Anomalous Continuous Translations

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Simons Collaboration on Ultra-Quantum Matter

The models

Certain continuum quantum field theories (will be defined below).

Examples:

- Dynamical $U(1)$ gauge field with constant charge density
$$\mathcal{L} = \rho a_t + \cdots$$
- Some nonlinear sigma models with a term with a single time derivative.
  - Example: continuum Lagrangian for a ferromagnet
$$\mathcal{L} = i\rho \, \frac{(z\partial_t \bar{z} - \bar{z}\partial_t z)}{1+|z|^2} + \cdots$$
- Many others.

We'll discuss them from the perspective of their symmetries and anomalies.

Classical particle on $\mathbb{T}^2$ with a constant magnetic field

$$x^1 \sim x^1 + \ell^1$$
$$x^2 \sim x^2 + \ell^2$$

The classical Lagrangian

$$\mathcal{L} = \frac{m}{2}\left((\dot{x}^1)^2 + (\dot{x}^2)^2\right) + Bx^1\dot{x}^2$$

is not globally well-defined and is not translation invariant.

However, the equations of motion

$$m\ddot{x}^1 = B\dot{x}^2 \quad , \qquad m\ddot{x}^2 = -B\dot{x}^1$$

are well-defined and are $U(1) \times U(1)$ translation invariant.

Quantum particle on $\mathbb{T}^2$ with a constant magnetic field [Landau (30), …]

$$\mathcal{L} = \frac{m}{2}\left((\dot{x}^1)^2 + (\dot{x}^2)^2\right) + Bx^1\dot{x}^2$$

In the quantum theory, the Lagrangian, (or more precisely, $\exp\left(\frac{i}{\hbar}\int dt\, \mathcal{L}\right)$), needs to be defined carefully.

(Can use differential cohomology and pre-quantization. We will not do it here.)

The careful definition leads to

- B is quantized

$$B = \hbar\, F_{12} = 2\pi \frac{\hbar k}{\ell^1 \ell^2} \quad , \qquad k \in \mathbb{Z}$$

- Dependence on $\exp\left(\oint dx^i A_i\right)$ of the background gauge field and not only B…

Quantum particle on $\mathbb{T}^2$ with a constant magnetic field [Landau (30), …]

- Dependence on $\exp\left(\oint dx^i A_i\right)$ of the background gauge field and not only B.

The classical $U(1) \times U(1)$ symmetry is explicitly (not spontaneously) broken to $\mathbb{Z}_k \times \mathbb{Z}_k$. Relatedly, $\mathcal{L}$ includes $\hbar\left(\frac{\theta_1}{\ell^1}\dot{x}^1 + \frac{\theta_2}{\ell^2}\dot{x}^2\right)$ with no preferred zero for θ_i.

Interpret it as an Adler-Bell-Jackiw (ABJ) anomaly in the classical symmetry.

Quantum particle on $\mathbb{T}^2$ with a constant magnetic field [Landau (30), …]

The "charges"

$$p_1 = m\dot{x}^1 - Bx^2 \quad , \qquad p_2 = m\dot{x}^2 + Bx^1$$

are conserved, but not well-defined.

The "charges"

$$\tilde{p}_1 = m\dot{x}^1 \quad , \qquad \tilde{p}_2 = m\dot{x}^2$$

are well-defined, but not conserved. (Common in ABJ-anomaly.)

$$T^1 = \exp\left(\frac{i}{\hbar k}\ell^1 p_1\right) \quad , \qquad T^2 = \exp\left(\frac{i}{\hbar k}\ell^2 p_2\right)$$

$$(T^1)^k = (T^2)^k = 1$$

are well-defined and generate $\mathbb{Z}_k \times \mathbb{Z}_k$.

't Hooft anomaly in $\mathbb{Z}_k \times \mathbb{Z}_k$

$$T^1 T^2 = e^{\frac{2\pi i}{k}} T^2 T^1$$

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Certain continuum quantum field theories (will be defined below).

Examples:

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$$\mathcal{L} = \rho a_t + \cdots$$

- Some nonlinear sigma models with a term with a single time derivative.
  – Example: continuum Lagrangian for a ferromagnet

$$\mathcal{L} = i\rho \, \frac{(z\partial_t \bar{z} - \bar{z}\partial_t z)}{1 + |z|^2} + \cdots$$

- Many others.

We'll discuss them from the perspective of their symmetries and anomalies.

