Higher Teichmüller spaces and Higgs bundles

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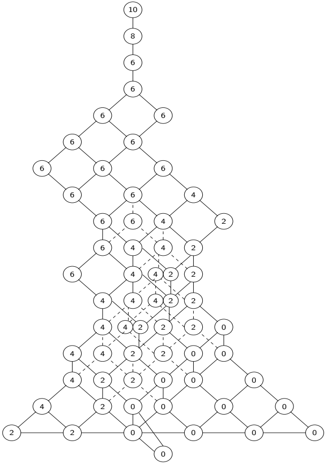


Character varieties

- S a closed oriented smooth surface of genus $g \geq 2$
- G a real or complex connected **simple** Lie group (e.g. $\mathrm{SL}_n\mathbb{C}$)

Define the G-character variety of S by

$$\mathcal{X}(S, \mathrm{G}) = \mathrm{Hom}^+(\pi_1 S, \mathrm{G})/\mathrm{G}$$

conjugacy classes of completely reducible reps $\rho : \pi_1 S \to \mathrm{G}$.

Questions for today:

- How many components does $\mathcal{X}(S, \mathrm{G})$ have?
- Are some components more interesting than others?

Example: For $\mathrm{PSL}_2\mathbb{R} = \mathit{Isom}^+(\mathbb{H}^2)$,

- $|\pi_0(\mathcal{X}(S, \mathrm{PSL}_2\mathbb{R}))| = 4g - 3$ (Goldman 88)
- two of them consist entirely of "discrete and faithful reps."

$\rightsquigarrow$ $\mathrm{Teich}(S) \cup \mathrm{Teich}(\overline{S}) \subset \mathcal{X}(S, \mathrm{PSL}_2\mathbb{R})$ open and closed subset

Some known results on components

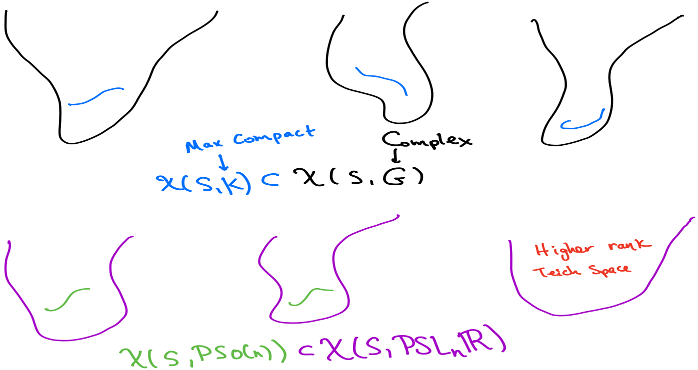
Given a representation $\rho : \pi_1 S \to \mathrm{G}$, we can build a flat G-bundle

$$E_\rho = \widetilde{S} \times_\rho \mathrm{G} \to S$$

$$\rightsquigarrow \tau : \ \pi_0(\mathcal{X}(S, \mathrm{G})) \longrightarrow H^2(S, \pi_1 \mathrm{G}) \cong \pi_1 G.$$

- If G is **compact**, then τ is a bijection. (Narasimhan-Sheshadri '65, Ramanathan '75)
- If G is **complex** then τ is a bijection. (J. Li '94)
  Corollary: If $\mathrm{K} < \mathrm{G}$ is maximal compact, then every $\rho : \pi_1 S \to \mathrm{G}$ can be deformed to $\rho' : \pi_1 S \to \mathrm{K} \hookrightarrow \mathrm{G}$.
- If G is a **split real** Lie group (e.g. $\mathrm{PSL}_n\mathbb{R}$), then there exists $\rho \in \mathcal{X}(\pi_1 S, \mathrm{G})$ which cannot be deformed to a compact representation. (Hitchin '91)

Picture of components of K < G



Definition

A higher rank Teichmüller space is a connected component of $\mathcal{X}(S, \mathrm{G})$ consisting entirely of discrete and faithful representations.

Higgs bundles and Nonabelian Hodge

Let X be a Riemann surface structure on S. Fixing this data, we get a moduli space $\mathcal{M}(X, \mathrm{G})$ of polystable G-Higgs bundles on X.

Theorem (Hitchin, Donaldson, Simpson, Corlette)

There is a real analytic isomorphism

$$\mathcal{T} : \mathcal{M}(X, \mathrm{G}) \to \mathcal{X}(S, \mathrm{G}) .$$

So, these spaces have the same number of components.
Provides more tools to study topology, but breaks symmetry.

Works by relating stability to existence of a special metric.

