1 Introduction

A basic and surprisingly rich question in graph theory is: how many graphs are there with a given degree sequence? That is, given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of nonnegative integers, we write $\mathcal{G}(\mathbf{d})$ for the set of all simple graphs on vertex set $[n]$ where vertex i has degree d_i, and we are interested in estimating or counting $|\mathcal{G}(\mathbf{d})|$. As a special case, we might ask how many (labelled) d-regular graphs there are on vertex set $[n]$ – this question turns out to be highly nontrivial for $d \geq 3$. Write $\mathcal{G}(n, d)$ for $\mathcal{G}(\mathbf{d})$ if $\mathbf{d}$ is the length-n sequence with d in every component.

This sort of counting problem comes up in a number of places. In random graph theory, for instance, we might want to know how likely a particular degree sequence is to appear in $G(n, m)$ or in $G(n, p)$. In applied settings, the degree sequence of a network is often easy to observe, and so many models fix the degrees and ask what a "typical" graph with those degrees looks like.

There are a few exact counting results, like the number of labelled trees (Cayley's formula), or the number of graphs with a fixed number of edges. For regular graphs, we know feasible exact formulae only when $d = 1$ or $d = 1$. The number of (labelled) 1-regular graphs, that is, perfect matchings, on vertex set $[n]$ is equal to the number of ways to partition the vertices into unordered pairs, namely, 0 if n is odd, and $(n-1)(n-3) \cdot \ldots \cdot 1 = \frac{n!}{2^{n/2}(n/2)!}$ if n is even. The number of 2-regular graphs corresponds to the number of ways to decompose the vertex set into disjoint cycles (of length at least 3):

$$|\mathcal{G}(n, 2)| = \sum_k \frac{1}{k!} \sum_{\substack{\ell_1 + \cdots + \ell_k = n \\ \ell_i \geq 3}} \prod_{i=1}^{k} \frac{(\ell_i - 1)!}{2}.$$

For regular graphs of higher degrees, Read in the 1950s gave a recurrence for the number of 3-regular graphs, and others extended this to higher degrees or used generating functions. Read extracted an asymptotic formula for $|\mathcal{G}(n, 3)|$ from his recurrence, but for larger d, this was not deemed feasible by various people, including Read, Wormald and Bollobás.

A major step forward came in the late 1970s and early 1980s with work by Bender & Canfield, by Wormald and by Bollobás, who introduced the configuration model and showed how to use it as an approximation for sparse regular graphs. In the 1980s, McKay introduced switchings that, together with the configuration model, give more precise estimates for $d = o(n^{1/3})$. Then in 1989, McKay and Wormald introduced more elaborate switchings that allowed them to find an asymptotic formula for $\mathcal{G}(n, d)$ for any $d = o(n^{1/2})$. This approach can be generalised to irregular sequences, and then the condition on d becomes a condition on the maximum degree. The approach via the configuration model does not seem to extend beyond $n^{1/2}$.

Also in 1989, McKay and Wormald gave an asymptotic formula for $\mathcal{G}(n, d)$ when $d = \Omega(n/\log n)$. This approach involves expressing the count as a multivariate coefficient in a generating function,

$$|\mathcal{G}(n, d)| = [x_1^d \cdots x_n^d] \prod_{1 \leq i < j \leq n} (1 + x_i x_j),$$

and evaluating it asymptotically using Cauchy's integral formula and the so-called saddle-point method.

Strikingly, McKay and Wormald observed that the formulae for the sparse range of d and for the dense range of d can be cast into a common form. Moreover, they showed that the existing enumeration formulae for $|\mathcal{G}(\mathbf{d})|$ implied that the degree sequence of the random graph $G(n, m)$ can be approximated by a product of independent binomials (conditioning on the correct degree sum).

The gap between $o(n^{1/2})$ and $\Omega(n/\log n)$ was patched recently using a completely new approach via recursive relations and contracting maps.

The goal of these lectures is to give an overview of these three methods, with an emphasis on the new method since it is applicable to the sparse range, as well as to other, similar, enumeration problems.

2 Notation

By $\mathbf{d}$ we usually denote a sequence $(d_1, \ldots, d_n)$, with n understood, of non-negative integers. The *total degree* of $\mathbf{d}$, denoted by $M_1 = M_1(\mathbf{d})$, is $\sum_i d_i$. We call $\mathbf{d}$ *even* if the total degree is even, and *odd* otherwise. Call a sequence *graphical* if there exists a graph G on vertex set $[n]$ such that $d_G(i) = d_i$. Being even is necessary for a sequence to be graphical, but we shall encounter odd sequences in this course as well.

Given a graph G on vertex set $[n]$, we denote by $D(G)$ its degree sequence.

3 The configuration model

Let $C_1, \dots, C_n$ be pairwise disjoint sets of size d, called *cells* or *degree boxes*. Elements of $V := \bigcup_i C_i$ are called *points*. A *configuration*, or a *pairing*, is a perfect matching P of the dn points in V.

Given a configuration P, form a multigraph $G(P)$ on vertex set $[n]$ by contracting every cell C_i into one vertex i and "keeping all the edges". Formally, for every pair xy in P, add an edge e_{xy} to $G(P)$ with endpoints i and j if $x \in C_i$ and $y \in C_j$.

Fact 3.1. *If $G \in \mathcal{G}(n, d)$, that is G is a simple (= no loops nor multiple edges) d-regular graph on n vertices, then $G = G(P)$ for exactly $(d!)^n$ configurations P.*

This is an important fact, since it implies that if we pick P uniformly at random (u.a.r.) from all configurations, then all simple graphs are equally likely to appear. That is, if we denote by Simple the event that the multigraph $G(P)$ is simple, then

$$|\mathcal{G}(n, d)| = \frac{\#\text{ pairings}}{(d!)^n} \cdot \Pr(\text{Simple}) = \frac{(dn)!}{\left(\frac{dn}{2}\right)!\, 2^{dn/2}\, (d!)^n} \cdot \Pr(\text{Simple}), \tag{1}$$

where the probability is over P chosen u.a.r. from all configurations.

