Multiparticale scattering amplitudes from lattice QCD

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3 (and 2)-particle scattering amplitudes from lattice QCD

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Collaborators

Underlying motivations

- Determine properties of strong interaction resonances from QCD
  - E.g. exotics such as $T_{cc}(3875)^+ \rightarrow DD^* \rightarrow DD\pi$
Cornucopia of exotics

62 new hadrons at the LHC

[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …
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  - E.g. exotics such as $T_{cc}(3875)^+ \rightarrow DD^* \rightarrow DD\pi$

- Determine three particle “forces” for $3n$, $3\pi$, $3K$, ...
  - Needed to understand neutron star EoS, ...

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  • Needed to understand neutron star EoS, …

• Calculate weak decay amplitudes within the Standard Model, in order to search for new physics
  • E.g. $K \rightarrow 2\pi$ (essentially done), $K \rightarrow 3\pi$ (method known), & $D \rightarrow \pi^+\pi^-$, $K^+K^-$ (open question)
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Will focus most of the discussion on $\mathcal{M}(3\pi \rightarrow 3\pi)$
Outline

- The fundamental issue: relating finite and infinite-volume quantities
  - Resolution uses two-step method involving intermediate K matrices ($\mathcal{K}_2, \mathcal{K}_{df,3}$)
- Formalism for $2 \rightarrow 2$ scattering
  - Example application: $\pi\pi \rightarrow \sigma/f_0(500) \rightarrow \pi\pi$
- Sketch derivation of the three-particle formalism for $3\pi^+$
  - Tests of formalism, and generalizations
- Status of applications of the three-particle formalism
  - Fitting $\mathcal{K}_2, \mathcal{K}_{df,3}$ to $\pi^+\pi^+K^+$ spectra from LQCD
  - Comparing $\mathcal{K}_{df,3}(3\pi \rightarrow 3\pi)$ to ChPT (Chiral Perturbation Theory)
  - Preliminary results for $M(3\pi \rightarrow 3\pi)$ at nearly physical quark masses from LQCD
    - (Results for $DD\pi$ scattering, relevant for $T_{cc}^+$)
- Outlook
The fundamental issue
On the one hand…

- LQCD determines energies and properties of finite-volume eigenstates
  - Obtained by fits to (numerically-evaluated) Euclidean correlation functions:

\[
\int_{L} d^{3}x \ e^{-i \vec{P} \cdot \vec{x}} _{L}\langle \Omega | \sigma_{3\pi} (\tau, \vec{x}) \sigma_{3\pi}^{\dagger} (0) | \Omega \rangle_{L} \propto \sum_{n} \left| _{L}\langle 0 | \sigma_{3\pi}^{\dagger} (0) | 3\pi, \vec{P}, n \rangle_{L} \right|^{2} e^{-E_{n}\tau}; \quad (\tau > 0)
\]

\( \vec{P} = 2\pi \vec{n}/L \)
Assuming \( L^{3} \) box with PBC

\( \sigma_{3\pi} \sim 3\pi^{+} \)
Lives on timeslice

\( \sigma_{3\pi} \sim 3\pi^{+} \)
Tower of finite-volume states with quantum numbers of \( 3\pi^{+} \), with momentum \( \vec{P} \), and living in irreps of cubic group

\( E_{n} \)
Energies of said states
On the one hand…

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\]

- \( \mathbf{P} = 2\pi \mathbf{n}/L \)
  - Assuming \( L^3 \) box with PBC

- \( \sigma_{3\pi} \sim 3\pi^+ \)
  - Lives on timeslice

- \( E_n \) are physical quantities!
  - Can determine 5-10 levels for each choice of quantum numbers \( (\mathbf{P}, \text{irrep}, \ldots) \)
  - Can now begin to calculate with physical quark masses
  - Results come with statistical & systematic errors (e.g. need \( a \to 0 \))
  - Mostly, we just assume here that the \( E_n \) are provided to us
...while on the other

- We want infinite-volume scattering amplitudes

\[ M_3 \sim \]

In state  Out state
...while on the other

- We want infinite-volume scattering amplitudes

\[ M_3 \sim \]

- How do we relate these? A finite-volume QFT problem.

Discrete energy spectrum

\[ E_0(L) \]
\[ E_1(L) \]
\[ E_2(L) \]

Scattering amplitudes

\[ iM_{n\rightarrow m} \]
A related question:

- LQCD can also calculate matrix elements between finite-volume states

\[
L\langle \Omega | \sigma_{3\pi}(\tau_f, \vec{P}) | \int_L d^3 x \mathcal{H}_W(0, \vec{x}) K^\dagger(\tau_i, \vec{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} L\langle 3\pi, \vec{P}, n' | \mathcal{H}_W(0) | K, n \rangle_L e^{E_{n'} \tau_i}
\]

A physical quantity if \(E_{n'} = E_n\)
A related question:

- LQCD can also calculate matrix elements between finite-volume states

\[ L\langle \Omega | \sigma_{3\pi}(\tau_f, \vec{P}) \rangle \int_L d^3x \mathcal{H}_W(0, \vec{x}) K^\dagger(\tau_i, \vec{P}) | \Omega \rangle_L \propto \sum_{n',n} c_{n',n} e^{-E_n \tau_f} L\langle 3\pi, \vec{P}, n' | \mathcal{H}_W(0) | K, n \rangle_L e^{E_{n'} \tau_i} \]

- How are these related to decay amplitudes?

\[ \mathcal{A}(K \to 3\pi) = \text{out}\langle 3\pi | \mathcal{H}_W(0) | K \rangle \]
Two-step method

2 & 3 particle Spectra from LQCD

Quantization conditions

\[ QC2: \det \left[ F^{-1} + \mathcal{K}_2 \right] = 0 \]
\[ QC3: \det \left[ F_3^{-1} + \mathcal{K}_{df,3} \right] = 0 \]

Integral equations in infinite volume

Scattering amplitude \( M_3 \)

[These are the RFT forms, and assume \( \mathbb{Z}_2 \) symmetry]
Two-step method

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Integral equations in infinite volume

Scattering amplitude \( \mathcal{M}_3 \)

