive integer. The following are equivalent for a function $f : I^m \to \mathbb{R}$.*

(1) *For each $N \geqslant 1$, and all tuples of $N \times N$ positive semidefinite matrices $(A^{(1)}, \dots, A^{(m)})$ with all entries in I, the matrix*
$$f[A^{(1)}, \dots, A^{(m)}] \in \mathbb{R}^{N\times N} \quad \text{with } (j,k) \text{ entry} \quad f(a_{jk}^{(1)}, \dots, a_{jk}^{(m)})$$
is positive semidefinite.

(2) *The previous statement holds, but restricted (for each $N \geqslant 1$) to tuples of Hankel $N \times N$ matrices of rank at most 3.*

(3) *The function f is the restriction to I^m of a real-analytic function on $(-\rho,\rho)^m$, with non-negative Maclaurin coefficients:*
$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}_+^m} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \qquad \text{for all } c_{\boldsymbol{\alpha}} \geqslant 0, \qquad \text{where } \mathbf{x}^{\boldsymbol{\alpha}} := \prod_{i=1}^m x_i^{\alpha_i}.$$

Such convergent power series F are also said to be *absolutely monotone functions* on the non-negative part of the domain, $[0,\infty)^m$.

For our Summer Collaborators project we explored a broader setting, wherein the matrices are allowed to have negative eigenvalues. We considered two kinds of test sets:

- $\mathcal{S}_N^{(k)}(I)$, the set of real symmetric $N \times N$ matrices with entries in I and exactly k negative eigenvalues, counted with multiplicity; and
- $\overline{\mathcal{S}_N^{(k)}}(I)$, the set of real symmetric $N \times N$ matrices with entries in I (not $\overline{I}$) and at most k negative eigenvalues, counted with multiplicity. That is,

$$\overline{\mathcal{S}_N^{(k)}}(I) = \bigsqcup_{j=0}^{k} \mathcal{S}_N^{(j)}(I).$$

If $k = 0$ then both test sets coincide, and equal the (previously studied) sets of positive semidefinite matrices.

We show in forthcoming work that the preservers of the sets $\bigcup_{N \geqslant 1} \mathcal{S}_N^{(k)}$ and of $\bigcup_{N \geqslant 1} \overline{\mathcal{S}_N^{(k)}}$ are far more restrictive when $k > 0$ – in contrast to the richer class of absolutely monotone functions (power series) above. The several-variables situation is also similar, as we show:

Theorem 2. *Let $I := (-\rho, \rho)$, $(0, \rho)$, or $[0, \rho)$, where $0 < \rho \leqslant \infty$, let k and m be positive integers, and suppose $f : I^m \to \mathbb{R}$.*

(1) *The entrywise transform $f[-]$ sends $\mathcal{S}_N^{(k)}(I)^m$ to $\mathcal{S}_N^{(k)}$ for all $N \geqslant k$ if and only if $f(\mathbf{x}) = c x_{p_0}$ for a constant $c > 0$ and some $p_0 \in [1, m]$, or, when $k = 1$, we may also have $f(\mathbf{x}) \equiv -c$ for some $c > 0$.*

(2) *The transform $f[-]$ sends $\overline{\mathcal{S}_N^{(k)}}(I)^m$ to $\overline{\mathcal{S}_N^{(k)}}$ for all $N \geqslant k$ if and only if $f(\mathbf{x}) = c x_{p_0} + d$ for some $p_0 \in [1, m]$, with either $c = 0$ and $d \in \mathbb{R}$, or $c > 0$ and $d \geqslant 0$.*

These characterizations follow from a stronger result that isolates the transforms such that

$$\mathcal{S}_N^{(k_1)}(I) \times \cdots \times \mathcal{S}_N^{(k_m)}(I) \to \overline{\mathcal{S}_N^{(l)}} \tag{1}$$

for every dimension N. Here the multiplicities of negative eigenvalues $k_1, \ldots, k_m \geqslant 0$ are allowed to differ. By permuting variables, we may assume that all zero k_i are promoted to the initial indices; thus,

$$k_1 = \cdots = k_{m_0} = 0 < k_{m_0+1}, \ldots, k_m \quad \text{for some } 0 \leqslant m_0 \leqslant m. \tag{2}$$

