Resonant Dynamics in Evolutionary Dispersive Equations

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- Nonresonant Forcing, let $\omega \in \mathbb{R} \setminus \{1\}$ and $|\omega| \gg 1$

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- Resonant Forcing

$$Re[b(t)] = Re[a(t)] + t\cos(t)$$

- Nonresonant Forcing, let $\omega \in \mathbb{R} \setminus \{1\}$ and $|\omega| \gg 1$

$$Re[b(t)] = Re[a(t)] + \varepsilon \sin(\omega t)$$

- Nearly Resonant Forcing, let $\omega \in \mathbb{R} \setminus \{1\}$ and $|\omega - 1| \ll 1$

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A trichotomy of types of asymptotic behavior

- Linear

$$b(t) \approx \tilde{a}_0 e^{it}$$

- Resonant

$$b(t) \approx \tilde{a}_0 (t+1) e^{it}$$

- Neither Linear nor Resonant

A trichotomy of types of asymptotic behavior

- Linear "Scattering"

$$b(t) \approx \tilde{a}_0 e^{it}$$

- Resonant "Modified Scattering"

$$b(t) \approx \tilde{a}_0 (t+1) e^{it}$$

- Neither Linear nor Resonant

Consider $a : [0, \infty) \to \mathbb{C}^N$ and $\lambda_k \in \mathbb{R}$ for $k \in \{1, ..., N\}$

$$\begin{cases} i\frac{d}{dt}a_k - \lambda_k a_k = \sum_{k_1+k_2-k_3 \equiv k} a_{k_1} a_{k_2} \bar{a}_{k_3} \\ a_k(0) = z_k \in \mathbb{C} \end{cases}$$

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How to identify resonant/nonresonant interactions of the polynomial nonlinearity?

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How to identify resonant/nonresonant interactions of the polynomial nonlinearity?Use iterative process.

Consider $a^j : [0, \infty) \to \mathbb{C}^N$ for $j = 0, 1, \dots$ and suppose the sequence satisfies

$$i \frac{d}{dt} a_k^0 - \lambda_k a_k^0 = 0$$
$$a_k^0(0) = z_k^0 \in \mathbb{C}$$

and

$$i \frac{d}{dt} a_k^{j+1} - \lambda_k a_k^{j+1} = \sum_{k_1 + k_2 - k_3 \equiv k} a_{k_1}^j a_{k_2}^j \bar{a}_{k_3}^j$$
$$a_k^{j+1}(0) = z_k^{j+1} \in \mathbb{C}$$

Observe

$$a_k^0(t) = z_k^0 e^{it\lambda_k}$$

Examination of the second step:

$$i\frac{d}{dt}a_k^1 - \lambda_k a_k^1 = \sum_{k_1+k_2-k_3\equiv k} z_{k_1}^0 z_{k_2}^0 \bar{z}_{k_3}^0 e^{it(\lambda_{k_1}+\lambda_{k_2}-\lambda_{k_3})}$$
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Examination of the second step:

$$i\frac{d}{dt}a_k^1 - \lambda_k a_k^1 = \varepsilon^3 \sum_{k_1+k_2-k_3\equiv k} z_{k_1}^0 z_{k_2}^0 \bar{z}_{k_3}^0 e^{it(\lambda_{k_1}+\lambda_{k_2}-\lambda_{k_3})}$$
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If $(\mathbb{C}^{2N}, \omega)$ is a symplectic manifold, then a is a solution to a Hamiltonian system with Hamiltonian

$$H(a) = H_0(a) + P(a) = \frac{1}{2}\sum_{k=1}^{N} \lambda_k |a_k|^2 + \frac{1}{4}\sum_{k_1+k_2 \equiv k_3+k_4} a_{k_1} a_{k_2} \bar{a}_{k_3} \bar{a}_{k_4}$$

Definition (Nonresonance)

Let $r \in \mathbb{N}$. A frequency vector, $\lambda \in \mathbb{R}^N$, is **nonresonant up to order r** if

$$k \cdot \lambda := \sum_{j=1}^{N} k_j \lambda_j \neq 0 \text{ for all } k \in \mathbb{Z}^N \text{ with } 0 < |k| \leq r$$

Definition (Normal Form)