We note here that a uniform configuration can be chosen cheaply: start with an arbitrary point $x \in V$ and choose a uniform random $y \in V \setminus \{x\}$ to be paired with x. Next, pick an arbitrary $w \in V \setminus \{x, y\}$ and choose $z \in V \setminus \{x, y, w\}$ uniformly at random to be paired with w. Keep going. This procedure will result in a uniform random matching on V. It also shows the following.

Lemma 3.2. *For an integer $k \ge 1$ and distinct points $x_1, \dots, x_k, y_1, \dots, y_k \in V$, the probability that all pairs $x_i y_i$ are pairs in a uniform random configuration P is equal to*

$$\prod_{i=0}^{k-1} \frac{1}{nd - 2i - 1} = \frac{1}{(nd)^k} \left(1 + O\left(\frac{k}{nd}\right)\right).$$

Proof. Exercise. □

3.1 A formula for constant d

Theorem 3.3 (Bollobás 1983)**.** *Let $d \ge 3$ be a fixed integer. Then for $n \in \mathbb{N}$ such that dn is even,*

$$\Pr(\text{Simple}) = e^{-\frac{d^2-1}{4} + o(1)}.$$

What is the meaning of this formula, how can we interpret it?

For a configuration P, let X_1 denote the number of pairs xy that create a loop in $G(P)$, and let X_2 denote the pairs of pairs (x_1y_1, x_2y_2) that create a double edge in $G(P)$ if they are present in P. Then $X := X_1 + X_2$ is the number of bad events of P, and

$$\Pr(\text{Simple}) = \Pr(X = 0).$$

Now, intuitively, the bad events − a pair forming a loop or a pair of pairs forming a double edge − are individually rare and approximately independent when d is constant and n is large. It is therefore conceivable that X behaves approximately like a Poisson random variable. The heuristic is this: If $\mathbb{E}(X) \to \lambda = O(1)$, then under suitable conditions, X converges in distribution to Poisson(λ), so $\Pr(X = 0) \to e^{-\lambda}$ as $n \to \infty$. Let us calculate the expectation of X. We have,

$$\mathbb{E}(X_1) = n\binom{d}{2}\frac{1}{dn-1} = \frac{d-1}{2}(1 + o(1)),$$

and

$$\mathbb{E}(X_2) = \binom{n}{2}\binom{d}{2}^2 \cdot \frac{2}{(dn-1)(dn-3)} = \frac{(d-1)^2}{4}(1+o(1)).$$

So indeed, $\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = \frac{d^2-1}{4} + o(1)$.

The above heuristic can be made rigorous using the *method of moments*.

Lemma 3.4 (Method of moments)**.** *Let $\lambda_1, \ldots, \lambda_k$ be some set of fixed non-negative reals, and let $X_{1,n}, \ldots, X_{k,n}$ be non-negative integer random variables defined on the same space Ω_n for each n. If*

$$\mathbb{E}\left(\prod_{i=1}^{k}[X_{i,n}]_{r_i}\right) \to \prod_{i=1}^{k}\lambda_i^{r_i}$$

as $n \to \infty$, for each fixed set of non-negative integers $r_1, \ldots, r_k$, then the variables $X_{1,n}, \ldots, X_{k,n}$ are asymptotically independent Poisson with means λ_i.

For more information on the method of moments, see, for example, Chapter 6 of [Random Graphs by Janson, Łuczak and Ruciński], where this lemma can be found, or Chapter 11 in [Random Graphs by Bollobás].

For $k \geq 1$, let $X_k = X_{k,n}$ denote the number of k-tuples of pairs in P that create a cycle of length k in P, where we use the convention that a cycle of length 1 is a loop, and a cycle of length 2 is a double edge.

Lemma 3.5 (Bollobás 1983)**.** *For fixed d and $k \geq 1$, the random variables $X_1, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{(d-1)^i}{2i}$.*

Proof. Sketch for $k = 2$ in the lecture. □

In particular, this implies that

$$\Pr(\text{Simple}) = \Pr(X_1 = 0 \text{ and } X_2 = 0) \to e^{-\lambda_1-\lambda_2} = e^{-(d^2-1)/4}.$$

3.2 How far can we stretch this method?

The proof of Bollobás works not only for constant d, but in fact for d tending to ∞ with n, as long as $d \leq \sqrt{2\log n}$). McKay introduced so called simple switchings which allowed him to give a refined estimate on Pr(Simple) which is asymptotically correct for $d = o(n^{1/3})$. More elaborate switchings due to McKay and Wormald yield a formula which is asymptotically correct for all $d = o(n^{1/2})$.

All proofs involving the configuration naturally adapt to irregular sequences, where the bound on d is replaced by an appropriate bound using the maximum degree $\Delta(\mathbf{d}) := \mathbf{d}_{\max}$. The most general asymptotic formula that has so far been obtained via the configuration model is the following.

For a degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ and fixed $i \geq 1$, write $M_k = M_k(\mathbf{d})$ for the kth factorial moment, that is, $M_k = \sum_i (d_i)_k = \sum_i d_i(d_i-1)\ldots(d_i-k+1)$. Note that M_1 is then just the total degree dn.