Common features

- The classical Lagrangian $\mathcal{L}$ is not well-defined. But the classical equations of motion are well-defined. They have continuous translation symmetry.
- In the quantum theory, we should define $\exp(i \int \mathcal{L})$ or its Euclidean version $\exp(-S_E)$, such that the equations of motion are unchanged.
- Often, this definition needs additional data (θ-parameters).
- In our case:
  - This data breaks explicitly the continuous translation symmetry to a discrete translation symmetry.
  - The discrete translation symmetry of the quantum theory is extended.

Common features

Many of our results are known. (Controversy about the interpretation.)

We will present a unified treatment as an Adler-Bell-Jackiw (ABJ)-like anomaly in translations

General setup and notation

Space is a flat torus $\mathbb{T}^d$, parameterized by $x^i \sim x^i + \ell^i$

Volume $V = \prod_i \ell^i$

Lorentzian time t

Euclidean time τ

Spacetime index μ is (t, i) or (τ, i)

Dynamical $U(1)$ gauge field a_μ with constant charge density

Related to [Fischler, Kogut, Susskind (79); Metlitski (07); ...]

The Lagrangian is

$$\mathcal{L} = \rho a_t + \cdots$$

The classical equations of motion are gauge invariant and translation invariant. They state that there is a constant charge density ρ.

In the quantum theory (with $\hbar = 1$), gauge invariance demands

$$\rho = \frac{k}{V} \qquad k \in \mathbb{Z}$$

V is the total volume and $k \in \mathbb{Z}$ is the total charge.

Naively, the quantum theory is gauge invariant and has continuous translation symmetry. In fact, this is not the case.

Dynamical $U(1)$ gauge field a_μ with constant charge density

$$\mathcal{L} = \frac{k}{V} a_t + \cdots$$

We need to define $\exp(i \int \mathcal{L})$ or its Euclidean version $\exp(-S_E)$.

The proper mathematical way to define it uses differential cohomology.

Alternatively, we can cover spacetime with patches with transition functions between them and define the action as an integral over that data. This leads to $S = \int \mathcal{L}$, but some "correction terms" need to be added.

Instead, we will take another approach…

Defining the action

Let us first set $k = 0$.

The theory has a magnetic $d-2$-form symmetry with current $\mathcal{J} = \frac{1}{2\pi} f = \frac{1}{2\pi} da$. (For our purposes, for $d = 1$, we can think of it as a "-1-form symmetry.")

We couple it to a classical background $d-1$-form gauge field A

$$A \wedge \mathcal{J} = \frac{1}{2\pi} A \wedge da$$

- For $d = 1$, this is $\frac{1}{2\pi} \theta da$.
- For $d = 2$, the symmetry is an ordinary symmetry and the coupling $\frac{1}{2\pi} A \wedge da$ is a standard Chern-Simons coupling.
- For $d > 2$, A is a higher-form gauge field.

Defining the action

$$\frac{1}{2\pi} A \wedge da$$

This Chern-Simons coupling has to be defined carefully.

Then, it is the same as the carefully defined term $\frac{1}{2\pi} a \wedge dA$

$$\exp\big(iCS(a, A)\big) = \exp\left(\frac{i}{2\pi} \int A \wedge da + \cdots\right)$$
$$= \exp\left(\frac{i}{2\pi} \int a \wedge dA + \cdots\right)$$

We will use the fact that, despite appearance, it depends on the constant modes of A and a. (More precisely, it depends on shifting A or a by an arbitrary flat gauge field.)

Defining the action

Consider a particular background field A with constant "magnetic field"

$$dA = \frac{2\pi k}{V} dx^1 \wedge dx^2 \wedge \cdots dx^d \quad , \qquad k \in \mathbb{Z}$$

- For $d = 1$, this is $d\theta = 2\pi \frac{k}{\ell} dx$.
- For $d = 2$, A is the gauge fields of a constant ordinary magnetic field (not the magnetic field of a).
- For $d > 2$, A is a higher-form gauge field.

It leads to our term

$$\frac{1}{2\pi} a \wedge dA = \frac{k}{V} a_t \, dt \wedge dx^1 \wedge \cdots dx^d$$

Broken translations

This means that through the explicit dependence of A on x^i, the properly defined

$$\exp\big(iCS(a,A)\big) = \exp\left(i\frac{k}{V}\int a_\tau\, d\tau \wedge dx^1 \wedge \cdots dx^d + \cdots\right)$$

is not translation invariant.

Shifting $x^i \to x^i + \epsilon^i$,

$$\exp\big(iCS(a,A)\big) \to \exp\left(iCS(a,A) - ik\sum_i \frac{\epsilon^i}{\ell^i}\left(\int dx^i d\tau f_{\tau i}\right)\right)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

All this does not affect the classical equations of motion and therefore does not affect the classical theory.