- (Hitchin, Simpson) On Higgs bundle side, a metric that solves a gauge theoretic equations $F_h + [\Phi, \Phi^{*_h}] = 0$.
- (Corlette, Donaldson) On character variety side, an equivariant map $h_\rho : \widetilde{X} \to \mathrm{G}/\mathrm{K}$ to the symmetric space which is *harmonic*.

What is a Higgs bundle

For $\mathsf{G} = \mathsf{GL}_n\mathbb{C}$, a Higgs bundle is a pair $(\mathcal{E}, \Phi)$, where

- $\mathcal{E} \to X$ is a rank n holomorphic vector bundle of degree 0,
- $\Phi \in \Omega^{1,0}(X, End(\mathcal{E}))$ is holomorphic, $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^{1,0}_X$.

For G complex, a Higgs bundle is a pair $(\mathcal{E}, \Phi)$, where

- $\mathcal{P}_\mathsf{G} \to X$ is a holomorphic principal G-bundle,
- $\Phi \in \Omega^{1,0}(X, \mathrm{ad}(\mathcal{P}_\mathsf{G}))$ is holomorphic.

Slope stability for Φ-invariant reductions $\rightsquigarrow$ moduli space $\mathcal{M}(X, \mathsf{G})$ of semistable G-Higgs bundles.

$(\mathcal{E}, \Phi)$ polystable if and only if there is a metric h on $\mathcal{E}$ such that

$$F_h + [\Phi, \Phi^{*_h}] = 0$$

$$\rightsquigarrow \quad A_h + \Phi + \Phi^{*_h} \text{ is a flat G connection}$$

Higgs for a real group G

For G real with maximal compact K, set $\mathfrak{p} = \mathfrak{k}^{\perp} \cong T_K G/K$,

$$\rightsquigarrow K_{\mathbb{C}} \text{ and } K_{\mathbb{C}}\text{-invariant} \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}.$$

A G-Higgs bundle is a pair $(\mathcal{E}_{K^{\mathbb{C}}}, \Phi)$, where

- $\mathcal{E}_{K^{\mathbb{C}}} \to X$ is a holomorphic principal $K^{\mathbb{C}}$-bundle
- $\Phi \in \Omega^{1,0}(\mathcal{E}_{K^{\mathbb{C}}} \times_{K^{\mathbb{C}}} \mathfrak{p}^{\mathbb{C}})$ which is holomorphic.
- Φ is identified with $dh_{\rho}^{1,0}$ of harmonic map.

For compact groups $\Phi = 0$, hence $\rho \in \mathcal{X}(G)$ factors through K if and only if $\mathcal{T}(\rho) \in \mathcal{M}(G)$ has $\Phi = 0$.

For $G = SL_n\mathbb{R}$, we have $K = SO(n)$ and $\mathfrak{sl}_n\mathbb{R} = \mathfrak{so}(n) \oplus sym_0(\mathbb{R}^n)$. So, an $SL_n\mathbb{R}$-Higgs bundle is tuple $(\mathcal{E}, Q, \Phi)$, where

- $\mathcal{E}$ is a holomorphic rank n bundle equipped with a symmetric isomorphism $Q : \mathcal{E} \to \mathcal{E}^*$,
- $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^{1,0}$ satisfying $\Phi^T Q = Q\Phi$ and $\bar{\partial}_E \Phi = 0$.

Teichmüller space from $SL_2\mathbb{R}$-Higgs bundles

An $SL_n\mathbb{R}$-Higgs bundle $(\mathcal{E}, Q, \Phi)$

- $\mathcal{E}$ is a holomorphic rank n bundle equipped with a symmetric isomorphism $Q : \mathcal{E} \to \mathcal{E}^*$,
- $\Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^{1,0}$ satisfying $\Phi^T Q = Q\Phi$ and $\bar{\partial}_E \Phi = 0$.

When $n = 2$, set $K = \Omega_X^{1,0}$ and consider $(\mathcal{E}, Q, \Phi)$ given by

$$\mathcal{E} = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathcal{E} \to \mathcal{E}^* \quad \Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : \mathcal{E} \to \mathcal{E} \otimes K$$

To this example, we can add $q \in H^0(K^2) \cong \mathbb{C}^{3g-3}$

$$(\mathcal{E}, Q, \Phi) \longrightarrow (\mathcal{E}, Q, \Phi + \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix})$$

Theorem (Hitchin '87)

- $H^0(K^2) \to \mathcal{M}(SL_2\mathbb{R})$ *is injective, open and closed map.*
- *Under the identification* $\mathcal{T} : \mathcal{M}(SL_2\mathbb{R}) \to \mathcal{X}(SL_2\mathbb{R})$, *this component identifies with Teichmüller space* $\text{Teich}(S)$.