Theorem 3.6 (McKay and Wormald 1991)**.** *Let $\mathbf{d} = (d_1, \ldots, d_n)$ be a sequence of non-negative integers such that M_1 is even and such that $\Delta(\mathbf{d}) := \max_i d_i = o(M_1^{1/3})$. Then $|\mathcal{G}(\mathbf{d})|$ is asymptotically equal to*

$$\frac{M_1!}{(M_1/2)!\,2^{M_1/2}\prod_{i=1}^{n} d_i!}\,\exp\left(-\frac{M_2}{2M_1} - \frac{M_2^2}{2M_1^2} - \frac{M_2^2M_3}{2M_1^4} + \frac{M_2^4}{4M_1^5} + \frac{M_3^2}{6M_1^3} + O\left(\frac{\Delta^3}{M}\right)\right). \tag{2}$$

4 The binomial approximation conjecture

McKay and Wormald realised in 1989 that the above formula can be rewritten in the following way. Assuming that $dn \to \infty$ and under the conditions of Theorem 3.6, the formula in (2) is equivalent to

$$|\mathcal{G}(\mathbf{d})| = \frac{\binom{\binom{n}{2}}{m}\prod_i\binom{n-1}{d_i}}{\binom{n(n-1)}{2m}} \cdot \exp\left(\frac{1}{4} - \frac{\gamma_2(\mathbf{d})^2}{4\mu^2(1-\mu)^2} + o(1)\right), \tag{3}$$

where $m = \sum_i d_i/2$ is the number of edges of a graph with degree sequence $\mathbf{d}$, $\mu = \mu(\mathbf{d}) = m/\binom{n}{2}$ is the edge density of such a graph, and

$$\gamma_2 = \gamma_2(\mathbf{d}) = \frac{\sum_i (d_i - d)^2}{(n-1)^2}$$

is a scaled variance.

This formula has a nice interpretation. Denote by $G_{n,m}$ a graph that is drawn uniformly at random from $\mathcal{G}(n,m)$, the set of all graphs on vertex set $[n]$ and with exactly m edges. Note that the probability that $G_{n,m}$ has degree sequence $\mathbf{d}$ is exactly $|\mathcal{G}(\mathbf{d})|/\binom{\binom{n}{2}}{m}$, assuming that $m = \sum_i d_i/2$.

Now let $\mathcal{B}_p(n)$ denote a vector $(X_1, \ldots, X_n)$ of independent random variables $X_i \sim \text{Bin}(n-1, p)$, for some $p \in [0,1]$, and let $\mathcal{B}_m(n)$ denote $\mathcal{B}_p(n)$ conditioned on $\sum_i X_i = 2m$. Note that the choice of p is irrelevant in $\mathcal{B}_m(n)$. Then

$$\Pr(\mathcal{B}_m(n) = \mathbf{d}) = \frac{\prod_i \binom{n-1}{d_i}}{\binom{n(n-1)}{2m}}.$$

With this notation, we see that (3) is equivalent to

$$\Pr(D(G_{n,m}) = \mathbf{d}) = \Pr(\mathcal{B}_m(n) = \mathbf{d}) \cdot \exp\left(\frac{1}{4} - \frac{\gamma_2(\mathbf{d})^2}{4\mu^2(1-\mu)^2} + o(1)\right). \tag{4}$$

Interestingly, McKay and Wormald showed via completely different methods that the same formula holds for dense and nearly-regular sequences, that is for even sequences $\mathbf{d}$ such that the average d satisfies $d = \Omega(n)$ and $|d_i - d| \le d^{1/2+\varepsilon}$, for some fixed $\varepsilon > 0$. Naturally, they conjectured that (4) should also hold for all sequences $\mathbf{d}$, as long as they are either sparse ($\Delta = o(M_1^3)$)) or they are nearly regular, meaning that $\max_i |d_i - d| \le \max\{n^\varepsilon, d^{1/2+\varepsilon}\}$, for some fixed $\varepsilon > 0$.

5 The intermediate range: Example when $d = o(n^{2/5}/\log n)$.

Our goal is to find an asymptotic formula for $|\mathcal{G}(d,n)|$ when d is polynomial in n. For this middle range, we follow a completely new approach. To explain the core ideas of the new method, we will first concentrate on $d = o(n^{2/5}/\log n)$, and then explain how these ideas can be extended to larger d. With this approach, we do not just find the number of d-regular n-vertex graphs asymptotically, but simultaneously $|\mathcal{G}(\mathbf{d})|$ for a whole set of sequences $\mathbf{d}$ that have average degree d.

We have seen above that for any m and a sequence $\mathbf{d}$ of length n and total degree $2m$ we have

$$\Pr(D(G_{n,m}) = \mathbf{d}) = \frac{|\mathcal{G}(\mathbf{d})|}{|\mathcal{G}(n,m)|}. \tag{5}$$

The degree sequence $D(G_{n,m})$ of the random graph defines a probability distribution on the set

$$\Omega_m := \{\mathbf{d} \in \mathbb{Z}^n_{\ge 0} : \sum_i d_i = 2m\}.$$

Some $\mathbf{d}$ in Ω_m may have probability 0 if it is not graphical. Now finding an (asymptotic) formula for the number of d-regular n-vertex graphs is equivalent to finding an (asymptotic) formula for the point probability of $\mathbf{d} = (d, \ldots, d)$ in this probability space.

Of course, this is just a reformulation of the original problem, but the probabilistic view can be helpful. The following simple lemma encapsulates the main idea. For a set S and $\xi > 0$, we say that $f(s) = O(\xi)$ *uniformly for all* $s \in S$ if there exists a constant $C > 0$, independent of $s \in S$, such that $|f(s)| \le C\xi$ for all $s \in S$.

Lemma 5.1. *Let $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$ be sets of non-negative real numbers, and let $\xi = \xi(N) > 0$. If $\sum_i x_i = (1 + O(\xi)) \sum_i y_i$, and*

$$\frac{x_i}{x_j} = \frac{y_i}{y_j}(1 + O(\xi))$$

uniformly for all $i, j \in [N]$, then $x_i = y_i(1 + O(\xi))$ uniformly for all $i \in [N]$.

The idea here is to approximate $x_{\mathbf{d}} = \Pr(D(G_{n,m}) = \mathbf{d})$ by some formula $y_{\mathbf{d}}$, for all $\mathbf{d}$ in some set $\mathcal{D}$ for which we can calculate the sum $\sum_{\mathbf{d}\in\mathcal{D}} \Pr(D(G_{n,m}) = \mathbf{d})$.