The subtlety in the quantum theory is with the action, not with the quantum loops.

Broken translations

$$\exp\big(iCS(a,A)\big) \to \exp\left(iCS(a,A) - ik\sum_i \frac{\epsilon^i}{\ell^i}\left(\int dx^i d\tau f_{\tau i}\right)\right)$$

The quantum theory does not have continuous $U(1)^d$ translation symmetry.

- $\int dx^i d\tau f_{\tau i} \in 2\pi\mathbb{Z}$ Hence, each $U(1) \to \mathbb{Z}_k$. $\left(T^i\right)^k = 1$.
- Interpret as an ABJ-like anomaly in translation.
- This is explicit (not spontaneous) breaking. Unlike a crystal, no phonons.
- For $d \geq 2$, translations do not commute

$$T^i T^j = e^{\frac{2\pi i}{k}\mathcal{Q}_{ij}} T^j T^i \quad , \qquad \mathcal{Q}_{ij} = \frac{1}{2\pi}\int dx^i dx^j f_{ij}$$

This is a non-central extension; not an 't Hooft anomaly.

Generalize: a theory with a topological $d-2$-form $U(1)$ global symmetry

- $d-2$-form $U(1)$ global symmetry: there is a conserved 2-form current $\mathcal{J}$ with integer charges $\int \mathcal{J} \in \mathbb{Z}$. (For $d=1$, we can think of it as a "-1-form symmetry.")
- Topological: $d\mathcal{J}=0$ without using the equations of motion. Locally, $\mathcal{J}=d\Phi$.

In the $U(1)$ gauge theory example, $\mathcal{J}=\frac{1}{2\pi}f$, $\Phi=\frac{1}{2\pi}a$.

Examples based on nonlinear sigma models:

- S^2 leads to the $O(3)$ sigma model.
- $\mathbb{T}^2$ leads to the $1+1$-dimensional $c=2$ CFT.
- Calabi-Yau manifolds lead to models on the string worldsheet.

Generalize: a theory with a topological $d-2$-form $U(1)$ global symmetry

Can repeat the previous discussion:

Couple to a classical background gauge field for this symmetry

$$\mathcal{J} \wedge A \quad \text{or} \quad \Phi \wedge dA$$

Specialize to A such that $dA = \frac{2\pi k}{V} dx^1 \wedge dx^2 \wedge \cdots dx^d$.

$$\mathcal{L} = \frac{2\pi k}{V} \Phi_t + \cdots$$

Classically, it is translation invariant. But since Φ_t is not well-defined:

- Anomalous translations – only discrete translations
- For $d \geq 2$ the translation symmetry is extended.

Special case: some nonlinear sigma models

- Can describe using a larger target space with a gauged $U(1)$. (A familiar example: the $O(3)$ sigma model presented as an S^3 target space with a $U(1)$ gauge field.) Then, this case is identical to the previous case with $\frac{k}{V} a_t$.
- Specialize further to the target space being a symplectic manifold (e.g., S^2).
  - Φ is the pullback of the Liouville potential to spacetime.
  - Adding Φ_t to the Lagrangian, terms with two time derivatives are negligible at low energies and can be dropped.

Example: from antiferromagnets to ferromagnets

$O(3)$ antiferromagnets are described in the continuum by the standard Lagrangian of the $O(3)$ sigma model.

It has a $d-2$-form symmetry associated with Skyrmions

$$\mathcal{J} = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \quad , \qquad \mathcal{Q} = \int \mathcal{J}$$

Coupling it to a background A as above, leads to the Berry term (Wess-Zumino term)

$$\frac{2\pi k}{V}\Phi_t = i\,\frac{k}{2V}\,\frac{(z\partial_t\bar{z} - \bar{z}\partial_t z)}{1+|z|^2}$$

Example: For $d=1$, include a θ-term, $\theta\mathcal{Q}$. A position-dependent $\theta = \frac{2\pi k}{\ell}x$, leads to $i\,\frac{k}{2\ell}\,\frac{(z\partial_t\bar{z} - \bar{z}\partial_t z)}{1+|z|^2}$.

Example: from antiferromagnets to ferromagnets

$$\frac{k}{V}\Phi_t = i\,\frac{k}{2V}\,\frac{(z\partial_t\bar{z} - \bar{z}\partial_t z)}{1+|z|^2}$$

Since the S^2 target space is symplectic, for nonzero k, at low energies, we can neglect the higher order terms in time derivatives and end up with a Lagrangian with first order time derivative.

This is the standard continuum description of a ferromagnet.

In this case, the breaking of continuous translation symmetry is known [Haldane (86); …].