Infinite-volume K matrix:
Obtained from Feynman diagrams using PV prescription for poles;
Real, free of unitary cuts

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Integral equations in infinite volume

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle interaction free of unitary cuts, and with physical divergences subtracted; unphysical since depends on cutoff

Scattering amplitude $M_3$

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Incorporates initial- and final-state interactions, and ensures unitarity

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Infinite-volume K matrix:
Obtained from Feynman diagrams using PV prescription for poles; Real, free of unitary cuts

Scattering amplitude \( \mathcal{M}_3 \)
If parametrize K matrices, can continue \( \mathcal{M}_3 \) into the complex plane & look for resonances, etc.
Two-particle formalism

[Lüscher, 1986-91 + many subsequent works by many authors]

I will follow approach of [Kim, Sachrajda, & SS, 2005], generalized to use time-ordered PT following [Blanton & SS, 2020]
Generic relativistic FT (RFT) approach

- Study Minkowski time, finite-volume correlator

\[ C_L(E, \vec{P}) \equiv \int_L d^4x \, e^{iEt - i\vec{P} \cdot \vec{x}} \langle \Omega | T \{ \sigma_{2\pi}(x) \sigma_{2\pi}^\dagger(0) \} | \Omega \rangle_L \]

- For fixed \( \vec{P} \), poles in \( C_L \) occur when \( E = E_n \)

- Analyze in generic EFT for pions, (kaons, …) working to all orders in (TO)PT

  - For simplicity, assume exact isospin symmetry

  - Restrict kinematic range to \( 0 < E^* = \sqrt{E^2 - P^2} < 4M_\pi \)
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\[ C_L(E, \vec{P}) = \frac{1}{E - \omega_1 - \omega_2} + \frac{1}{E - \omega_1 - \omega_2} + \frac{1}{E - \omega_1 - \omega_2} + \frac{1}{E - \omega_1 - \omega_2} + \ldots \]

No need for \( i\epsilon \) in finite volume

Can go on shell

Cannot go on shell

Can go on shell

Can go on shell
Generic relativistic FT (RFT) approach

\[ C_L(E, \overrightarrow{P}) = \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]

\[ + \]

\[ + \ldots \]
Generic relativistic FT (RFT) approach

\[ C_L(E, \vec{P}) = \]

- Relevant—can go on shell
- Irrelevant—cannot go on shell

\[ \cdot \]

S. Sharpe, ``Multiparticle scattering from LQCD,'' Amplitudes24, 6/12/24
$C_L(E, \vec{P}) =$ \[ E, \vec{P} \]

Cuts divide into:
- Relevant—can go on shell
- Irrelevant—cannot go on shell

Three-momenta in loops are summed over finite-volume set
Use Poisson summation formula

\[
\frac{1}{L^3} \sum_k g(k) = \int \frac{d^3k}{(2\pi)^3} g(k) + \sum_{l \neq 0} \int \frac{d^3k}{(2\pi)^3} e^{i\tilde{L} \cdot \tilde{k}} g(k)
\]
Use Poisson summation formula

\[ \frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{l \neq 0} \int \frac{d^3k}{(2\pi)^3} e^{i \vec{l} \cdot \vec{k}} g(\vec{k}) \]

Exp. suppressed if \( g(\vec{k}) \) is smooth and \( g' \sim g/M_\pi \)
Use Poisson summation formula

$$\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3 k}{(2\pi)^3} g(\vec{k}) + \sum_{l \neq 0} \int \frac{d^3 k}{(2\pi)^3} e^{i L \vec{l} \cdot \vec{k}} g(\vec{k})$$

Exp. suppressed if $g(\vec{k})$ is smooth and $g' \sim g/M_\pi$

- Replace loop sums with integrals where summand/integrand is nonsingular
  - Drop exponentially suppressed terms ($e^{-M_\pi L}$, $e^{-(M_\pi L)^2}$, etc.) while keeping power-law dependence

S. Sharpe, "Multiparticle scattering from LQCD," Amplitudes24, 6/12/24
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- Replace loop sums with integrals where summand/integrand is nonsingular
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Expansion in relevant cuts

\[
C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + (A') A + (A') B_2 A + (A') B_2 B_2 A + \ldots
\]

- \( B_2 \) is the TOPT version of a Bethe-Salpeter kernel (2PI in s-channel)
  - \( A' \) and \( A \) are corresponding “endcaps”

\[
B_2 = \ldots
\]
Dealing with relevant cuts

\[ \frac{1}{L^3} \sum_{\vec{k}} f(E, \vec{P}, \vec{k}) \frac{1}{2} \frac{1}{4\omega_k \omega_{P-k}} \frac{1}{E - \omega_k - \omega_{P-k}} g(E, \vec{P}, \vec{k}) \]

\[ = \text{PV} \int \frac{d^3k}{(2\pi)^3} f(E, \vec{P}, \vec{k}) \frac{1}{2} \frac{1}{4\omega_k \omega_{P-k}} \frac{1}{E - \omega_k - \omega_{P-k}} g(E, \vec{P}, \vec{k}) \]

\[ + \sum_{\ell',m';\ell,m} f^{\text{on}}_{\ell' m'}(E^*) F_{\ell' m';\ell m}(E, \vec{P}, L) g^{\text{on}}_{\ell m}(E^*) \]