Goal:

Find a set $\mathcal{D}$ and formulae $y_{\mathbf{d}}$ for all $\mathbf{d} \in \mathcal{D}$ such that

$$\frac{\Pr(D(G_{n,m}) = \mathbf{d})}{\Pr(D(G_{n,m}) = \mathbf{d}')} = \frac{y_{\mathbf{d}}}{y_{\mathbf{d}'}}(1 + o(1)) \text{ for all } \mathbf{d}, \mathbf{d}' \in \mathcal{D}, \tag{6}$$

such that

$$\sum_{\mathbf{d}\in\mathcal{D}} \Pr(D(G_{n,m}) = \mathbf{d}) = (1 + o(1)) \sum_{\mathbf{d}\in\mathcal{D}} y_{\mathbf{d}}, \tag{7}$$

and such that $\mathbf{d}_{\text{reg}} = (d, \dots, d) \in \mathcal{D}$.

If we reach this goal, then we can deduce from the lemma above that $\Pr(D(G_{n,m}) = \mathbf{d}_{\text{reg}}) = y_{\mathbf{d}_{\text{reg}}}(1 + o(1))$, which is equivalent to

$$|\mathcal{G}(\mathbf{d}_{\text{reg}})| = \binom{\binom{n}{2}}{m} y_{\mathbf{d}_{\text{reg}}}(1 + o(1)),$$

that is we get an asymptotic formula for the number of d-regular graphs on n vertices.

Of course, we could start with $\mathcal{D}$ to be the set of all sequences of total degree $2m$. There are at least two difficulties with this choice. Firstly, we'd have to handle very rare events in $G_{n,m}$, and that can be hard. The second reason will become apparent in a little while, and we will point this out when it is appropriate.

5.1 Concentration for $D(G_{n,m})$

Lemma 5.2. *Let N, K, m be integers and let $X \sim \text{Hypergeometric}(N, K, m)$ then for all $\delta > 0$*

$$\Pr(|X - \mathbb{E}(X)| \geq \delta\,\mathbb{E}(X)) \leq 2\exp\left(-\min\{\delta, \delta^2\}\,\mathbb{E}(X)/4\right).$$

We note that if $\mathbf{d}$ is the degree sequence of $G_{n,m}$ then $d_v \sim \text{Hypergeometric}\big(\binom{n}{2}, n-1, m\big)$; and if $\mathbf{d}$ is the random sequence $\mathcal{B}_m(n))$ then $d_v \sim \text{Hypergeometric}\big(n(n-1), n-1, 2m\big)$. Using the union bound together with the above concentration bound we therefore get the following.

Corollary 5.3. *Let $n \log n \ll m \leq n^2/2$ and let $\mathbf{d}$ be either $D(G_{n,m})$ or $\mathcal{B}_m(n)$. Then with probability $1 - o(n^{-23})$ we have that $|d_i - d| \leq 10\sqrt{d \log n}$ for all $1 \leq i \leq n$, where $d = \frac{1}{n}\sum_i d_i = 2m/n$ is the average degree.*

Proof. Exercise. □

Definition 5.4. Given n and $1 \leq m \leq \binom{n}{2}$, define

$$\mathcal{D}_m := \left\{ \mathbf{d} \in \mathbb{Z}^n_{\geq 0} \;:\; \sum_i d_i = 2m \text{ and } |d_i - d| \leq 10\sqrt{d \log n} \text{ for all } i \in [n] \right\}.$$

Corollary 5.3 implies that $\sum_{\mathbf{d}\in\mathcal{D}_m} \Pr(D(G_{n,m}) = \mathbf{d}) = 1 - o(1)$. In terms of our main goal we will be done if we achieve the following two subgoals.

Subgoal 1:

Find a good approximation for the ratio

$$R_{\mathbf{d},\mathbf{d}'} := \frac{\Pr(D(G_{n,m}) = \mathbf{d})}{\Pr(D(G_{n,m}) = \mathbf{d}')}$$

for all $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_m$.

> **Subgoal 2:**
> For all $\mathbf{d} \in \mathcal{D}_m$, find a formula $y_{\mathbf{d}}$ such $\sum_{\mathbf{d}\in\mathcal{D}_m} y_{\mathbf{d}} = 1 - o(1)$ and such that $R_{\mathbf{d},\mathbf{d}'} = y_{\mathbf{d}}/y_{\mathbf{d}'}(1+o(1))$ for all $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_m$.

Subgoal 1 is the crux of the whole argument. Subgoal 2 can be tough to achieve in general as well, but we have the luxury of having a conjecture for what this formula should be.

5.2 The local change graph and adjacent sequences

The core idea of our approach is to understand how the probability $\Pr(D(G_{n,m}) = \mathbf{d})$ changes when we make small, local modifications to the degree sequence $\mathbf{d}$. One such modification – though certainly not the only one – is what we call a *degree switch*: choose two vertices $a, b \in [n]$, decrease the degree of a by 1, and increase the degree of b by 1. This transforms $\mathbf{d}$ into the sequence $\mathbf{d} - \mathbf{e}_a + \mathbf{e}_b$.

This local change corresponds to a simple operation on a graph with degree sequence $\mathbf{d}$: removing an edge incident to vertex a and adding an edge incident to vertex b, while keeping all other vertex degrees unchanged. Figure 1 illustrates this: the solid edge av is removed, and the dashed edge bv is inserted.

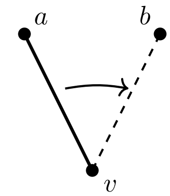

Figure 1: Degree switch.

By studying these local transitions, we can estimate the ratio of probabilities (or counts) between any two degree sequences in $\mathcal{D}_m$ by chaining together small, controlled ratio estimates for these degree switches.