The energy-momentum tensor

In all these cases, Noether procedure leads to a conserved energy-momentum current

$$\partial^\mu \Theta_{\mu\nu} = 0$$

The energy current $\Theta_{\mu t}$ is well-defined.

The conserved momentum current (recall, $\mathcal{J} = d\Phi$)

$$\Theta_{ti} = \Theta_{ti}^{(0)} + \frac{2\pi k}{V} \Phi_i \quad , \qquad \Theta_{ji} = \Theta_{ji}^{(0)} + \delta_{ji} \frac{2\pi k}{V} \Phi_t$$

is not well-defined. ($\Theta_{\mu i}^{(0)}$ depends on the other terms in the Lagrangian and is well-defined.)

Alternatively, when $\Theta_{\mu i}^{(0)}$ is nonzero, view $\Theta_{\mu i}^{(0)}$ as a non-conserved, but a well-defined current

$$\partial^\mu \Theta_{\mu i}^{(0)} = \frac{2\pi k}{V} \mathcal{J}_{ti}$$

The energy-momentum tensor

$$\Theta_{ti} = \Theta_{ti}^{(0)} + \frac{2\pi k}{V}\Phi_i \quad , \qquad \Theta_{ji} = \Theta_{ji}^{(0)} + \delta_{ji}\frac{2\pi k}{V}\Phi_t$$

$$\partial^\mu \Theta_{\mu i}^{(0)} = \frac{2\pi k}{V}\mathcal{J}_{ti} \quad , \qquad \partial^\mu \Theta_{\mu i} = 0$$

This is common in ABJ-anomaly.

The $\mathbb{Z}_k$ translation generators

$$T^i = \exp\left(\frac{i\ell^i}{k}\int \Theta_{ti} d^d x\right)$$

are well-defined (with the proper definition of the RHS).

The translation symmetry is discrete and is extended

$$\left(T^i\right)^k = 1$$

$$T^i T^j = e^{\frac{2\pi i}{k}\mathcal{Q}_{ij}} T^j T^i \quad , \qquad \mathcal{Q}_{ij} = \frac{1}{2\pi}\int dx^i dx^j \mathcal{J}_{ij}$$

Resurrecting the continuous translations

In special cases, the theory also has $d-1$-form symmetries, with charges $\mathcal{W}_i^a$.

- For example, for a $\mathbb{T}^2$ targe space, there are two such charges $\mathcal{W}_i^a$ associated with winding of the spatial direction i around the $a=1,2$ cycle of the target $\mathbb{T}^2$.

This leads to a $d-2$-form charge

$$\mathcal{Q}_{ij}=\mathcal{W}_i^1\mathcal{W}_j^2-\mathcal{W}_j^1\mathcal{W}_i^2$$

- In the $\mathbb{T}^2$ example, this is the wrapping number of the torus.

Coupling to a background field for $\mathcal{Q}_{ij}$ as above, the translation symmetry is explicitly broken and extended by $\mathcal{Q}_{ij}$.

Resurrecting the continuous translations

$$\mathcal{Q}_{ij} = \mathcal{W}_i^1 \mathcal{W}_j^2 - \mathcal{W}_j^1 \mathcal{W}_i^2$$

Coupling to a background field as above, the translation symmetry is explicitly broken and extended by $\mathcal{Q}_{ij}$.

Following [Choi, Lam, Shao; Cordova, Ohmori (22)], consider a projection operator $\mathcal{P}$ on states with vanishing winding charges $\mathcal{W}_i^a = 0$.

In the projected space, $\mathcal{Q}_{\mu\nu} = 0$.

Therefore, in this projected space, the translation symmetry is not broken and is not extended.

Continuous translation combined with $\mathcal{P}$, is an exact noninvertible continuous translation symmetry.

It is not invertible because of the use of the projection.

Infinite volume limit

The hero of the story is the term

$$\mathcal{L} = \frac{k}{V}\Phi_t + \cdots$$

$$V = \prod_i \ell^i \qquad , \qquad k \in \mathbb{Z}$$

Nonstandard fact: the Lagrangian density depends on the total volume.

- $V \to \infty$ with fixed k: the effect disappears
- $V, k \to \infty$ with fixed $\rho = \frac{k}{V}$:
  - for $d = 1$, the translation is $\mathbb{Z}$
  - for $d \geq 2$, it depends on the limit

Starting with a lattice model

Consider the lattice Heisenberg model on a square lattice with L^i sites in direction i. (Easy to generalize.)

The Hilbert space is

$$\mathcal{H} = \bigotimes_{\vec{r}} \mathcal{C}_{\vec{r}}$$

$\mathcal{C}_{\vec{r}}$ is a $2s+1$ dimensional Hilbert space – spin s of $SU(2)$.