● \( F \) is a known, calculable kinematic finite-volume function

\[ F_{\text{PV;\ell' m';\ell m}}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \text{PV} \int \frac{d^3k}{(2\pi)^3} \right) \frac{\mathcal{Y}_{\ell' m'}(\vec{k}^*) \mathcal{Y}_{\ell m}(\vec{k}^*) h(\vec{k})}{2\omega_k 2\omega_{P-k}(E - \omega_k - \omega_{P-k})} \]

\[ \text{On-shell projected in pair CM frame, and decomposed into harmonics} \]

\[ \text{CM frame relative momentum} \]

\[ \text{UV cutoff} \]

\[ \text{Harmonic polynomial} \]
Key move

\[ \Sigma = \int + F \]

\[ B_2 \quad B_2 \quad \rightarrow \quad B_2 \int B_2 \quad + \quad B_2 \quad B_2 \]
Resummations

\[ C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + (A') A + (A') B_2 A + (A') B_2 B_2 A + \ldots \]
Resummations

\[ C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + A' A + A' B_2 A + \ldots \]

\[ = C_L^{(0)}(E, \vec{P}) + A' \int^+ F A + A' \int^+ F B_2 \int^+ F A + \ldots \]
Resummations

\[ C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + \frac{A'}{A} + \frac{A'}{B_2} \frac{A}{A} + \frac{A'}{B_2} \frac{B_2}{B_2} \frac{A}{A} + \ldots \]

\[ = C_L^{(0)}(E, \vec{P}) + \frac{A'}{A} \int + F \frac{A}{A} + \frac{A'}{B_2} \int + F \frac{B_2}{B_2} F \frac{A}{A} + \frac{A'}{B_2} \int + F \frac{B_2}{B_2} F \frac{B_2}{B_2} \int + F \frac{A}{A} + \ldots \]

\[ = C_\infty(E, \vec{P}) + \frac{A'}{\bar{A}} \cdot iF \cdot \bar{A} + \frac{A'}{\bar{A}} \cdot iF \cdot \bar{\mathcal{K}}_2 \cdot iF \cdot \bar{A} + \frac{A'}{iF} \cdot \bar{A} + \frac{A'}{iF} \cdot \bar{\mathcal{K}}_2 \cdot iF \cdot \bar{\mathcal{K}}_2 \cdot iF \cdot \bar{A} + \ldots \]

\[ \mathcal{K}_2 = B_2 + \frac{B_2}{B_2} \int \frac{B_2}{B_2} \ldots \]

\[ \bar{A} = A + \frac{B_2}{A} \int + \frac{B_2}{A} + \ldots \]
Resummations

\[ C_L(E, \vec{P}) = C_L^{(0)}(E, \vec{P}) + A' A + A' B_2 A + A' B_2 B_2 A + \ldots \]

\[ = C_L^{(0)}(E, \vec{P}) + A' \int + F A + A' \int + F B_2 \int + F A + A' \int + F B_2 \int + F B_2 \int + F A + \ldots \]

\[ = C_\infty(E, \vec{P}) + \mathcal{A}' \cdot iF \cdot \mathcal{A} + \mathcal{A}' \cdot iF \cdot i \mathcal{H}_2 \cdot iF \cdot \mathcal{A} + \mathcal{A}' \cdot iF \cdot i \mathcal{H}_2 \cdot iF \cdot i \mathcal{H}_2 \cdot iF \cdot \mathcal{A} + \ldots \]

\[ = C_\infty(E, \vec{P}) + \mathcal{A}' \cdot iF \cdot \frac{1}{1 + \mathcal{H}_2 \cdot F} \cdot \mathcal{A} \]

\[ \mathcal{H}_2 = B_2 + B_2 \int B_2 + \ldots \]

\[ \mathcal{A} = A + B_2 \int A + \ldots \]
Quantization condition (QC2)

\[
C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \vec{A}' \cdot iF \cdot \frac{1}{1 + \mathcal{K}_2 \cdot F} \cdot \vec{A}
\]

Has no L-dependent poles

Only source of L-dependent poles
Quantization condition (QC2)

\[ C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \vec{A}' \cdot iF \cdot \frac{1}{1 + H_2 \cdot F} \cdot \vec{A} \]

- Has no L-dependent poles
- Only source of L-dependent poles

**QC2:** finite-volume energies occur when

\[ \det(F^{-1} + H_2) = 0 \]

- Matrix indices are CM-frame \( \ell, m \)
- \( H_2 \) is an infinite-volume quantity: diagonal in \( \ell, m \)
- \( F \) depends on finite-volume size & geometry, mixes \( \ell, m \)
- In practical applications, must truncate in \( \ell \)
Step 2: relating $K_2$ to $M_2$

- Consider “finite-volume scattering amplitude”

$$M^{(off)}_{2,L} = B_2 + \ldots$$
Step 2: relating $\mathcal{K}_2$ to $\mathcal{M}_2$

- Consider “finite-volume scattering amplitude”

$$\mathcal{M}^{\text{(off)}}_{2,L} = B_2 + B_2 \cdot B_2 + \ldots$$

- Use similar steps as for $C_{2,L}$: project on $\ell, m$, project on shell, use “key move”

$$i\mathcal{M}_{2,L} = i\mathcal{K}_2 + i\mathcal{K}_2 \cdot iF \cdot i\mathcal{K}_2 + \ldots = i\mathcal{K}_2 \frac{1}{1 + F\mathcal{K}_2}$$
Step 2: relating $\mathcal{K}_2$ to $\mathcal{M}_2$

- Consider “finite-volume scattering amplitude”

$$\mathcal{M}_{2,L}^{(\text{off})} = \begin{array}{c}
\begin{array}{c}
\mathcal{B}_2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\mathcal{B}_2 \mathcal{B}_2
\end{array}
\end{array} + \ldots$$

- Use similar steps as for $C_{2,L}$: project on $\ell, m$, project on shell, use “key move”

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- Take $L \to \infty$ limit, regularizing integrals with $i\epsilon$ prescription

$$\mathcal{M}_{2,L} \to \mathcal{M}_2, \quad F_{\ell',m';\ell,m} \to -i\delta_{\ell',\ell}\delta_{m',m}\rho, \quad \rho = -i\sqrt{q^*}^2/16\pi E^*$$
Step 2: relating $\mathcal{H}_2$ to $\mathcal{M}_2$

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$$\mathcal{M}^{(\text{off})}_{2,L} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
B_2
\hline
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
B_2
\hline
\end{array}
\end{array}
\end{array} + \ldots \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hline
B_2
\hline
\end{array}
\end{array}
\end{array}$$

- Use similar steps as for $C_{2,L}$: project on $\ell, m$, project on shell, use “key move”

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- Leads to standard relation between $\mathcal{M}_2$ & $\mathcal{H}_2$, showing that $\mathcal{H}_2$ is the standard, relativistic two-particle K matrix

$$\mathcal{M}_2 = \mathcal{H}_2 \frac{1}{1 - i\rho\mathcal{H}_2}$$
Applications of QC\(\Phi\) are well developed