To formalize this, we define an auxiliary graph S, called the *local change graph* whose vertex set is $\mathcal{D}_m$, the set of degree sequences that satisfy the concentration bounds from Corollary 5.3. We place an edge between two sequences $\mathbf{d}, \mathbf{d}'$ in S if they differ by a degree switch; that is, if there exist $a, b \in [n]$ such that $\mathbf{d}' = \mathbf{d} - \mathbf{e}_a + \mathbf{e}_b$. Call two sequences $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_m$ *adjacent*, if they are adjacent in S.

The following captures the crucial property of S.

Claim 5.5. *The local change graph S is connected and the diameter* $\operatorname{diam}(S)$ *is at most* $20\, n\sqrt{d \log n}$.

Proof. Fix $d := 2m/n$ and assume for simplicity that d is an integer. Recall that $\mathbf{d}_{\text{reg}} = (d, \ldots, d)$ denotes the d-regular sequence. Clearly, $\mathbf{d}_{\text{reg}} \in \mathcal{D}_m$ (see the definition of $\mathcal{D}_m$).

Now take any $\mathbf{d} \in \mathcal{D}_m$. Choose an $a \in [n]$ with $d_a > d$ and $b \in [n]$ with $d_b < d$, and apply a degree switch from a to b. The resulting sequence $\mathbf{d}' = \mathbf{d} - \mathbf{e}_a + \mathbf{e}_b$ is a neighbour of $\mathbf{d}$ in S. We claim that $\mathbf{d}' \in \mathcal{D}_m$ as well. Indeed, the quantities $|d_a - d|$ and $|d_b - d|$ decrease by 1, and all other $|d_i - d|$ remain unchanged. Thus, the condition for containment in $\mathcal{D}_m$ is clearly satisfied. Moreover,

$$\sum_i |d_i' - d| = \sum_i |d_i - d| - 2,$$

so the total deviation decreases by 2 in each step. Repeating this process, we reach $\mathbf{d}_{\text{reg}}$ after at most $\sum_i |d_i - d| \le 10n\sqrt{d \log n}$ steps. Since $\mathbf{d}$ was arbitrary in $\mathcal{D}_m$ it follows that S is connected and that the diameter is at most $20n\sqrt{d \log n}$. □

Lemma 5.6 (Crucial Ratio Lemma)**.** *Let n, m be integers such that $n \log n \ll m \ll n^{7/5}/(\log n)^{1/5}$. Then for all $\mathbf{d} \in \mathcal{D}_m$ and all $a, b \in [n]$ such that $\mathbf{d} - \mathbf{e}_a + \mathbf{e}_b \in \mathcal{D}_m$ we have that*

$$\frac{\Pr(D(G_{n,m}) = \mathbf{d})}{\Pr(D(G_{n,m}) = \mathbf{d} - \mathbf{e}_a + \mathbf{e}_b)} = \frac{d_b + 1}{d_a}\left(1 + \frac{d_b - d_a}{dn}\left(1 + \frac{M_2(\mathbf{d})}{dn}\right) + o\left(\frac{1}{n\sqrt{d \log n}}\right)\right),$$

where $d = 2m/n$ and $M_2(\mathbf{d}) = \sum_i d_i(d_i - 1)$, and where the error is uniform for all $\mathbf{d} \in \mathcal{D}_m$.

Let us first argue briefly why the Crucial Ratio Lemma suffices to approximate all pairwise ratios in $\mathcal{D}_m$.

Let $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_m$ be arbitrary. Since the auxiliary graph S on $\mathcal{D}_m$ is connected, there exists a path of adjacent degree sequences

$$\mathbf{d} = \mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_r = \mathbf{d}'$$

with $r \le \operatorname{diam}(S)$, such that each $\mathbf{d}_{i+1} = \mathbf{d}_i - \mathbf{e}_{a_i} + \mathbf{e}_{b_i}$ for some $a_i, b_i \in [n]$, and $\mathbf{d}_i \in \mathcal{D}_m$ for all i. Applying the Crucial Ratio Lemma to each adjacent pair, we get

$$\frac{\Pr(D(G_{n,m}) = \mathbf{d})}{\Pr(D(G_{n,m}) = \mathbf{d}')} = \prod_{i=0}^{r-1} \frac{\Pr(D(G_{n,m}) = \mathbf{d}_i)}{\Pr(D(G_{n,m}) = \mathbf{d}_{i+1})} = \prod_{i=0}^{r-1} \left[\text{formula}_i \cdot \left(1 + o\left(\frac{1}{\operatorname{diam}(S)}\right)\right)\right].$$

Since the product involves at most $\operatorname{diam}(S)$ terms, each with a relative error $o(1/\operatorname{diam}(S))$, the total accumulated relative error is $o(1)$. Therefore, the pairwise ratio

$$\frac{\Pr(D(G_{n,m}) = \mathbf{d})}{\Pr(D(G_{n,m}) = \mathbf{d}')}$$

is approximated by the product of the corresponding formula terms, up to relative error $o(1)$, uniformly over all $\mathbf{d}, \mathbf{d}' \in \mathcal{D}_m$.

This completes the argument that the Crucial Ratio Lemma implies uniform pairwise ratio estimates for all degree sequences in $\mathcal{D}_m$. Thus, Subgoal 1 reduces to proving the Crucial Ratio Lemma.

5.2.1 Developing recursive formulae for ratios

Definition 5.7. Let $\mathbf{d} \in \mathbb{Z}^n_{\ge 0}$ be an odd sequence, that is, such that $\sum_i d_i$ is odd, and let $a, b \in [n]$. Assuming that $\mathbf{d} - \mathbf{e}_b$ is graphical we define

$$R_{(a,b)}(\mathbf{d}) := \frac{|\mathcal{G}(\mathbf{d} - \mathbf{e}_a)|}{|\mathcal{G}(\mathbf{d} - \mathbf{e}_b)|}.$$

The quantities that we are interested in in the Crucial Ratio Lemma are thus $R_{(b,a)}(\mathbf{d} + \mathbf{e}_b)$ for $\mathbf{d} \in \mathcal{D}_m$. This change in perspective is not necessary, but results in symmetric formulae.

