Intersecting Families in Downsets

Park City Experimental Mathematics Lab

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Park City Mathematics Institute

Summer 2025

Downset

A **downset** is a family of sets closed under $\subseteq$. One can view a downset $\mathcal{F}$ as the union of power sets of some basis sets $A_1, \ldots, A_k$; $\mathcal{F} = \bigcup_{i=1}^{k} 2^{A_i}$. We say the A_i's **generate** $\mathcal{F}$.

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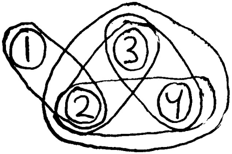
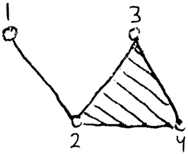
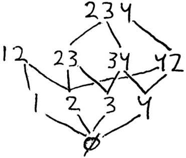
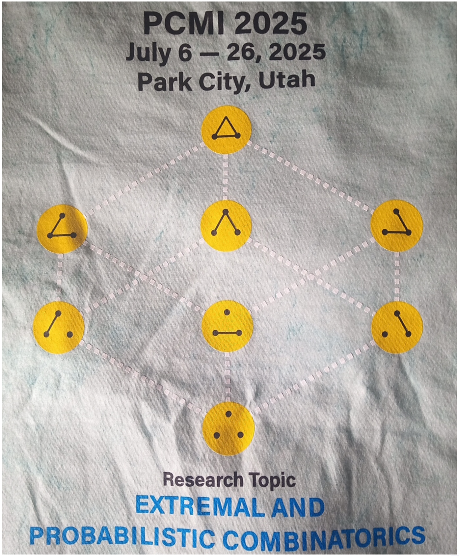






Above are three different pictures of the downset generated by $\{1, 2\}$ and $\{2, 3, 4\}$.

Downset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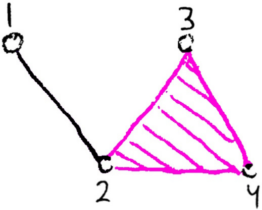
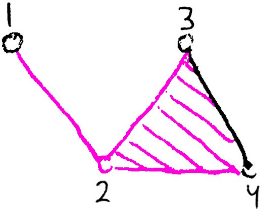


$23, 24, 34, 234$



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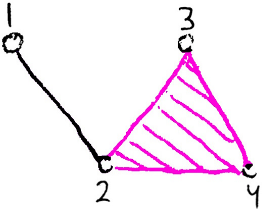
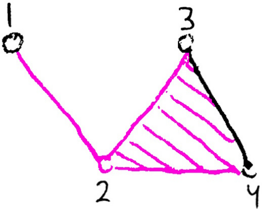


$23, 24, 34, 234$ (non-star)



$2, 21, 23, 24, 234$ (star)

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Intersecting Family Example

Stars

Theorem

A maximal intersecting family in $2^{[n]}$ can be attained by considering a star with size 2^{n-1}.

Proof is by considering (A, A^c) for $A \in 2^{[n]}$.

Theorem (Erdös-Ko-Rado)

Let r such that $2r \leq n$. Then the maximal intersecting family consisting only of r-element subsets of $[n] = \{1, 2, \ldots, n\}$ is attained by a 'star' with size $\binom{n-1}{r-1}$.

Proofs via induction/double-counting/Kruskal-Kotana Theorem.

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Conjecture (Chvátal 1972)

Among the intersecting families of maximum size of a downset over a finite set, there is a star.

Many special cases are known:

Chvátal 1972	$\mathcal{F}$ is "left-compressed".
Schönheim 1976	$\bigcap_{i=1}^{n} A_i \neq \varnothing$ (counting).
Wang-Wang 1978	$A_i \cap A_j \cap A_k \neq \varnothing$ for all i, j, k distinct.
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$\mathcal{F}$ has two maximal elements A, B with $A \cap B \neq \varnothing$
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where $a_{i_1,\ldots,i_p} := |\bigcap_{j=1}^{p} A_{i_j}|$.

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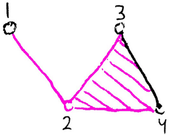
Sizes of stars

Theorem (Berge 1976)

The size of an ∩-family in a downset $\mathcal{F}$ is at most $\frac{1}{2}|\mathcal{F}|$.



In its full generality, the problem appears to be quite difficult. I am rather skeptical about the use of counting arguments. It would be interesting to prove the conjecture for independence systems whose maximal sets are lines of a projective plane.

