Higher Teichmüller spaces and Higgs bundles

Brian Collier
University of California Riverside

$E_7$:
Character varieties

- $S$ a closed oriented smooth surface of genus $g \geq 2$
- $G$ a real or complex connected simple Lie group (e.g. $\text{SL}_n\mathbb{C}$)

Define the $G$-character variety of $S$ by

$$\mathcal{X}(S, G) = \text{Hom}^+(\pi_1 S, G)/G$$

conjugacy classes of completely reducible reps $\rho : \pi_1 S \to G$.

Questions for today:
- How many components does $\mathcal{X}(S, G)$ have?
- Are some components more interesting than others?

Example: For $\text{PSL}_2\mathbb{R} = \text{Isom}^+(\mathbb{H}^2)$,
  - $|\pi_0(\mathcal{X}(S, \text{PSL}_2\mathbb{R}))| = 4g - 3$ (Goldman 88)
  - two of them consist entirely of “discrete and faithful reps.”

$\sim \text{Teich}(S) \cup \text{Teich}(\overline{S}) \subset \mathcal{X}(S, \text{PSL}_2\mathbb{R})$ open and closed subset
Some known results on components

Given a representation $\rho : \pi_1 S \to G$, we can build a flat $G$-bundle

$$E_\rho = \tilde{S} \times_{\rho} G \to S$$

$$\sim\tau : \pi_0(\mathcal{X}(S, G)) \to H^2(S, \pi_1 G) \cong \pi_1 G.$$ 

- If $G$ is **compact**, then $\tau$ is a bijection. (Narasimhan-Sheshadri ’65, Ramanathan ’75)

- If $G$ is **complex** then $\tau$ is a bijection. (J. Li ’94)

**Corollary:** If $K < G$ is maximal compact, then every $\rho : \pi_1 S \to G$ can be deformed to $\rho' : \pi_1 S \to K \hookrightarrow G$.

- If $G$ is a **split real** Lie group (e.g. $\text{PSL}_n \mathbb{R}$), then there exists $\rho \in \mathcal{X}(\pi_1 S, G)$ which cannot be deformed to a compact representation. (Hitchin ’91)
Definition

A higher rank Teichmüller space is a connected component of \( \mathcal{X}(S, G) \) consisting entirely of discrete and faithful representations.
Higgs bundles and Nonabelian Hodge

Let $X$ be a Riemann surface structure on $S$. Fixing this data, we get a moduli space $\mathcal{M}(X, G)$ of polystable $G$-Higgs bundles on $X$.

Theorem (Hitchin, Donaldson, Simpson, Corlette)

*There is a real analytic isomorphism*

$$\mathcal{T} : \mathcal{M}(X, G) \rightarrow \mathcal{X}(S, G).$$

*So, these spaces have the same number of components. Provides more tools to study topology, but breaks symmetry.*

Works by relating stability to existence of a special metric.

- (Hitchin, Simpson) On Higgs bundle side, a metric that solves a gauge theoretic equations $F_h + [\Phi, \Phi^*h] = 0$.

- (Corlette, Donaldson) On character variety side, an equivariant map $h_\rho : \tilde{X} \rightarrow G/K$ to the symmetric space which is harmonic.
What is a Higgs bundle

For \( G = \text{GL}_n \mathbb{C} \), a Higgs bundle is a pair \((\mathcal{E}, \Phi)\), where

- \( \mathcal{E} \rightarrow X \) is a rank \( n \) holomorphic vector bundle of degree 0,
- \( \Phi \in \Omega^{1,0}(X, \text{End}(\mathcal{E})) \) is holomorphic, \( \Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^{1,0}_X \).

For \( G \) complex, a Higgs bundle is a pair \((\mathcal{E}, \Phi)\), where

- \( \mathcal{P}_G \rightarrow X \) is a holomorphic principal \( G \)-bundle,
- \( \Phi \in \Omega^{1,0}(X, \text{ad}(\mathcal{P}_G)) \) is holomorphic.

Slope stability for \( \Phi \)-invariant reductions \( \sim \) moduli space \( \mathcal{M}(X, G) \) of semistable \( G \)-Higgs bundles.