In order to obtain a formula for $R_{(a,b)}(\mathbf{d})$ we construct an auxiliary bipartite graph B between the two sets $\mathcal{G}(\mathbf{d} - \mathbf{e}_b)$ and $\mathcal{G}(\mathbf{d} - \mathbf{e}_a)$, by placing an edge between $G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)$ and $G' \in \mathcal{G}(\mathbf{d} - \mathbf{e}_a)$ if and only if G' can be obtained from G by a degree switch from a to b, that is, if there is some $v \in N_G(a) \setminus N_G(b)$ such that the edges of G' are exactly $(E(G) \cup \{vb\}) \setminus \{va\}$.

Then counting the number of edges in this bipartite graph B implies that

$$\sum_{G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)} \deg_B(G) = \sum_{G' \in \mathcal{G}(\mathbf{d} - \mathbf{e}_a)} \deg_B(G')$$

which is equivalent to

$$|\mathcal{G}(\mathbf{d} - \mathbf{e}_b)| \, \mathbb{E}_{G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)} \deg_B(G) = |\mathcal{G}(\mathbf{d} - \mathbf{e}_a)| \, \mathbb{E}_{G' \in \mathcal{G}(\mathbf{d} - \mathbf{e}_a)} \deg_B(G'). \tag{8}$$

The vertex a has exactly d_a neighbours in a graph $G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)$. A neighbour v of a can be used for a degree switch, unless

(L) ab is an edge in G and $v = b$, or

(D) vb is an edge in G.

Definition 5.8. Let $\mathrm{Bad}_{(a,b)}(\mathbf{d} - \mathbf{e}_b)$ denote the probability that one of the two events (L) or (D) happens if $G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)$ is picked u.a.r. and if v is picked u.a.r. among all neighbours of a in G.

We will obtain a formula for $\mathrm{Bad}_{(a,b)}$ shortly. With this notation, we obtain that

$$\mathbb{E}_{G \in \mathcal{G}(\mathbf{d} - \mathbf{e}_b)} \deg_B(G) = d_a \Big(1 - \mathrm{Bad}_{(a,b)}(\mathbf{d} - \mathbf{e}_b)\Big).$$

Now, we only need to note that we can just swap the roles of a and b in order to get an analogous expression for $\mathbb{E}_{G' \in \mathcal{G}(\mathbf{d} - \mathbf{e}_a)} \deg_B(G')$. With these observations, (8) is equivalent to

$$R_{(a,b)}(\mathbf{d}) = \frac{|\mathcal{G}(\mathbf{d} - \mathbf{e}_a)|}{|\mathcal{G}(\mathbf{d} - \mathbf{e}_b)|} = \frac{d_a\big(1 - \mathrm{Bad}_{(a,b)}(\mathbf{d} - \mathbf{e}_b)\big)}{d_b\big(1 - \mathrm{Bad}_{(b,a)}(\mathbf{d} - \mathbf{e}_a)\big)}. \tag{9}$$

Next, we find a formula for the bad probability $\mathrm{Bad}_{(a,b)}(\mathbf{d} - \mathbf{e}_b)$ in terms of edge and 2-path probabilities.

Definition 5.9. For distinct $a, v, b \in [n]$ and a graphical sequence $\mathbf{d} \in \mathbb{Z}^n_{\geq 0}$ we define

$$P_{av}(\mathbf{d}) := \frac{|\{G \in \mathcal{G}(\mathbf{d}) \mid av \in E(G)\}|}{|\mathcal{G}(\mathbf{d})|}, \text{ and } Y_{avb}(\mathbf{d}) := \frac{|\{G \in \mathcal{G}(\mathbf{d}) \mid av, bv \in E(G)\}|}{|\mathcal{G}(\mathbf{d})|}.$$

Equivalently, $P_{av}(\mathbf{d})$ and $Y_{avb}(\mathbf{d})$ are the probabilities that the edge av is present, or that both edges av and bv are present, respectively, in a graph drawn u.a.r from $\mathcal{G}(\mathbf{d})$.

Lemma 5.10. *If* $\mathbf{d}-\mathbf{e}_b$ *is graphical and* $d_a > 0$ *then* $\mathrm{Bad}_{(a,b)}(\mathbf{d}-\mathbf{e}_b) = \frac{1}{d_a}\Big(P_{ab}(\mathbf{d}-\mathbf{e}_b)+\sum_{v\neq a,b} Y_{avb}(\mathbf{d}-\mathbf{e}_b)\Big)$.

Proof. First note that the two events (L) and (D) are disjoint. The probability of (L) is exactly $P_{ab}(\mathbf{d}-\mathbf{e}_b)/d_a$, and the probability of (D) is $\frac{1}{d_a}\sum_{v\neq a,b} Y_{avb}(\mathbf{d}-\mathbf{e}_b)$ (exercise). □

5.2.2 First approximation of $R_{(a,b)}(\mathbf{d})$

We need an initial bound on the edge probability which we obtain via a simple switching as introduced by McKay for graph enumeration.

