Park City Experimental Mathematics Lab

Dependent Random Choice

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Summer 2025

Main idea

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Given a reasonably dense graph G, find a subset U where every d-tuple has lots of common neighbors.

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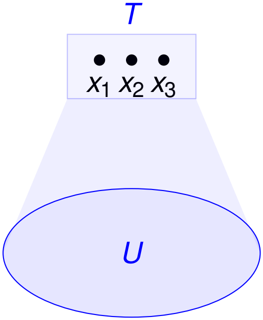
- Choose a random set of vertices T and look at its common neighborhood $U = \bigcap_{x \in T} N(x)$.



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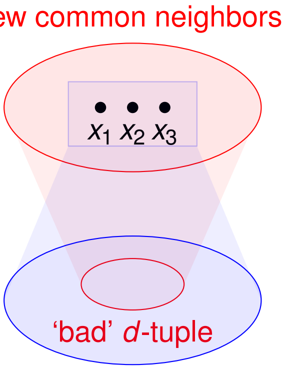
Given a reasonably dense graph G, find a subset U where every d-tuple has lots of common neighbors.

- Choose a random set of vertices T and look at its common neighborhood $U = \bigcap_{x \in T} N(x)$.
- Any d-tuple with few common neighbors is unlikely to land in U, since this would require all vertices in T to come from its small common neighborhood.

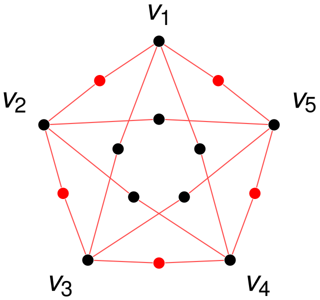


Problem Statement

Problem (Fox and Sudakov, *Dependent Random Choice*)

If G is a graph with n vertices and εn^2 edges, then G contains a 1-subdivision of a complete graph with $a = \varepsilon^{3/2} n^{1/2}$ vertices.

(A 1-subdivision of H is formed by replacing each edge in H with a path with 1 internal vertex.)



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Idea: It is sufficient to find a set of *a* vertices where every pair ($d = 2$) has at least $\binom{a}{2} + a \leq a^2$ common neighbors.

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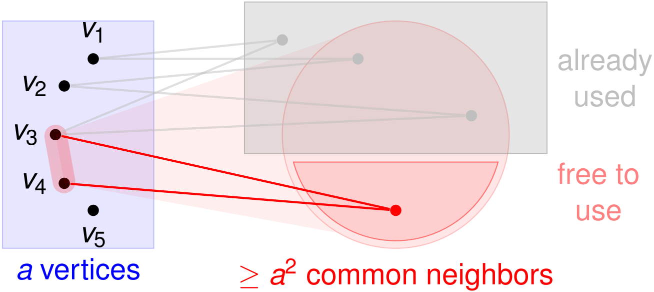

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- For v to land in U, it should be adjacent to all of $x_1, \ldots, x_t$.
- Each x_i has probability $\frac{d(v)}{n}$ of being adjacent to v, so

$$\mathbb{E}[X] = \sum_{v \in G} \left(\frac{d(v)}{n} \right)^t \geq n \left(\frac{2\varepsilon n}{n} \right)^t = (2\varepsilon)^t n.$$

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- If (u, v) is bad, this has probability at most $\left(\frac{a^2}{n}\right)^t$.
- Hence,

$$\mathbb{E}[Y] \leq \binom{n}{2} \left(\frac{a^2}{n}\right)^t \leq \frac{n^2}{2} \varepsilon^{3t}.$$

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- When we delete a vertex from each bad pair from U, we're left with at least $X - Y$ vertices; we want this to be at least $a = \varepsilon^{3/2}\sqrt{n}$.

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- We computed $\mathbb{E}[X] \geq (2\varepsilon)^t n$ and $\mathbb{E}[Y] \leq \frac{n^2}{2}\varepsilon^{3t}$, so

$$\mathbb{E}[X - Y] \geq (2\varepsilon)^t n - \frac{n^2}{2}\varepsilon^{3t}.$$

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- Set t so that $\varepsilon^t = n^{-1/2}$. Then

$$(2\varepsilon)^t n \geq 2\sqrt{n} \quad \text{and} \quad \frac{n^2}{2}\varepsilon^{3t} \leq \frac{n^2}{2} n^{-3/2} = \frac{1}{2}\sqrt{n}.$$

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- So $\mathbb{E}[X - Y] \geq \sqrt{n} \geq a$, so there's a realization where our final set U' has at least a vertices.

Reference

Fox, Jacob and Benny Sudakov. *Dependent Random Choice*. 2010. arXiv: 0909.3271 [math.CO]. URL: https://arxiv.org/abs/0909.3271.