\((\mathcal{E}, \Phi)\) polystable if and only if there is a metric \( h \) on \( \mathcal{E} \) such that

\[
F_h + [\Phi, \Phi^*h] = 0
\]

\( \sim \) \( A_h + \Phi + \Phi^*h \) is a flat \( G \) connection
Higgs for a real group G

For G real with maximal compact K, set $\mathfrak{p} = \mathfrak{k}^\perp \cong T_K G/K$,

$\rightsquigarrow K_C$ and $K_C$-invariant $\mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_C$.

A G-Higgs bundle is a pair $(\mathcal{E}_K^C, \Phi)$, where

$\Rightarrow \mathcal{E}_K^C \to X$ is a holomorphic principal $K_C^\mathbb{C}$-bundle

$\Rightarrow \Phi \in \Omega^{1,0}(\mathcal{E}_K^C \times_K \mathfrak{p}_C^\mathbb{C})$ which is holomorphic.

$\Rightarrow \Phi$ is identified with $dh_1^{1,0}$ of harmonic map.

For compact groups $\Phi = 0$, hence $\rho \in \mathcal{X}(G)$ factors through $K$ if and only if $T(\rho) \in \mathcal{M}(G)$ has $\Phi = 0$.

For $G = \text{SL}_n\mathbb{R}$, we have $K = \text{SO}(n)$ and $\mathfrak{sl}_n\mathbb{R} = \mathfrak{so}(n) \oplus \text{sym}_0(\mathbb{R}^n)$.

So, an $\text{SL}_n\mathbb{R}$-Higgs bundle is tuple $(\mathcal{E}, Q, \Phi)$, where

$\Rightarrow \mathcal{E}$ is a holomorphic rank $n$ bundle equipped with a symmetric isomorphism $Q : \mathcal{E} \to \mathcal{E}^*$,

$\Rightarrow \Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega^{1,0}_X$ satisfying $\Phi^T Q = Q\Phi$ and $\bar{\partial}_E \Phi = 0$. 
Teichmüller space from SL\(_{2\mathbb{R}}\)-Higgs bundles

An SL\(_n\mathbb{R}\)-Higgs bundle \((E, Q, \Phi)\)
- \(E\) is a holomorphic rank \(n\) bundle equipped with a symmetric isomorphism \(Q : E \to E^*\),
- \(\Phi : E \to E \otimes \Omega_X^{1,0}\) satisfying \(\Phi^T Q = Q \Phi\) and \(\bar{\partial}_E \Phi = 0\).

When \(n = 2\), set \(K = \Omega_X^{1,0}\) and consider \((E, Q, \Phi)\) given by
\[
E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : E \to E^* \quad \Phi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : E \to E \otimes K
\]

To this example, we can add \(q \in H^0(K^2) \cong \mathbb{C}^{3g-3}\)
\[
(E, Q, \Phi) \longrightarrow (E, Q, \Phi + \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix})
\]

Theorem (Hitchin '87)
- \(H^0(K^2) \to \mathcal{M}(\text{SL}_2\mathbb{R})\) is injective, open and closed map.
- Under the identification \(\mathcal{T} : \mathcal{M}(\text{SL}_2\mathbb{R}) \to \mathcal{X}(\text{SL}_2\mathbb{R})\), this component identifies with Teichmüller space \(\text{Teich}(S)\).
Aside on nilpotents and Slodowy Slices

Consider $\mathfrak{g}$ a complex simple Lie algebra and the nilpotent cone

$$N_\mathfrak{g} \subset \mathfrak{g}$$

Jacobson-Morozov Thm implies every $f \in N_\mathfrak{g} \setminus \{0\}$ can be completed to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$, where

$$[h, f] = -2f \quad [h, e] = 2e \quad [e, f] = h$$

**Slodowy slice** through $f$ is a linear slice for through $G \cdot f$ parameterized by the vector space $V_e = \ker(\text{ad}_e)$.

$$S_f = f + V_e.$$ 

For $\mathfrak{sl}_2 \mathbb{C}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and

$$S_f = f + \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$
Hitchin component

Idea: Use principal embedding of $\text{SL}_2 \mathbb{C} \to G^\mathbb{C}$ to embed $\text{Teich}(S)$ Higgs bundles into $G^\mathbb{C}$-Higgs and consider a “Slodowy slice”

For $\text{SL}_n \mathbb{C}$, take the irr. action of $\text{SL}_2 \mathbb{C}$ on $\mathbb{C}^n = \text{Sym}^{n-1}(\mathbb{C}^2)$.