Lemma 5.11. *Let* $\mathbf{d}$ *be a graphical sequence of length* n *with* $\sum_i d_i = dn$ *such that* $\Delta = \Delta(\mathbf{d})$ *satisfies* $\Delta^2 = o(dn)$. *Then for* $a, v \in [n]$ *we have*

$$P_{av}(\mathbf{d}) \leq \frac{\Delta^2}{dn}\left(1 + O\left(\frac{\Delta^2}{dn}\right)\right).$$

Proof. For each graph G with degree sequence $\mathbf{d}$ and an edge joining a and v, we perform an edge switching by removing both av and another randomly chosen $a'v'$, and inserting aa' and vv', provided no loops or multiple edges are created. There are at least

$$dn - O(\Delta^2)$$

such switchings for G. On the other hand, for any G' in which av is not an edge, the number of ways that G' is created by such a switching is at most Δ^2 (choose a neighbour of a and a neighbour of v in G'). Thus, if we let A denote the set of graphs $G \in \mathcal{G}(\mathbf{d})$ containing the edge av, and B the set of graphs $G \in \mathcal{G}(\mathbf{d})$ not containing the edge av, then

$$P_{av}(\mathbf{d}) = \frac{|A|}{|A|+|B|} = \frac{|A|}{|B|(1+|A|/|B|)} \leq \frac{\Delta^2}{dn}\left(1 + O\left(\frac{\Delta^2}{dn}\right)\right). \qquad \square$$

In particular, for $\mathbf{d} \in \mathcal{D}_m$, this gives that $P_{av}(\mathbf{d}) = O\big(\frac{d}{n}\big)$, since $\Delta \leq 2d$ for such $\mathbf{d}$. A similar argument gives an upper bound on the 2-path probabilities:

$$Y_{avb}(\mathbf{d}) = O(d^2/n^2) \text{ for } \mathbf{d} \in \mathcal{D}_m.$$

We do not provide an argument for this, since we will see a more precise description of Y_{avb} shortly.

Plugging these two crude upper bounds into the recursive formula for $\mathrm{Bad}_{(a,b)}$ in Lemma 5.10, we obtain that

$$\begin{aligned}\mathrm{Bad}_{(a,b)}(\mathbf{d}-\mathbf{e}_b) &= \frac{1}{d_a}\Big(P_{ab}(\mathbf{d}-\mathbf{e}_b) + \sum_{v\neq a,b} P_{avb}(\mathbf{d}-\mathbf{e}_b)\Big) \\ &= O\left(\frac{1}{n}\right) + \frac{1}{d_a}\cdot O\left(\frac{d^2}{n}\right) = O\left(\frac{d}{n}\right). \qquad (10)\end{aligned}$$

Feeding this expression into (9) we obtain that

$$R_{(a,b)}(\mathbf{d}) = \frac{d_a}{d_b}\Big(1 + O\Big(\frac{d}{n}\Big)\Big).$$

That's not quite good enough yet to achieve Lemma 5.6, since $\frac{d}{n} \neq o\left(\frac{1}{n\sqrt{d\log n}}\right)$ (for $d \geq 1$).

5.2.3 Developing recursive formulae for edge and path probabilities

Instead of trying to find better approximations for the edge and 2-path probabilities P_{ab} and Y_{avb}, we will now express those quantities in terms of edge and 2-path probabilities, and of ratios R_{ab} of slightly shifted sequences (with different indices).

Definition 5.12. Write $\mathcal{N}(\mathbf{d})$ for $|\mathcal{G}(\mathbf{d})|$, $\mathcal{N}_{av}(\mathbf{d})$ for the number of graphs in $\mathcal{G}(\mathbf{d})$ that contain the edge av, and $\mathcal{N}_{av,bv}(\mathbf{d})$ for the number of graphs in $\mathcal{G}(\mathbf{d})$ that contain both av and bv as edges.

We start with a simple observation.

Observation 5.13. *Let $a, v \in [n]$ be distinct and let $\mathbf{d}$ be a sequence of non-negative integers such that both $\mathbf{d}$ and $\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v$ are graphical. Then*

$$\begin{aligned}\mathcal{N}_{av}(\mathbf{d}) &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big(1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big)\\ \mathcal{N}_{av,bv}(\mathbf{d}) &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big(P_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - Y_{avb}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big).\end{aligned}$$

The first equality follows by noticing that there is a natural bijection between the set of graphs $\mathcal{G}(\mathbf{d})$ that contain the edge av and the set of graphs with reduced degree sequence $\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v$ that *do not* contain the edge av. The second equality follows similarly.

Lemma 5.14. *Let $\mathbf{d}$ be a graphical sequence of length n.*

(*a*) *Let $a, v \in [n]$. If $\mathcal{N}_{bv}(\mathbf{d}) > 0$ for all $b \in [n]$ then*

$$P_{av}(\mathbf{d}) = d_v \left(\sum_{b \in [n] \setminus \{a\}} R_{(b,a)}(\mathbf{d} - \mathbf{e}_v) \frac{1 - P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \right)^{-1}.$$

(*b*) *Let a, v, b be distinct elements of $[n]$. If $\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) > 0$ then*

$$\mathrm{Y}_{avb}(\mathbf{d}) = \frac{P_{av}(\mathbf{d})\big(P_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - \mathrm{Y}_{avb}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big)}{1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)}.$$

Proof. For (a), pick G in $\mathcal{G}(\mathbf{d})$ uniformly at random and let $a, v \in [n]$ be distint. Then

$$\begin{aligned} d_v = \mathbb{E}_G\, d(v) = \sum_{b \neq v} P_{bv}(\mathbf{d}) = \sum_{b \neq v} \frac{\mathcal{N}_{bv}(\mathbf{d})}{\mathcal{N}(\mathbf{d})} &= \frac{\mathcal{N}_{av}(\mathbf{d})}{\mathcal{N}(\mathbf{d})} \sum_{b \neq v} \frac{\mathcal{N}_{bv}(\mathbf{d})}{\mathcal{N}_{av}(\mathbf{d})}\\ &= P_{av}(\mathbf{d}) \sum_{b \neq v} \frac{\mathcal{N}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)\big(1 - P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)\big)}{\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big(1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)\big)}\\ &= P_{av}(\mathbf{d}) \sum_{b \neq v} R_{(b,a)}(\mathbf{d} - \mathbf{e}_v) \frac{1 - P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)}.\end{aligned}$$

The second part also follows easily from Observation 5.13.

$$Y_{avb}(\mathbf{d}) = \frac{\mathcal{N}_{av,bv}(\mathbf{d})}{\mathcal{N}(\mathbf{d})} = P_{av}(\mathbf{d}) \frac{\mathcal{N}_{av,bv}(\mathbf{d})}{\mathcal{N}_{av}(\mathbf{d})}.$$

Plugging in the identities from Observation 5.13 for $\mathcal{N}_{av,bv}(\mathbf{d})$ and $\mathcal{N}_{av}(\mathbf{d})$ yields the result. □

5.2.4 Second approximation of $R_{(a,b)}(\mathbf{d})$

From the recursive identities in (9), Lemma 5.10 and Lemma 5.14 we are able to deduce a more precise formula for the ratios and prove Lemma 5.6. All these lemmas have conditions on $\mathbf{d}$ or some nearby sequence being graphical. So let us address graphicality first.