Aside on nilpotents and Slodowy Slices

Consider $\mathfrak{g}$ a **complex** simple Lie algebra and the nilpotent cone

$$N_{\mathfrak{g}} \subset \mathfrak{g}$$

Jacobson-Morozov Thm implies every $f \in N_{\mathfrak{g}} \setminus \{0\}$ can be completed to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$, where

$$[h, f] = -2f \qquad [h, e] = 2e \qquad [e, f] = h$$

Slodowy slice through f is a linear slice for through $G \cdot f$ parameterized by the vector space $V_e = \ker(\mathrm{ad}_e)$.

$$\mathcal{S}_f = f + V_e.$$

For $\mathfrak{sl}_2\mathbb{C}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$$\mathcal{S}_f = f + \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

Hitchin component

Idea: Use principal embedding of $\mathrm{SL}_2\mathbb{C} \to \mathrm{G}^{\mathbb{C}}$ to embed Teich(S) Higgs bundles into $\mathrm{G}^{\mathbb{C}}$-Higgs and consider a "Slodowy slice"

For $\mathrm{SL}_n\mathbb{C}$, take the irr. action of $\mathrm{SL}_2\mathbb{C}$ on $\mathbb{C}^n = Sym^{n-1}(\mathbb{C}^2)$.

For $n = 3$, $f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$$S^2(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}), S^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = K \oplus \mathcal{O} \oplus K^{-1}, \begin{pmatrix} 0 & q_2 & q_3 \\ 2 & 0 & q_2 \\ 0 & 2 & 0 \end{pmatrix}$$

Now add to highest weight spaces, this is an $\mathrm{SL}_3\mathbb{R}$-Higgs bundle

Hitchin '91

$H^0(K^2) \oplus \cdots \oplus H^0(K^n) \to \mathcal{M}(\mathrm{SL}_n\mathbb{R})$ is injective, open and closed.
For general complex G, this gives an injective, open, closed map

$$\bigoplus_{j=1}^{\mathrm{rk}\,\mathfrak{g}} H^0(K^{m_j+1}) \to \mathcal{M}(\mathrm{G}^{\mathbb{R}}_{split}).$$

The idea of Labourie's Anosov condition

This component is called the **Hitchin component**.
How much does the Hitchin component generalize Teich(S)?

Theorem (Labourie '06, Fock-Goncharov '06)

The Hitchin component consists entirely of discrete and faithful representations, and is hence a higher rank Teichmüller space.

Labourie developed the notion of Anosov representations.
Facts: There is a class of reps $\mathcal{A} \subset \mathcal{X}(S, G)$ called Anosov which generalize many features of Teich(S):

- $\rho \in \mathcal{A} \Rightarrow$ discrete and faithful
- $\rho \in \mathcal{A} \Rightarrow$ holonomies of geometric structures
- $\mathcal{A}$ is open in $\mathcal{X}(S, G)$ BUT not necessarily closed (generalization of quasi-Fuchsian reps)
- key tool given by a boundary map to a flag variety

$$\xi_\rho : \partial\pi_1 S \to \mathrm{G/P}$$

The idea of Positivity

Key tool given by boundary map $\xi_\rho : \partial\pi_1 S \to \mathsf{G}/\mathsf{P}$

Guichard-Wienhard '18

For some very special and classified pairs $(\mathsf{G}, \mathsf{P}_\Theta)$, generic triples in $\mathsf{G}/\mathsf{P}_\Theta$ have a "cyclic order"

$$\rightsquigarrow \text{Positive Anosov reps} : \mathcal{A}^{\Theta\text{-}pos} \subset \mathcal{X}(S, \mathsf{G})$$

open and conjectured to also be closed.

Conjecture: Guichard-Labourie-Wienhard

The set $\mathcal{A}^{\Theta\text{-}pos}$ is closed and defines all higher Teichmüller spaces.

Slightly stronger and now known to be true in many cases:

- $\mathcal{A}^{\Theta\text{-}pos}$ are exactly the higher Teich spaces
- Every component of $\mathcal{X}(S, \mathsf{G}) \setminus \mathcal{A}^{\Theta\text{-}pos}$ is labeled by the topological invariant $\tau \in \pi_1 \mathsf{G}$.

Possible strategy: Translate notion of positivity into the language of Higgs bundles prove closed. Too hard...

Alternative

Come up with a different Lie theory notion (magical $\mathfrak{sl}_2$-triples), adapted to Higgs bundle language and prove a theorem.