Sizes of stars

Theorem

Let S be the size of a random star in a random downset. Then

$$\mathbb{E}(S) = \sum_{i_1=0}^{\binom{i_0}{1}} \sum_{i_2=0}^{\binom{i_1}{2}} \cdots \sum_{i_n=0}^{\binom{i_{n-1}}{n}} \left(1 + \sum_{j=1}^{n} i_j\right) S_{i_1,\ldots,i_n}\,,$$

where

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Future work:

- Is there a simpler way to express the above?
- Compute some examples
- $\mathbb{E}$(size of a *largest* star)
- $\mathbb{E}$(size of a largest ∩-family)
- Make the following idea work

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$$\begin{aligned}
\mathbb{P}(S=0) &= 0\\
\mathbb{P}(S=1) &= q_1^n\\
\mathbb{P}(S=2) &= np_1q_1^{n-1}\\
\mathbb{P}(S=3) &= \binom{n}{2}p_1^2q_1^{n-2}\\
\mathbb{P}(S=4) &= \binom{n}{3}p_1^3q_1^{n-3}+\binom{n}{2}p_1^2q_1^{n-2}p_1p_2\\
\mathbb{P}(S=5) &= \binom{n}{4}p_1^4q_1^{n-4}+\binom{n}{3}p_1^3q_1^{n-3}\binom{3}{1}p_1p_2(q_1+p_1q_2)^2\,.
\end{aligned}$$

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We have the upper bound of $1/2$ for an intersecting family, and we have some slight progress on expected sizes of stars.

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Entropy

Recently used by Gilmer (2022) and more recently Sawin (2023) to prove that for a union-closed family $\mathcal{F}$, there exists an element $i \in [n]$ such that $\frac{\#\text{ of sets with } i \text{ in } \mathcal{F}}{|\mathcal{F}|} \geq c$, where Gilmer had $c = 0.01$ and Sawin had $c = \frac{3-\sqrt{5}}{2} \approx 0.382$.

Lemma

Let $u \in [0, 1]$ (Gilmer used 0.01), let p, q be independent identically distributed random variables on $[0, 1]$ such that $\mathbb{E}[p] \leq u$. Then, letting H be the binary entropy function,

$$\mathbb{E}[H(p + q - pq)] \geq c\mathbb{E}[H(p)]$$

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Entropy methods

Note that downsets are intersection-closed. The natural analogue of that proof method requires the following technical lemma: Let $u \in [0, 1]$, p, q be i.i.d. on $[0, 1]$, such that $\mathbb{E}[p] \leq u$. Then

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This does not seem true, as $pq < p$, and if $u < 1/2$ we expect $H(pq) < H(p)$ as H is increasing on $(0, 1/2)$.

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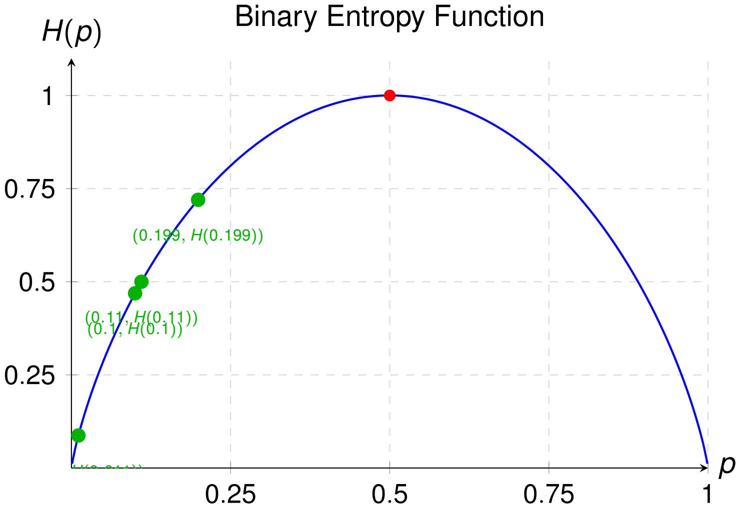
Binary Entropy



Failure of Proportions

Consider the down-set generated by $\{1\}, \{2\}, \dots, \{n\}$. The proportion of each star decreases as n^{-1}, so you cannot hope for a constant!

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Complements of intersecting families

Note that the complement of an downset $\mathcal{D}$ is necessarily union closed:

$$A, B \in \mathcal{D} \implies A^c \cup B^c = (A \cap B)^c \in \mathcal{D}^c$$

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Other questions

- What about modular intersections? E.g., in a downset, what is the biggest family $\mathcal{I}$ such that $|A \cap B|$ is even for all $A, B \in \mathcal{I}$? (Oddtown generalization)
- What about t-intersecting families? Specifically, for a downset $\mathcal{F}$, what is the largest $\mathcal{S} \subseteq \mathcal{F}$ such that $|A \cap B| \geq t$ for all $A, B \in \mathcal{S}$?
- When are the only maximal intersecting families a star? (not the nonempty intersection case.)

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