For $n = 3$, $f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Now add to highest weight spaces, this is an $\text{SL}_3 \mathbb{R}$-Higgs bundle

Hitchin ’91

$H^0(K^2) \oplus \cdots \oplus H^0(K^n) \to M(\text{SL}_n \mathbb{R})$ is injective, open and closed.

For general complex $G$, this gives an injective, open, closed map

$$\bigoplus_{j=1}^{r_k q} H^0(K^{m_j+1}) \to M(G_{\text{split}}^\mathbb{R})$$
The idea of Labourie’s Anosov condition

This component is called the **Hitchin component**. How much does the Hitchin component generalize Teich(S)?

**Theorem (Labourie ’06, Fock-Goncharov ’06)**

*The Hitchin component consists entirely of discrete and faithful representations, and is hence a higher rank Teichmüller space.*

Labourie developed the notion of Anosov representations.

Facts: There is a class of reps $\mathcal{A} \subset \mathcal{X}(S, G)$ called **Anosov** which generalize many features of Teich(S):

- $\rho \in \mathcal{A} \Rightarrow$ discrete and faithful
- $\rho \in \mathcal{A} \Rightarrow$ holonomies of geometric structures
- $\mathcal{A}$ is open in $\mathcal{X}(S, G)$ BUT not necessarily closed (generalization of quasi-Fuchsian reps)
- key tool given by a boundary map to a flag variety

$$\xi_\rho : \partial \pi_1 S \to G/P$$
The idea of Positivity

Key tool given by boundary map $\xi_\rho : \partial \pi_1 S \to G/P$

Guichard-Wienhard ’18
For some very special and classified pairs $(G, P_\Theta)$, generic triples in $G/P_\Theta$ have a “cyclic order”

$\leadsto$ Positive Anosov reps : $\mathcal{A}^{\Theta-pos} \subset \mathcal{X}(S, G)$

open and conjectured to also be closed.

Conjecture: Guichard-Labourie-Wienhard
The set $\mathcal{A}^{\Theta-pos}$ is closed and defines all higher Teichmüller spaces.

Slightly stronger and now known to be true in many cases:
- $\mathcal{A}^{\Theta-pos}$ are exactly the higher Teich spaces
- Every component of $\mathcal{X}(S, G) \setminus \mathcal{A}^{\Theta-pos}$ is labeled by the topological invariant $\tau \in \pi_1 G$. 
Possible strategy: Translate notion of positivity into the language of Higgs bundles prove closed. Too hard...

**Alternative**

Come up with a different Lie theory notion (magical $\mathfrak{sl}_2$-triples), adapted to Higgs bundle language and prove a theorem.

Two results joint with Bradlow, Garcia-Prada, Gothen, Oliveira:

- Classification of magical $\mathfrak{sl}_2$-triples, which agrees with classification of positive structures,
- Higgs Slodowy slice construction defines components of $\mathcal{M}(G)$ containing positive representations.

Using these Higgs bundles results, Guichard-Labourie-Wienhard proved the Higgs components are higher Teichmüller spaces.
What’s a magical $\mathfrak{sl}_2$-triple

Let $\mathfrak{g}_C$ be a complex simple Lie algebra, and $\{f, h, e\} \subset \mathfrak{g}_C$ be an $\mathfrak{sl}_2$-triple.

$V = \ker(\text{ad}_e) \subset \mathfrak{g}_C$ highest weight spaces

$$V = V_0 \oplus V_+,$$

0 and positive $\text{ad}_h$-weight spaces.