Two results joint with Bradlow, Garcia-Prada, Gothen, Oliveira:

- Classification of magical $\mathfrak{sl}_2$-triples, which agrees with classification of positive structures,
- Higgs Slodowy slice construction defines components of $\mathcal{M}(\mathsf{G})$ containing positive representations.

Using these Higgs bundles results, Guichard-Labourie-Wienhard proved the Higgs components are higher Teichmüller spaces.

What's a magical $\mathfrak{sl}_2$-triple

- Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra, and $\{f, h, e\} \subset \mathfrak{g}_{\mathbb{C}}$ be an $\mathfrak{sl}_2$-triple.
- $V = \ker(\mathrm{ad}_e) \subset \mathfrak{g}_{\mathbb{C}}$ highest weight spaces

$$V = V_0 \oplus V_+,$$

  0 and positive ad_h-weight spaces.
- Define a **vector space** involution $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ by

$$\sigma_e(f) = -f \qquad \sigma_e|_{V_0} = +\mathrm{Id}$$

  and $\sigma_e(\mathrm{ad}_f^j(v)) = (-1)^{j+1}\mathrm{ad}_f^j(v)$ for $v \in V_+$.

Definition

$\{f, h, e\} \subset \mathfrak{g}$ is magical if σ_e is a Lie algebra homomorphism.

Note, magical defines a real form $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$

$$f + V_+ \subset \mathfrak{p}_{\mathbb{C}} \qquad \text{and} \qquad V_0 \subset \mathfrak{k}_{\mathbb{C}}\ .$$

Theorems

Theorem (Bradlow, C, Garcia-Prada, Gothen, Oliveira '21)

For each magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}_{\mathbb{C}}$ with associated real form G*, there are components $\mathcal{P}_e(\mathrm{G}) \subset \mathcal{X}(S, \mathrm{G})$, such that*

1. $\mathcal{P}_e(\mathrm{G})$ *contains positive representations.*
2. $\mathcal{P}_e(\mathrm{G})$ *does not contain compact representations.*
3. $\mathcal{P}_e(\mathrm{G})$ *does not contain representations factoring through proper parabolic subgroups.*

The components are constructed via a Higgs bundle 'Slodowy slice' through magical $\mathfrak{sl}_2$-triples. Proofs are harder because the parameter space is itself a moduli space.

Theorem (Guichard, Labourie, Wienhard '21)

Properties 1. and 3. above imply the spaces $\mathcal{P}_e(\mathrm{G}^{\mathbb{R}})$ are higher Teichmüller spaces.

Classification of magical $\mathfrak{sl}_2$

Theorem (BCGGO '21)

There are four families of magical $\mathfrak{sl}_2$-triples, the associated real groups are

1. G *is a split real group (e.g,* $\mathsf{SL}_n\mathbb{R}$*)*
2. G *is a Hermitian group of tube type (e.g.* $\mathsf{SU}(n,n)$*)*
3. $\mathsf{G} \cong \mathsf{SO}(p,q)$ *for* $1 < p < q$
4. G *is a quaternionic real form of* $\mathsf{F}_4, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$.

This is the same as Guichard-Wienhard's list of groups which have a notion of positivity.

The $\mathfrak{sl}_2's$

1. Principal $\mathfrak{sl}_2$ in $\mathfrak{g}$
2. $\mathfrak{sl}_2$ given holomorphic $\mathbb{D} \to \mathsf{G}/\mathsf{K}$ with maximal holomorphic sectional curvature.
3. Principal $\mathfrak{sl}_2$ in $\mathfrak{so}(2p+1,\mathbb{C}) \subset \mathfrak{so}(p+q,\mathbb{C})$
4. Principal in $\mathfrak{g}_2 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$.

Example: $e = \begin{pmatrix} 0 & \mathrm{Id}_n \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_{2n}\mathbb{C}$

$$\rightsquigarrow \{f, h, e\} = \left\{ \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_n & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{Id}_n & \\ & -\mathrm{Id}_n \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id}_n \\ 0 & 0 \end{pmatrix} \right\}$$

	$\mathfrak{g}_{-2}$	$\mathfrak{g}_0$	$\mathfrak{g}_2$
W_2 $n_2 = n^2$	$\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ $-$	$\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$ $+$	$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = V_+$ $-$
W_0 $n_0 = n^2 - 1$		$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$ $+$ V_0	

$$\sigma_e : \mathfrak{sl}_{2n}\mathbb{C} \to \mathfrak{sl}_{2n}\mathbb{C} \rightsquigarrow \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

$$\dim(\mathfrak{h}^{\mathbb{C}}) = 2n^2 - 1 \qquad \Rightarrow \qquad \mathfrak{g} = \mathfrak{su}_{n,n}.$$