- LQCD gives spectrum, fit to QC\(\Phi\) with parametrized, truncated \(\mathcal{H}_2\), determine \(\mathcal{M}_2\), look for poles in complex plane

- State-of-the-art involves multiple channels, particles with spin, as well as decay and transition amplitudes

- Nice recent example [Rodas et al., 2304.03762 (PRD)] for \(\pi\pi \rightarrow \sigma/f_0(500) \rightarrow \pi\pi\) where crossing symmetry/dispersion relations restrict parametrizations of \(\mathcal{H}_2\)
Applications of QC2 are well developed

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- State-of-the-art involves multiple channels, particles with spin, as well as decay and transition amplitudes

- Nice recent example [Rodas et al., 2304.03762 (PRD)] for \( \pi\pi \rightarrow \sigma/f_0(500) \rightarrow \pi\pi \) where crossing symmetry/dispersion relations restrict parametrizations of \( \mathcal{H}_2 \)

91 \( \pi\pi \) levels
\( M_\pi \approx 240 \) MeV
Applications of QC\(2\) are well developed

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- Nice recent example [Rodas et al., 2304.03762 (PRD)] for \(\pi\pi \rightarrow \sigma f_0(500) \rightarrow \pi\pi\)

\[
\pi\pi \rightarrow \sigma f_0(500) \rightarrow \pi\pi
\]

\[
\text{“good agreement” & reduced spread}
\]

\[
m_\pi \sim 239 \text{ MeV}
\]

\[
\text{“poor agreement”}
\]
Three-particle formalism

[Hansen & SS, 2014 & 2015]

[Blanton & SS, 2020]
RFT approach

- Study Minkowski time, finite-volume correlator, and look for poles

\[ C_L(E, \vec{P}) \equiv \int_L d^4x e^{iEt-i\vec{P} \cdot \vec{x}} \langle \Omega | T \left\{ \sigma_{3\pi}(x)\sigma_{3\pi}^\dagger(0) \right\} | \Omega \rangle_L \]

- Restrict kinematic range to \( M_\pi < E^* = \sqrt{E^2 - P^2} < 5M_\pi \)

- Use TOPT, and decompose into kernels, separated by relevant (3 particle) cuts

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S. Sharpe, "Multiparticle scattering from LQCD," Amplitudes24, 6/12/24
New features

\[ C_L(E, \vec{P}) = \ldots + \sum_{\ell, m} \hat{A}' B_2 B_3 \hat{A} + \ldots \]

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum \( \Rightarrow \) Indices are \( \vec{k}, \ell, m \)
**New features**

\[ C_L(E, \vec{P}) = \ldots + \]

\[ \hat{A} \]

\[ \ell, m \]

\[ \vec{k} \]

\[ B_2 \]

\[ B_3 \]

\[ \hat{A} \]

\[ + \ldots \]

- **Sum over 3 momenta:** when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are \( \vec{k}, \ell, m \)

- **Sum over spectator momentum** leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over \( \vec{k} \), and avoids left-hand cut in \( B_2 \)
\[ C_L(E, \overrightarrow{P}) = \ldots + \frac{(\ell, m)}{k} \quad \frac{\hat{A}}{B_2} \quad \frac{\hat{B}_2}{B_3} \quad \frac{\hat{A}}{+ \ldots} \]

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum \( \Rightarrow \) Indices are \( \vec{k}, \ell, m \)

- Sum over spectator momentum leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over \( \vec{k} \), and avoids left-hand cut in \( B_2 \)

- Switches between spectators: leads to two types of finite-volume kinematic function, \( F \) and \( G \)
New features

\[ C_L(E, \vec{P}) = \ldots + \]

\[ \begin{array}{c}
\hat{A}' \\
\hat{k}
\end{array} \quad \begin{array}{c}
B_2 \\
\ell, m
\end{array} \quad \begin{array}{c}
B_2 \\
B_3
\end{array} \quad \begin{array}{c}
\hat{A}
\end{array} \quad + \ldots
\]

- Sum over 3 momenta: when project a pair on shell, have additional finite-volume spectator momentum ⇒ Indices are \( \vec{k}, \ell, m \)

- Sum over spectator momentum leads to subthreshold pair when project on shell
  - Introduce smooth cutoff function so pair cannot go too far below threshold
  - Truncates sum over \( \vec{k} \), and avoids left-hand cut in \( B_2 \)

- Switches between spectators: leads to two types of finite-volume kinematic function, \( F \) and \( G \)

- Tree particle Bethe-Salpeter kernel \( B_3 \): once “dressed” it will become \( \mathcal{H}_{df,3} \)
...skipping over details...

- Can reorganize into geometric series and sum to find poles
  - Involves $\mathcal{H}_3$ that is neither Lorentz invariant nor symmetric under particle exchange
- Nasty algebraic reorganization brings $\mathcal{H}_3$ into symmetric, Lorentz-invariant form

$$\text{QC3: } \det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right] = 0 \quad \text{[cf. QC2: } \det(F^{-1} + \mathcal{H}_2) = 0]$$

$$F_3 = \frac{1}{2\omega L^3} \left[ \frac{\widetilde{F}}{3} - \frac{1}{1/\mathcal{H}_{2,L} + \widetilde{F} + \widetilde{G}} \right]$$
Explicit forms

- $F$ & $G$ are known geometrical functions, containing cutoff function $H(k)$

$$
\widetilde{F}_{p\ell'\ell'; k\ell m} = \delta_{pk} H(\vec{k}) F_{\ell'\ell'; \ell m}(E - \omega_k, \vec{P} - \vec{k}, L)
$$

$$
F_{\ell'\ell'; \ell m}(E, \vec{P}, L) = \frac{1}{2} \left( \frac{1}{L^3} \sum_{\vec{k}} - \text{PV} \int \frac{d^3 k}{(2\pi)^3} \right) \frac{\mathcal{Y}_{\ell'\ell m}(\vec{k}^*) \mathcal{Y}_{\ell'\ell m}(\vec{k}^*) H(\vec{k})}{2\omega_k 2\omega_{P-k} (E - \omega_k - \omega_{P-k})}
$$

$$
\mathcal{Y}_{\ell m}(\vec{k}^*) = \sqrt{4\pi} \left( \frac{k^*}{q^*} \right)^\ell Y_{\ell m}(\hat{k}^*)
$$

$$
G_{p\ell'\ell'; k\ell m} = \left( \frac{k^*}{q^*_p} \right)^\ell' \frac{4\pi \mathcal{Y}_{\ell'\ell m}(\vec{k}^*) H(\vec{p}) H(\vec{k}) Y_{\ell m}(\hat{p}^*)}{(P - \vec{k} - \vec{p})^2 - m^2} \left( \frac{p^*}{q^*_k} \right)^\ell \frac{1}{2\omega_k L^3}
$$