The Hamiltonian is

$$H = \lambda \sum_{\langle rr' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'}$$

$\vec{S}_{\vec{r}}$ a spin s operator acting on $\mathcal{C}_{\vec{r}}$

- $\lambda > 0$ antiferromagnet
- $\lambda < 0$ ferromagnet

Heisenberg model

$$H = \lambda \sum_{\langle rr' \rangle} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}'}$$

- $\lambda > 0$ antiferromagnet. At low energies, $O(3)$ sigma model $\mathcal{L} \sim \frac{\partial_\mu z \partial^\mu \bar{z}}{(1+|z|^2)^2}$. (For $d = 1$, there is also a $\theta = 2\pi s$ term and the model is gapped/gapless depending on $s \; mod \; 1$.)
- $\lambda < 0$, ferromagnet. Can be described as in the discussion above, by adding $\frac{k}{V}\Phi_t = i\,\frac{k}{2V}\,\frac{(z\partial_t\bar{z} - \bar{z}\partial_t z)}{1+|z|^2}$ with $\frac{k}{2}$ the total spin

$$k = 2s \prod_i L^i$$

Heisenberg model

$$\frac{k}{V}\Phi_t \quad , \qquad\qquad k = 2s\prod_i L^i$$

- Lattice translation symmetry $\otimes\, \mathbb{Z}_{L^i}$, generated by $T^i_{Lattice}$.
- In the continuum, naively the translation symmetry is $U(1)^d$. Instead, it is discrete, generated by T^i with an extension by $\mathcal{Q}_{ij}$. (No $\mathcal{Q}_{ij}$ on the lattice.)

The relation between them,

$$T^i_{Lattice} \to \left(T^i\right)^{\frac{k}{L^i}}$$

Using the continuum relations, $\left(T^i\right)^{\frac{k}{L^i}}$ commute – no extension. (Otherwise, an inconsistency.)

Heisenberg model

$$T^i_{Lattice} \to \left(T^i\right)^{\frac{k}{L^i}} = \left(T^i\right)^{2s\prod_{j\neq i} L^j}$$

- $d = 1$, $T_{Lattice} \to T^{2s}$.
  - $s = \frac{1}{2}$ the continuum translation is the same as the lattice translation.
  - $s > \frac{1}{2}$ the continuum symmetry is larger than the lattice symmetry.
- $d > 1$, $T^i_{Lattice} \to \left(T^i\right)^{2s\prod_{j\neq i} L^j}$ the continuum symmetry is much larger than the lattice symmetry.

Heisenberg model

In modern terms, the famous Lieb-Schultz-Mattis theorem states that when $s \in \mathbb{Z} + \frac{1}{2}$, there is a mixed ‘t Hooft anomaly between the internal $SO(3)$ symmetry and lattice translation [Cheng, Zaletel, Barkeshli, Vishwanath, Bonderson (15); Metlitski, Thorngren (17); Cheng, NS (22)].

How is this anomaly realized in the continuum? (No anomaly in continuum translation.)

For $d = 1$ in the antiferromagnetic phase, a discrete internal $\mathbb{Z}_2^C$ (charge conjugation symmetry) emanates from lattice translation. The lattice anomaly is realized as an ’t Hooft anomaly of internal symmetries […, Metlitski, Thorngren (17); Cheng, NS (22)].

For $d \geq 1$, it is more subtle [Metlitski, Thorngren (17); NS (??)].

Heisenberg model

In the ferromagnetic phase

- No emanant internal charge conjugation symmetry $\mathbb{Z}_2^C$
- The low-energy Lagrangian depends only on k and not on s.

How is the lattice anomaly matched by the continuum theory? (No anomaly in continuum translation.)

The continuum theory has a discrete translation symmetry. The lattice translation symmetry is a subgroup of it.

For $s \in \mathbb{Z} + \frac{1}{2}$, that subgroup has a mixed 't Hooft anomaly with the internal $SO(3)$ symmetry.

For $s \in \mathbb{Z}$, there is no such anomaly.

Conclusions

We discussed continuum field theories with a topological $d - 2$-form $U(1)$ global symmetry in a particular background field.

Special cases:

- Dynamical $U(1)$ gauge field with constant charge density
- Certain nonlinear sigma models with a term with a single time derivative. Example: continuum Lagrangian for a ferromagnet.

The classical theory has $U(1)^d$ translation symmetry. In the quantum theory, it is broken by an ABJ-like anomaly.

- Anomalous translations – only discrete translations
- For $d \geq 2$ the translation symmetry is extended.

Thank you