In forthcoming work, carried out in part at the IAS, we have obtained the classification of the transforms (1) (which was used to prove Theorem 2). The following result covers most cases of this classification:

Theorem 3. *Let $I := (-\rho, \rho)$, $(0, \rho)$, or $[0, \rho)$, where $0 < \rho \leqslant \infty$. Also let $k_1, \ldots, k_m \geqslant 0$ be integers, not all zero, and satisfying (2) for some integer $m_0 < m$.*

Suppose $l = 1$ if $k_p = 1$ for some $p \in [1, m]$ and $l \in [1, 2K - 2]$ otherwise, where $K = \min\{k_p : p \in [1, m],\ k_p > 0\}$. Given a function $f : I^m \to \mathbb{R}$, the following are equivalent.

(1) *The map $f[-]$ sends $\mathcal{S}_N^{(k_1)}(I) \times \cdots \times \mathcal{S}_N^{(k_m)}(I) \to \overline{\mathcal{S}_N^{(l)}}$ for all $N \geqslant \max\{k_i\}$.*

(2) *There exist an index $p_0 \in [m_0 + 1, m]$, a function $F : (-\rho, \rho)^{m_0} \to \mathbb{R}$, and a constant $c \geqslant 0$ such that*
   (a) *we have the representation*
   $$f(\mathbf{x}) = F(x_1, \ldots, x_{m_0}) + c x_{p_0} \qquad \text{for all } \mathbf{x} \in I^m, \tag{3}$$
   (b) *the function $\mathbf{x}' \mapsto F(\mathbf{x}') - F(\mathbf{0})$ is absolutely monotone on $[0, \rho)^{m_0}$,*
   (c) *if $c > 0$ then p_0 is unique and $l \geqslant k_{p_0}$, and*
   (d) *if $c > 0$ and $l = k_{p_0}$ then $F(\mathbf{0}) \geqslant 0$.*

Thus, the characterization reveals a combination of the rich class of preservers in Theorem 1 with the restricted, rigid class in Theorem 2.

2. Fixed-dimension setting: strict positive definiteness and strict monotonicity of Schur polynomial ratios. When restricted to a fixed dimension, the study of positivity preservers is more challenging: to date, a complete characterization of the preservers of positive semidefiniteness on 3×3 matrices remains unknown. When restricted to power series, those with non-negative Maclaurin coefficients preserve positive semidefiniteness in all dimensions, by the Schur product theorem, but no other examples were found until recently. In [*Adv. Math.* 2016], we discovered the first entrywise polynomial positivity preservers with negative coefficients. This work revealed quite unexpected connections between analysis and symmetric function theory. It was taken forward for larger classes of functions by one of us with Tao in [*Amer. J. Math.* 2021]; this latter work also showed a Schur-polynomial characterization of weak majorization of real tuples. Both of these facts relied upon the following "Schur monotonicity lemma" from *loc. cit.*:

Theorem 4. *Fix tuples* $\mathbf{m} = (m_0, \dots, m_{N-1})$ *and* $\mathbf{n} = (n_0, \dots, n_{N-1})$ *in* $\mathbb{Z}_+^N$ *with strictly increasing entries such that* $m_j \leqslant n_j$ *for all* j*. The ratio of Schur polynomials*

$$(0,\infty)^N \to (0,\infty);\ \mathbf{u} \mapsto \frac{s_{\mathbf{n}}(\mathbf{u})}{s_{\mathbf{m}}(\mathbf{u})}$$

is coordinatewise non-decreasing.

(Here, the Schur polynomial $s_{\mathbf{n}}(\mathbf{u})$ is defined to be $\det(u_i^{n_{j-1}})_{i,j=1}^N / \det(u_i^{j-1})_{i,j=1}^N$ when $\mathbf{u}$ has distinct coordinates, and extended by continuity to $(0,\infty)^N$. While this is an algebraic object from representation theory, it is being studied in this setting as a function on the positive orthant.) In turn, Theorem 4 helped show the following preserver result in *loc. cit.*:

Theorem 5. *Fix a dimension* $N \geqslant 1$ *and integer exponents* $0 \leqslant n_0 < \cdots < n_{N-1} < M$*. For real coefficients* $c_0, \dots, c_{N-1}$ *and* c'*, let*

$$h(x) := c_0 x^{n_0} + \cdots + c_{N-1} x^{n_{N-1}} \qquad \text{and} \qquad f_{c'}(x) := h(x) + c' x^M.$$