Define a vector space involution $\sigma_e : \mathfrak{g} \to \mathfrak{g}$ by

$$\sigma_e(f) = -f \quad \sigma_e|_{V_0} = +\text{Id}$$

and $\sigma_e(\text{ad}_f^j(v)) = (-1)^{j+1}\text{ad}_f^j(v)$ for $v \in V_+$.

Definition

$\{f, h, e\} \subset \mathfrak{g}$ is magical if $\sigma_e$ is a Lie algebra homomorphism.

Note, magical defines a real form $\mathfrak{g} \subset \mathfrak{g}_C$ with $\mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_C$

$$f + V_+ \subset \mathfrak{p}_C \quad \text{and} \quad V_0 \subset \mathfrak{k}_C.$$
Theorems

Theorem (Bradlow, C, Garcia-Prada, Gothen, Oliveira '21)

For each magical $\mathfrak{sl}_2$-triple $\{f, h, e\} \subset \mathfrak{g}_{\mathbb{C}}$ with associated real form $G$, there are components $\mathcal{P}_e(G) \subset \mathcal{X}(S, G)$, such that

1. $\mathcal{P}_e(G)$ contains positive representations.
2. $\mathcal{P}_e(G)$ does not contain compact representations.
3. $\mathcal{P}_e(G)$ does not contain representations factoring through proper parabolic subgroups.

The components are constructed via a Higgs bundle ‘Slodowy slice’ through magical $\mathfrak{sl}_2$-triples. Proofs are harder because the parameter space is itself a moduli space.

Theorem (Guichard, Labourie, Wienhard '21)

Properties 1. and 3. above imply the spaces $\mathcal{P}_e(G^\mathbb{R})$ are higher Teichmüller spaces.
Classification of magical $\mathfrak{sl}_2$

Theorem (BCGGO '21)

There are four families of magical $\mathfrak{sl}_2$-triples, the associated real groups are

1. $G$ is a split real group (e.g. $\text{SL}_n\mathbb{R}$)
2. $G$ is a Hermitian group of tube type (e.g. $\text{SU}(n, n)$)
3. $G \cong \text{SO}(p, q)$ for $1 < p < q$
4. $G$ is a quaternionic real form of $F_4, E_6, E_7, E_8$.

This is the same as Guichard-Wienhard’s list of groups which have a notion of positivity.

The $\mathfrak{sl}_2's$

1. Principal $\mathfrak{sl}_2$ in $\mathfrak{g}$
2. $\mathfrak{sl}_2$ given holomorphic $\mathbb{D} \to G/K$ with maximal holomorphic sectional curvature.
3. Principal $\mathfrak{sl}_2$ in $\mathfrak{so}(2p + 1, \mathbb{C}) \subset \mathfrak{so}(p + q, \mathbb{C})$
4. Principal in $\mathfrak{g}_2 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$. 
Example: $e = \begin{pmatrix} 0 & \text{Id}_n \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_{2n}\mathbb{C}$

$\rightsquigarrow \{f, h, e\} = \left\{ \begin{pmatrix} 0 & 0 \\ \text{Id}_n & 0 \end{pmatrix}, \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix}, \begin{pmatrix} 0 & \text{Id}_n \\ 0 & 0 \end{pmatrix} \right\}$

<table>
<thead>
<tr>
<th></th>
<th>$\mathfrak{g}_{-2}$</th>
<th>$\mathfrak{g}_0$</th>
<th>$\mathfrak{g}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_2$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ C &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} B &amp; 0 \ 0 &amp; -B \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; A \ 0 &amp; 0 \end{pmatrix} = V_+$</td>
</tr>
<tr>
<td>$n_2 = n^2$</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$\begin{pmatrix} D &amp; 0 \ 0 &amp; D \end{pmatrix}$</td>
<td></td>
<td>$V_0$</td>
</tr>
<tr>
<td>$n_0 = n^2 - 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\sigma_e : \mathfrak{sl}_{2n}\mathbb{C} \rightarrow \mathfrak{sl}_{2n}\mathbb{C} \rightsquigarrow \mathfrak{h}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C}$

$\dim(\mathfrak{h}^\mathbb{C}) = 2n^2 - 1 \Rightarrow \mathfrak{g} = \mathfrak{su}_{n,n}$. 