Relativistic form introduced in [BHS17]
\( \mathcal{K}_{\text{df},3} \) has known, complicated expression; can crudely represent as

\[
\mathcal{K}_{\text{df},3} = B_3 + \left( B_2 \right) \int B_3 + \left( B_3 \right) \int B_3 + \ldots
\]
\[ K_{\text{df},3} \]

- \( K_{\text{df},3} \) has known, complicated expression; can crudely represent as

\[
K_{\text{df},3} = B_3 + \int B_2 B_3 + \int B_3 B_3 + \ldots
\]

- Key properties:
  - Infinite-volume quantity, with same symmetries as \( M_3 \)
  - Unlike \( M_3 \), does not contain one-particle exchange singularities
  - Real, smooth function of momenta, aside from possible three-particle poles
  - Relativistically invariant, so can expand about threshold in “effective-range exp.”
  - Unphysical since depends on cutoff function \( H(\vec{k}) \)

- Can think of \( K_{\text{df},3} \) as a quasi-local three-particle interaction
Step 2: relating $\mathcal{K}_{df,3}$ to $M_3$

- Consider “finite-volume scattering amplitude” in TOPT

$M^{(\text{off})}_{23,L} = B_2 + B_2 B_2 + B_3 + B_2 B_3 + B_3 B_3 + \ldots$

- Resum geometric series; project onto $\vec{k}, \ell, m$; project on shell; use “key move”; algebraic reorganization; take $L \to \infty$ (i.e.) limit

- Result is set of integral equations relating $M_3$ to $M_2$ and $\mathcal{K}_{df,3}$ (all on shell)

\[
M_3 = \lim_{L \to \infty} \mathcal{S} \left\{ \mathcal{D}^{(u,u)}_L + M^{(u,u)}_{\text{df,3},L} \right\}, \quad \mathcal{S} \Rightarrow \text{symmetrization}
\]

\[
i \mathcal{D}^{(u,u)}_L = i M_{2,L} \tilde{G} i M_{2,L} \frac{1}{1 - i \tilde{G} i M_{2,L}}, \quad M_{2,L} = 2 \omega L^3 M_2
\]

\[
i M^{(u,u)}_{\text{df,3},L} = \mathcal{L}^{(u)}_L i \mathcal{K}_{df,3} \frac{1}{1 - i F_3 i \mathcal{K}_{df,3}} \mathcal{L}^{(u)\dagger}_L
\]

\[
\mathcal{L}^{(u)}_L = \frac{1}{3} + \frac{1}{1 - i M_{2,L} i \tilde{G}} i M_{2,L} i \tilde{F}
\]
Step 2: relating $K_{df,3}$ to $M_3$

$$M_3 = \lim_{L \to \infty} S \left\{ D^{(u,u)}_L + M^{(u,u)}_{df,3,L} \right\} = D + M_{df,3}$$

- $D$ contains all divergent contributions to $M_3$, but depends on cutoff function $H(\vec{k})$

$$D = S \left\{ \frac{M_2}{G} \frac{G}{M_2} \right\} + \left\{ \frac{M_2}{G} \frac{G}{M_2} \frac{M_2}{G} \right\} + \ldots $$

- $M_{df,3}$ is divergence-free, equals $K_{df,3}$ at leading order, and is also cutoff-dependent

$$M_{df,3} = K_{df,3} + S \left\{ \frac{M_2}{\rho} K_{df,3} \right\} + \left\{ K_{df,3} \frac{\rho}{M_2} \right\} + \ldots $$

- “Decorations” ensure that $M_3$ is unitary

- Methods for solving integral equations, and analytically continuing to complex momenta, are now well established [Briceño, Dawid, Hansen, Islam, Jackura, 2020-23]

  - In practice, project on definite overall $J^P$
Tests of formalism [Refs. at end]
Tests of formalism [Refs. at end]

- Expansion in $L^{-1}$ of ground-state 3-particle energy agrees with NRQM through $L^{-5}$
  - Agreement extended to $L^{-6}$ in relativistic $\phi^4$ theory at 3-loop order
- Volume dependence of energy and form factor of Efimov “trimer” matches NRQM
- s-channel unitarity of $M_3$
- Decomposition into $D + M_{df,3}$ checked at NLO in ChPT for $3\pi$
  - Leads to NLO ChPT prediction for $H_{df,3}$
- Three approaches to deriving formalism lead to equivalent results
Status: formalism

- 3 identical spinless particles [Hansen & SRS 14, 15 (RFT); Hammer, Pang, Rusetsky 17 (NREFT); Mai, Döring 17 (FVU)]
  - Applications: $3\pi^+$, $3K^+$, as well as $\phi^4$ theory
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS 17]
  - Step on the way to $N(1440) \to N\pi, N\pi\pi$, etc.
- 3 degenerate but distinguishable spinless particles, e.g. $3\pi$ with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS 20]; $I = 1$ case in FVU approach [Mai et al., 21]
  - Potential applications: $\omega(782), a_1(1260), h_1(1170), \pi(1300), \ldots$
- 3 nondegenerate spinless particles [Blanton, SRS 20]
  - Potential applications: $D^+_sD^0\pi^-$
- 2 identical +1 different spinless particles [Blanton, SRS 21]
  - Applications: $\pi^+\pi^+K^+, K^+K^+\pi^+$
- 3 identical spin-$\frac{1}{2}$ particles [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $3n, 3p, 3\Lambda$
- $DD\pi$ for all isospins (also $BB\pi, KK\pi$) [Draper, Hansen, Romero-López, SRS 23]
  - Potential applications: $T_{cc} \to D^*D$ incorporating LH cut
- Multiple three-particle channels: $\eta\pi\pi + K\bar{K}\pi$ [Draper & SRS 24]
  - Potential applications: $b_1(1235), \eta(1295)$
Applications of 3-particle formalism:

Fitting $\mathcal{K}_2, \mathcal{K}_{df,3}$ to $\pi^+\pi^+K^+$ spectra from LQCD

[Draper, Hanlon, Hörz, Morningstar, Romero-López & SRS, 2302.13587 (JHEP)]
π⁺π⁺K⁺ interactions

• System with weakly repulsive interactions
  • No resonances in two-particle subchannels or in three-particle system
• Simultaneously fit to several spectra to QC2/QC3 to obtain \( \mathcal{K}_2 \) and \( \mathcal{K}_{df,3} \)

• Parametrize \( \mathcal{K}_{df,3} \) (and \( \mathcal{K}_2 \)) as the most general smooth functions consistent with particle interchange, time-reversal and parity symmetries, using an expansion about threshold
  • s-wave interactions in \( \pi^+\pi^+ \) (sub)channel, s- and p-wave in \( \pi^+K^+ \); 9 or 10 parameters in all
Lattices used in pilot calculation

- Improved Wilson fermions at \( a = 0.064 \) fm (CLS lattices)

<table>
<thead>
<tr>
<th>((L/a)^3 \times (T/a))</th>
<th>(M_\pi) [MeV]</th>
<th>(M_K) [MeV]</th>
<th>(N_{cfg})</th>
<th>(t_{src}/a)</th>
<th>(N_{ev})</th>
<th>dilution</th>
<th>(N_r(\ell/s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>N203</td>
<td>48(^3) × 128</td>
<td>340</td>
<td>440</td>
<td>771</td>
<td>32, 52</td>
<td>192</td>
<td>(LI12, SF)</td>
</tr>
<tr>
<td>D200</td>
<td>64(^3) × 128</td>
<td>200</td>
<td>480</td>
<td>2000</td>
<td>35, 92</td>
<td>448</td>
<td>(LI16, SF)</td>
</tr>
</tbody>
</table>

D200 configurations

\(L = 4.1\) fm

\(\pi^+\) \(M_\pi L = 4.1\)

\(K^+\) \(M_K L = 10\)
Example of fit

**Lattice QCD spectrum**

\[ \frac{E_{\text{CM}}}{M_\pi}, \pi \pi K, \text{N203, } a \simeq 0.063 \text{ fm} \]

\[ M_\pi = 340 \text{ MeV, } M_K = 440 \text{ MeV} \]

**O(50) energy levels!**

Simultaneous fit to 27 \( \pi^+ \pi^+ \), 19 \( \pi^+ K^+ \), & 36 \( \pi^+ \pi^+ K^+ \) \( \Rightarrow 82 \) levels with 9 parameters

\[ \chi^2/\text{DOF} = 119/(82 - 9) \]
Results: scattering lengths

- 2-particle s-wave scattering lengths are well determined
- All are repulsive and consistent with ChPT
  - Evidence for small discretization errors

\[ L_{r_i}(\mu_2) = L_{r_i}(\mu_1) + i \frac{16}{5} f_{i2} \ln \frac{\mu_1}{\mu_2} \]

\[ (5.2) \]

Varying the choice of \( f_i \) to take for the initial scale (using the physical value of \( f_i \), or the value on either of the ensembles) leads to changes in \( L_{5} \) that are significantly smaller than the error. Our result for \( L_{5} \) is in agreement with all values in the literature, although we note that our error is much larger than that in the other values.

Figure 5. Results for \( M_{fifi} \), \( M_{fifK} \) and \( M_{KfK} \) as a function of \( M_{f}^{2}/F_{f}^{2} \), where \( M_{fi} = (M_{fi} + M_{K})/2 \). The LO ChPT result is shown, along with a fit to NLO SU(3) ChPT.

The shaded bands show the uncertainties in the fit.

Next, we discuss our results for the effective range parameters, which are presented in table 18 in the combination \( M_{XY}a_{XY} \). For the case of identical particles (\( X = Y = fi \) or \( K \)), the LO ChPT prediction from section 3.3 is that this quantity equals 3. For two pions, the results lie 15% and 25% below this prediction on the D200 and N203 ensembles, respectively, which is consistent with being due to an NLO correction. For two kaons, the results lie very far away from the LO prediction. Both findings are qualitatively similar to those obtained in Ref. [49].

For the \( fiK \) channel, which is a novel result of this work, the LO ChPT prediction—given in eq. (3.22)—depends on the ensemble:

\[ \begin{align*}
  M_{fi}^{2}a_{fiK}^{0}r_{fiK}^{0} & = 1.597, \quad \text{LO ChPT D200} \\
  M_{fi}^{2}a_{fiK}^{0}r_{fiK}^{0} & = 2.395. \quad \text{LO ChPT N203}
\end{align*} \]

Our results in table 18 lie \( \geq 25\% \) and \( \geq 30\% \), respectively, below the LO ChPT prediction. Again we view this as reasonable consistency, given the absence of NLO corrections.
s-wave contributions to $\mathcal{K}_{df,3}$

- Evidence for nonzero values ($2-5\sigma$)
- Overall effect of $\mathcal{K}_{df,3}$ is repulsive
- LO ChPT predicts opposite sign (but see later)

S. Sharpe, “Multiparticle scattering from LQCD,” Amplitudes24, 6/12/24
Applications of 3-particle formalism:

Calculating $\mathcal{H}_{df,3}$ for $3\pi \rightarrow 3\pi$ in ChPT

[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206 (JHEP) & 2401.14293 (JHEP) ]
2π/3π K matrices vs ChPT

2π⁺ scattering length

3π⁺ K matrix

[Results from Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]

- LO ChPT describes 2-pion sector well
- Large discrepancy in 3-pion sector!
NLO ChPT for $\mathcal{K}_{df,3}$