Let $H = H_0 + P$ where $P \in C^\infty(\mathbb{C}^{2N}, \mathbb{R})$, which is at least cubic such that P is a perturbation of H_0. We say that P is in **normal form** with respect to H_0 if it Poisson commutes with H_0:

$$\{P, H_0\} = 0$$

Theorem (Moser '68)

Let $H = H_0 + P$ *where*

- $H_0 = \sum_{j=1}^{N} \lambda_j \frac{|a_j|^2}{2}$
- $P \in C^\infty(\mathbb{C}^{2N}, \mathbb{R})$ *having a zero of order 3 at the origin*

Fix $M \geq 3$ *an integer. There exists* $\tau : \mathcal{U} \ni (a', \bar{a}') \mapsto (a, \bar{a}) \in \mathcal{V}$ *a real analytic canonical transformation from a nbhd of the origin to a nbhd of the origin which puts* H *in normal form up to order* M *i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

1. Z *is a polynomial of order* r *and is in normal form*
2. $R \in C^\infty(\mathbb{C}^{2N}, \mathbb{R})$ *and* $R(a, \bar{a}) = O(\|(a, \bar{a})\|^{M+1})$
3. τ *is close to the identity:* $\tau(a, \bar{a}) = (a, \bar{a}) + O(\|(a, \bar{a})\|^2)$

Corollary

Assume λ is nonresonant. For each $M \geq 3$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\|(a_0, \bar{a}_0)\| = \varepsilon < \varepsilon_0$ the solution $(a(t), \bar{a}(t))$ of the Hamiltonian system associated to H which takes value $(a_0, \bar{a}_0)$ at $t = 0$ satisfies

$$\|(a(t), \bar{a}(t)\| \leq 2\varepsilon \text{ for } |t| \leq \frac{c}{\varepsilon^{M-1}}.$$

Nonlinear Schrödinger Equation

Consider the Cauchy problem

$$\begin{cases} iu_t + \Delta_{\mathcal{M}} u = \mathcal{N}(u), & x \in \mathcal{M}, t \geq 0 \\ u(x,0) = u_0(x) \in H^s(\mathcal{M};\mathbb{C}), & \end{cases} \quad (1)$$

Consider the Cauchy problem

$$\begin{cases} iu_t + \Delta_{\mathcal{M}} u = |u|^2 u, & x \in \mathcal{M}, t \geq 0 \\ u(x,0) = u_0(x) \in H^s(\mathcal{M}; \mathbb{C}), & \end{cases} \qquad (2)$$

Infinite-Dimensional Nonresonance Condition

Definition (Nonresonance Condition)

There exists $\gamma = \gamma_M > 0$ and $\tau = \tau_M > 0$ such that for any N large enough, one has

$$\left| \sum_{1 \leq j \leq N} m_j \lambda_j \right| \geq \frac{\gamma}{N^\tau} \qquad \text{for } \|m\|_1 \leq M, \tag{3}$$

where $m \in \mathbb{Z}^\infty \setminus \{0\}$.

Normal Form Theorem

Theorem (Bambusi-Grebert 2006)

Consider the equation

$$i\dot{x} = \lambda x + \sum_{k\geq 2} f_k(x). \tag{4}$$

and assume the nonresonance condition (3). *For any* $M \in \mathbb{N}$, *there exists* $s_0 = s_0(M, \tau)$ *such that for any* $s \geq s_0$ *there exists* $r_s > 0$ *such that for* $r < r_s$, *there exists an analytic canonical change of variables*

$$y = \Phi^{(M)}(x)$$
$$\Phi^{(M)} : B_s(r) \to B_s(3r)$$

which puts (4) *into the normal form*

$$i\dot{y} = \lambda y + \mathcal{R}^{(M)}(y) + \mathcal{X}^{(M)}(y). \tag{5}$$

Normal Form Theorem continued

Theorem (Theorem cont.)

Moreover there exists a constant $C = C_s$ such that:

- $$\sup_{x \in B_s(r)} \|x - \Phi^{(M)}(x)\|_s \leq Cr^2$$
- $\mathcal{R}^{(M)}$ *is at most of degree $M + 2$, is resonant,*
- *the following bound holds*

$$\|\mathcal{X}^{(M)}\|_{s,r} \leq Cr^{M+\frac{3}{2}}$$

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    - Uses normal form method of Shatah (1985) among other methods.