Definition 5.15. For a set of degree sequences $\mathcal{D}$ of length n and an integer r, let $\mathcal{B}_r(\mathcal{D})$ denote the ball of radius r around $\mathcal{D}$ in Hamming distance, that is

$$\mathcal{B}_r(\mathcal{D}) := \left\{\mathbf{d} \in \mathbb{Z}^n_{\geq 0} \,:\, |\mathbf{d}-\mathbf{d}'|_1 \leq r \text{ for some } \mathbf{d}' \in \mathcal{D}\right\}.$$

Recall the definition of $\mathcal{D}_m$ as the set of all sequences $\mathbf{d}$ of total degree $2m$ such that $\max_i |d_i - d| \leq 10\sqrt{d\log n}$. Also recall that a sequence is *even* if $\sum_i d_i$ is even, and *odd* otherwise.

Lemma 5.16 (Graphicality)**.** *Let m and n be integers such that $n\log n \ll m \leq n^2/10$. Then $\mathbf{d}$ is graphical for all even sequences in $\mathcal{B}_8(\mathbf{d})$.*

Proof. Standard application of the Erdős-Gallai conditions, see Problem Set 1. Exercise. □

There is of course nothing special about the radius 8 in this lemma. One could choose a larger radius such as $\log n$ if m is sufficiently large, say larger than $n\log^2 n$. However, 8 is enough for our second ratio approximation.

Proof of Lemma 5.6. Fix m as in the lemma, and let $d = 2m/n$. First, we use the previous lemma to say that $\mathcal{N}(\mathbf{d}) > 0$ for all even $\mathbf{d} \in \mathcal{B}_8(\mathcal{D}_m)$. We note that the average degree of a sequence $\mathbf{d} \in \mathcal{B}_8(\mathcal{D}_m)$ is not exactly d, but $d + O(1)$. Now, Lemma 5.11 implies that

$$P_{av}(\mathbf{d}) = O(d/n) = o(1)$$

for all even $\mathbf{d} \in \mathcal{B}_8(\mathcal{D}_m)$ and distinct $a, v \in [n]$. Next, we use this bound on P_{av}, Lemma 5.14(b) and the graphicality of all even $\mathbf{d} \in \mathcal{B}_8(\mathcal{D}_m)$ to deduce that

$$Y_{avb}(\mathbf{d}) = O(d^2/n^2)$$

for all even $\mathbf{d} \in \mathcal{B}_6(\mathcal{D}_m)$ and all distinct $a, v, b \in [n]$. Thus, as in the first approximation of $R_{a,b}$, we deduce that $\mathrm{Bad}_{(a,b)}(\mathbf{d}) = O\left(\frac{d}{n}\right)$ for all even $\mathbf{d} \in \mathcal{B}_6(\mathcal{D}_m)$ and all distinct $a, b \in [n]$. And then that

$$R_{(a,b)}(\mathbf{d}) = \frac{d_a}{d_b}\Big(1 + O\Big(\frac{d}{n}\Big)\Big)$$

for all odd $\mathbf{d} \in \mathcal{B}_5(\mathcal{D}_m)$ and all $a, b \in [n]$ (note that we do not need to restrict to distinct a, b since the ratio is trivially 1 if $a = b$.

In the second round, we feed these approximations into Lemma 5.14(a) and obtain that

$$P_{av}(\mathbf{d}) = \frac{d_a d_v}{dn}\left(1 + O\left(\frac{d}{n}\right)\right)$$

for all even $\mathbf{d} \in \mathcal{B}_4(\mathcal{D}_m)$ and distinct $a, v \in [n]$. With this improved approximation, we obtain from Lemma 5.14(b) that

$$Y_{avb}(\mathbf{d}) = \frac{d_a (d_v)_2 d_b}{(dn)^2}(1 + O(d/n)$$

for all even $\mathbf{d} \in \mathcal{B}_2(\mathcal{D}_m)$ and all distinct $a, v, b \in [n]$. Now the calculations start to become a bit more involved. From Lemma 5.10 we deduce that

$$\mathrm{Bad}_{(a,b)}(\mathbf{d}) = \frac{d_b}{dn} + \frac{d_b M_2(\mathbf{d})}{(dn)^2} + O\left(\frac{d^2}{n^2}\right)$$

for all even $\mathbf{d} \in \mathcal{B}_2(\mathcal{D}_m)$ and all distinct $a, b \in [n]$. Finally, we feed this into (9) and, with a bit of patience for calculations, derive that

$$R_{a,b}(\mathbf{d}) = \frac{d_a}{d_b}\left(1 + \frac{d_a - d_b}{dn}\left(1 + \frac{M_2(\mathbf{d})}{dn}\right) + O\left(\frac{d^2}{n^2}\right)\right)$$

for all odd $\mathbf{d} \in \mathcal{B}_1(\mathcal{D}_m)$ and all $a, b \in [n]$.

Now, since $d = o(n^{2/5}/\log n)$ then

$$\frac{d^2}{n^2} = o\left(\frac{1}{n\sqrt{d\log n}}\right),$$

which proves the lemma by rearranging and identifying error terms. □

As explained right after the statement of Lemma 5.6, this achieves Subgoal 1.