- Integral equations simplify to:

$$\mathcal{K}_{df,3}^{\text{NLO}} = \text{Re} \, \mathcal{M}_{df,3}^{\text{NLO}}$$

- One-particle-irreducible diagrams
- Bull's-head subtraction
- One-particle-exchange diagrams
- One-particle-exchange subtraction

NLO 6-pion amplitude computed in
- [Bijnens, Husek 2107.06291]
- [Bijnens, Husek, Sjö, 2206.14212]
Comparison to LQCD

- (Very) large NLO corrections
- Discrepancy with LO ChPT resolved!
- ChPT not trustworthy for $\kappa_1$
Applications of 3-particle formalism:

Results for $\mathcal{M}(3\pi \rightarrow 3\pi)$ at nearly physical quark masses

[Dawid, Draper, Hanlon, Hörz, Skinner, Morningstar, Romero-López & SRS, in progress]
Example of complete application

- First calculation used $M_\pi \approx 390$ MeV, $a \approx 0.12$ fm, $L \approx 2.5$ & 2.9 fm
  - [Hansen, Briceño, Dudek, Edwards, Wilson (HADSPEC collaboration), 2009.04931, PRL 21]
- We use almost physical quark masses (E250 CLS ensemble, 500 configurations)
  - $96^3 \times 192, a = 0.064$ fm, $M_\pi = 130(1)$ MeV, $M_K = 488(5)$ MeV (isosymmetric)

$L = 6.2$ fm
$2\pi^+$ amplitudes

$\mathcal{M}^{(s)}_{2\pi^+}$

$E_{\text{CM}}^2/M_\pi^2$

Adler zero

Threshold

Left-hand cut

Preliminary

s and d waves

$\text{Re } \mathcal{M}^{(s=0)}_{2\pi^+}$

$10 \text{ Im } \mathcal{M}^{(s=0)}_{2\pi^+}$

$10^2 \text{ Re } \mathcal{M}^{(s=2)}_{2\pi^+}$

$10^7 \text{ Im } \mathcal{M}^{(s=2)}_{2\pi^+}$
3π⁺ amplitude ($J^P = 0^-$)

Equilateral triangle configuration

$M_\pi^2 M_{3\pi^+}$

Threshold

$E_{CM}/M_\pi$

$M_\pi^2 M_{df,3}$

$E_{CM}/M_\pi$

$M_3 = D + M_{df,3}$

$D = S \left\{ M_2 G + M_2 G M_2 + \cdots \right\}$

$s & d$ waves in $\mathcal{K}_2$ & $\mathcal{K}_{df,3}$

Preliminary

S. Sharpe, "Multiparticle scattering from LQCD," Amplitudes 24, 6/12/24

47/50
Summary & outlook
Summary & Outlook

- Two-particle sector is entering precision phase
  - Frontier is two nucleons, and form factors of mesonic resonances

- Major steps have been taken in the three-particle sector
  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Initial discrepancy with LO ChPT explained by large NLO contributions
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open

- Next steps in implementation
  - $T^{+}_{cc} \to D^*D \to DD\pi$
  - $3\pi(I = 2) \leftrightarrow \rho\pi$ ; $3\pi(I = 0) \leftrightarrow \omega(782) \leftrightarrow K\bar{K}(I=0)$ (WZW term)
  - $N\pi\pi \leftrightarrow \Delta\pi$ ; $N\pi\pi + N\pi$ [Roper]

- Next steps in formalism
  - $NNN(I = \frac{1}{2}), N\pi\pi + N\pi$ [for Roper] & $NN\pi + NN$ (all underway)
  - Four particles!
The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.
Thank you!

Questions?
References
(Highly-)selected 2-particle refs

★ Original papers


★ Generalizations


RFT 3-particle papers

Max Hansen & SRS:
“Relativistic, model-independent, three-particle quantization condition,”
arXiv:1408.5933 (PRD) [HS14]
“Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,”
arXiv:1504.04028 (PRD) [HS15]
“Perturbative results for 2- & 3-particle threshold energies in finite volume,”
arXiv:1509.07929 (PRD) [HSPT15]
“Threshold expansion of the 3-particle quantization condition,”
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“Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,”
arXiv: 1609.04317 (PRD) [HSBS16]
“Lattice QCD and three-particle decays of Resonances,”
Raúl Briceño, Max Hansen & SRS:
“Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles,”
arXiv:1701.07465 (PRD) [BHS17]

“Numerical study of the relativistic three-body quantization condition in the isotropic approximation,”
arXiv:1803.04169 (PRD) [BHS18]


SRS
“Testing the threshold expansion for three-particle energies at fourth order in $\phi^4$ theory,”
arXiv:1707.04279 (PRD) [SPT17]

Tyler Blanton, Fernando Romero-López & SRS:
“Implementing the three-particle quantization condition including higher partial waves,” arXiv:1901.07095 (JHEP) [BRS19]

“I=3 three-pion scattering amplitude from lattice QCD,”
arXiv:1909.02973 (PRL) [BRS-PRL19]

“Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems” 2111.12734 (JHEP)
Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:


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“Generalizing the relativistic quantization condition to include all three-pion isospin channels”, arXiv:2003.10974 (JHEP) [HRS20]

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“Equivalence of relativistic three-particle quantization conditions,”
arXiv:2007.16190 (PRD) [BS20b]
“Relativistic three-particle quantization condition for nondegenerate scalars,”
“Three-particle finite-volume formalism for \(\pi^+\pi^+K^+\) & related systems,” arXiv:2105.12904 (PRD)

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“3\(\pi^+\) & 3\(K^+\) interactions beyond leading order from lattice QCD,” arXiv:2106.05590 (JHEP)

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“Interactions of \(\pi K\), \(\pi\pi K\) and \(KK\pi\) systems at maximal isospin from lattice QCD,” arXiv:2302.13587 (JHEP)
Zach Draper, Max Hansen, Fernando Romero-López & SRS:
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of the T_{cc}(3875)^+,” arXiv:2401.06609 (JHEP)

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Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820, PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitudes]
- A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]

★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ Other numerical simulations