Given $\rho \in (0,\infty]$*, the following are equivalent.*

1. *The entrywise map* $f[-]$ *preserves positive semidefiniteness on the set* $\mathcal{P}_N\big([0,\rho]\big)$ *of all* $n \times n$ *positive matrices with entries in* $[0,\rho]$*.*
2. *The map* $f[-]$ *preserves positive semidefiniteness on the subset of* $\mathcal{P}_N\big([0,\rho]\big)$ *consisting of rank-one matrices.*
3. *Either* $c_0, \dots, c_{N-1}$ *and* c' *are non-negative, or* $c_0, \dots, c_{N-1} > 0$ *and* $c' \geqslant -\mathcal{C}^{-1}$*, where*

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_j}$$

and the Vandermonde determinant

$$V(\mathbf{m}) := \prod_{0 \leqslant k < l \leqslant N-1} (m_l - m_k) \qquad \text{for any } \mathbf{m} = (m_0, \dots, m_{N-1}),$$

with N*-tuples*

$$\mathbf{n}_j := (n_0, \dots, \widehat{n_j}, \dots, n_{N-1}, M) \qquad \text{and} \qquad \mathbf{n} := (n_0, \dots, n_{N-1}),$$

where $\widehat{n_j}$ *indicates that* n_j *is omitted.*

In particular, Theorem 5 identified a sharp negative threshold $-\mathcal{C}^{-1}$, above which every value of c' yields a positivity preserver. Such fixed-dimension polynomial preservers with negative coefficients were not previously known.

We now describe two results from another forthcoming work – also partly carried out at the IAS – in which we took a closer look at the two preceding results. First, we have shown that the entrywise polynomial map $f_{c'}$ for $c' > -\mathcal{C}^{-1}$, which preserves positive semidefiniteness, in fact yields strict positive definiteness (that is, all eigenvalues positive instead of non-negative):

Theorem 6. *With notation as in Theorem 5, let $n_0 = 0$, $c_1, \ldots, c_{N-1} > 0$, and $c' > -\mathcal{C}^{-1}$. If all rows of $A \in \mathcal{P}_N([0, \rho])$ are distinct then $f_{c'}[A]$ is positive definite.*

That A has distinct rows is an obvious necessary condition for $f_{c'}[A]$ to be non-singular. Theorem 6 says that this condition is also sufficient.

Akin to the above results, Theorem 6 is proved using a Schur-monotonicity phenomenon which we have now shown, and which similarly enhances Theorem 4 to strict monotonicity:

Theorem 7. *The function $\mathbf{u} \mapsto s_{\mathbf{n}}(\mathbf{u})/s_{\mathbf{m}}(\mathbf{u})$ from Theorem 4 is coordinatewise strictly increasing on $(0, \infty)^N$.*

More strongly, this strict monotonicity is now shown over parts of the boundary of the orthant.

Remark 8. Given previous works (cited above), we have also shown "strict" counterparts of both Theorems 6 and 7 for non-integer powers.

In order to obtain the results above, we used two techniques which may be of independent interest. The latter was proved during our IAS visit.

(1) Compression and inflation of matrices with isogenic block structure. A real symmetric matrix $A = (a_{ij})_{i,j=1}^N$ respects the block structure associated to a partition $\pi = \{I_1, \ldots, I_m\}$ of $\{1, \ldots, N\}$ if the entry a_{ij} is independent of $i, j \in I_k$ for some k. The compression map collapses each cell I_k (and hence each $I_j \times I_k$) to a single entry, sending the matrix A to the $m \times m$ matrix with entries given by the constant values along the fibres of the projection. The reverse inflation map restores the repetitions of matrix entries in A. These are linear, mutually inverse maps that preserve rank, positive semidefiniteness, and the entrywise product.

(2) Rank-one lower bounds with positive and distinct entries. If $A \in \mathcal{P}_N\big([0, \infty)\big)$ has distinct rows, then there exists $\mathbf{u} \in [0, \infty)^N$ with distinct entries such that $A \geqslant \mathbf{u}\mathbf{u}^T$ and $\mathbf{u}$ has a zero entry if and only if A has a zero row.

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