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4. Arbitrarily small "divisors" are very difficult to control

Idea for argument

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$$iu_t = \mathcal{R}(u,t) + \mathcal{E}(u,t)$$
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Consider

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Integrative Methods

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Then

$$\|v-u\| \lesssim \left\| \int_0^t \mathcal{E}(u,s)\, ds \right\| \lesssim \varepsilon(t) \left\| \int_0^t \mathcal{R}(v,s)\, ds \right\|$$

Using Strichartz estimates and integration by parts. Where $\varepsilon(t)$ is small for all time or $\varepsilon(t) \to 0$.

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Let $1 < s$, $K \gg 1$, *and* $0 < \delta < 1$ *be given parameters. Then there exists a global smooth solution* $u(t,x)$ *to* (2) *and a time* $T > 0$ *with*

$$\|u(0)\|_{H^s} \leq \delta \qquad \text{and} \qquad \|u(T)\|_{H^s} \geq K.$$

Integrative Methods for $\mathcal{M} = \mathbb{T}^d$

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Theorem (Guardia, Kaloshin (2015))

Let $s > 1$. Then there exists $c > 0$ with the following property: for any large $K \gg 1$ there exists a a global solution $u(t;x)$ of (2) and a time T satisfying $0 < T \leq \exp(K^c)$ such that

$$\|u(T)\|_{H^s} \geq K\|u(0)\|_{H^s}$$

Proposition (Guardia, Kaloshin (2015))

Let $K \gg 1$. Then, there exists $N \gg 1$ and a set $\Lambda \subset \mathbb{Z}^2$, with

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N,$$

which satisfies I-Team conditions and also

$$\frac{\sum_{n \in \Lambda_{N-1}} |n|^{2s}}{\sum_{n \in \Lambda_3} |n|^{2s}} \geq \frac{1}{2} 2^{(s-1)(N-4)} \geq K^2.$$

Moreover, given any $R > 0$ (which may depend on K), one can ensure that each generation Λ_j has 2^{N-1} disjoint frequencies n satisfying $|n| \geq R$.

Integrative Methods for $\mathcal{M} = \mathbb{T}_\theta^d$

Proposition (Staffilani, W. (2018))

Let $N \in \mathbb{Z}_+$. Consider a set $\Lambda \subset \mathbb{Z}^2$, with

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N,$$

which satisfies I-Team conditions with irrational torus frequencies, Then

$$\frac{\sum_{n \in \Lambda_k} |n|^{2s}}{\sum_{n \in \Lambda_j} |n|^{2s}} \leq 2^s$$

for any $j, k \in \{1, ..., N\}$.

Integrative Methods for $\mathcal{M} = \mathbb{T}^d_\theta$

Proposition (Giuliani, Guardia 2021)

Let $s > 1$. Then there exists $c > 0$ with the following property: for $0 < \delta \ll 1$ and for $K \gg 1$ there exists a a global solution $u(t;x)$ of (2) and a time T satisfying $0 < T \leq \exp(c_0(K/\delta)^c)$ such that

$$\|u(0)\|_{H^s} \leq \delta \qquad \textit{and} \qquad \|u(T)\|_{H^s} \geq K.$$

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  - Yu, W. (2022) (small data Modified Scattering on $\mathbb{R} \times \mathbb{T}^d_\theta$)
    - Can use better resonant structure to demonstrate boundedness of solutions.

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5. One can still lose regularity.
6. Often requires small data.

When analyzing NLSE nonlocally, one uses the diagonalization with respect to the linearized equation (Fourier transform):

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\begin{aligned}
iu_t + \Delta_{\mathbb{T}^d} u &= |u|^2 u \\
i\frac{d}{dt} u_k - \lambda_k u_k &= \sum_{k_1+k_2-k_3=k} u_{k_1} u_{k_2} \bar{u}_{k_3}
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This works perfectly for completely solving the linear equation, but not as well for the nonlinear equation. An important part of the inverse scattering method is a diagonalization w.r.t. the nonlinear equation.

Schrödinger Operator (Example for KdV equation)
First Step: Given a solution $u(t) \in \mathcal{S}(\mathbb{R})$, for each t, find solutions $(\psi_\lambda(t), \lambda(t)) \in C^\infty(\mathbb{R}) \times \mathbb{C}$ to

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Third Step: Demonstrate that the given constants of motion are enough to put equation into action-angle coordinates

Inverse Scattering and Integrability

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Allows for the application to geometric Schrödinger equations.

Thank you for listening