- F. Romero-López, A. Rusetsky, C. Urbach, 1806.02367, JHEP [2- & 3-body interactions in $\phi^4$ theory]
- M. Fischer et al., 2008.03035, Eur.Phys.J.C [2$\pi^+$ & 3$\pi^+$ at physical masses]
- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in $\phi^4$ theory]
Other work

★ Other RFT (and related) derivations

- A. Jackura, 2208.10587, PRD [3-body scattering and quantization conditions from S-matrix unitarity]


★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating 3π⁺ spectrum and using to determine three-particle scattering amplitude]

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  - J-Y. Pang, R. Bubna, F. Müller, A. Rusetsky, J-J. Wu, 2312.04391 [Lellouch-Lüscher factor for $K \rightarrow 3\pi$ decays]
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Alternate 3-particle approaches

★ Finite-volume unitarity (FVU) approach

- M. Mai & M. Döring, 1709.08222, EPJA [formalism]
- M. Mai et al., 1706.06118, EPJA [unitary parametrization of $M_3$ involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]
- D. Seivert, M. Mai, U-G. Meißner, 2212.02171, JHEP [Particle-dimer approach for the Roper resonance]

★ HALQCD approach

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]
Backup slides
Matrix structure in QC3

- All quantities are infinite-dimensional matrices with indices $k \ell mi$ describing 3 on-shell particles

  [finite volume “spectator” momentum: $k = (2\pi/L)n$] \times [2-particle CM angular momentum: $\ell, m$] \times [spectator flavor: $i$]

\[ (\omega_k, \vec{k}) \]

\[ (E - \omega_k, \vec{P} - \vec{k}) \]

\[ \text{BOOST} \]

\[ \hat{a}^* \rightarrow \ell, m \]

- For large $k$ (at fixed $E, L$), the other two particles are below threshold

- Must include such configurations, by analytic continuation, up to a cut-off at $k \approx m$ [Polejaeva & Rusetsky, `12]
**Divergence-free K matrix**

- $K_{df,3}$ has the same symmetries as $M_3$: relativistic invariance, particle interchange, $T$-reversal

**M$_2$, K$_2$**

- 12 momentum components
- 10 Poincaré generators

2 degrees of freedom

$s = E^2 + \theta$

**M$_3$, K$_{df,3}$**

- 18 momentum components
- 10 Poincaré generators

8 degrees of freedom

$s = E^2 + 7 \text{ “angles”}$

- Need more parameters to describe $K_{df,3}$ than $K_2$ (will be discussed in lecture 3)

- Why $K_2$ and $K_{df,3}$ appear in QC3, rather than $M_2$ and $M_{df,3}$, will be explained shortly

S. Sharpe, “Multiparticle scattering from LQCD,” Amplitudes24, 6/12/24
Threshold expansion for $\mathcal{K}_{df,3}$

- $\mathcal{K}_{df,3}$ is a real, smooth function which is Lorentz, P and T invariant
- Expand about threshold in powers of $\Delta = (s - 9M^2_\pi)/9M^2_\pi$, $\bar{t}_{ij} = (p'_i - p_j)^2/9M^2_\pi$, ...

$$\mathcal{K}_{df,3} = \mathcal{K}^{\text{iso},0}_{df,3} + \mathcal{K}^{\text{iso},1}_{df,3} \Delta + \mathcal{K}^{\text{iso},2}_{df,3} \Delta^2 + \mathcal{K}_A \Delta_A + \mathcal{K}_B \Delta_B + \mathcal{O}(\Delta^3)$$

- Depend on CM energy
- Angular dependence

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathcal{K}_B$ contributes to nontrivial irreps
Sensitivity to $K_{df3}$

Simultaneous fit to 28 $K^+K^+$, 16 $\pi^+K^+$, & 29 $K^+K^+\pi^+$ levels with 10 parameters on D200: $\chi^2/\text{DOF} = 162/(73-10)$

S. Sharpe, "Multiparticle scattering from LQCD," Amplitudes24, 6/12/24
NLO ChPT results for $\mathcal{K}_{df,3}$

$$\kappa = \frac{1}{(16\pi^2)}$$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[ -3\kappa(35 + 12 \log 3) - \mathcal{D}_0 + 111L + \ell^r_{(0)} \right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[ -\frac{\kappa}{20}(1999 + 1920 \log 3) - \mathcal{D}_1 + 384L + \ell^r_{(1)} \right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \frac{207\kappa}{1400} (2923 - 420 \log 3) - \mathcal{D}_2 + 360L + \ell^r_{(2)},$$

$$\mathcal{K}_A = \left(\frac{M_\pi}{F_\pi}\right)^6 \frac{9\kappa}{560} (21809 - 1050 \log 3) - \mathcal{D}_A - 9L + \ell^r_{(A)},$$

$$\mathcal{K}_B = \left(\frac{M_\pi}{F_\pi}\right)^6 \frac{27\kappa}{1400} (6698 - 245 \log 3) - \mathcal{D}_B + 54L + \ell^r_{(B)}.$$

Numerical coefficients depend on cutoff $H(\mathbf{k})$.

$\mu$-dependence cancels.

LEC dependence cancels.

$L \equiv \kappa \log \left(\frac{M_\pi^2}{\mu^2}\right)$.
Comparison to LQCD

- $\mathcal{K}_B$ first appears at NLO in ChPT
- Discrepancy may be resolved by NNLO terms?
Finite- and infinite-volume analysis of the tetraquark $T_{cc}^{+}(3875)$

SUMMARY

1) We lay out a strategy for a rigorous determination of $T_{cc}$ and related systems from Lattice QCD
2) We propose resolution of the so-called "left-hand cut problem"
Available lattice results

Signature of a doubly charm tetraquark pole in $D D^*$ scattering on the lattice
Padmanath, Prelovsek, PRL 129, 032002 (2022)

Towards the quark mass dependence of from lattice QCD
Collins, Nefediev, Padmanath, Prelovsek, PRD 109 (2024) 9, 094509

$m_\pi \approx 280$ MeV

Thresholds are inverted but the three-body effects still play an important role in the analysis

$m_\pi \approx 146$ MeV
$m_\pi \approx 350$ MeV
$m_\pi \approx 391$ MeV

Lyu et al., PRL 131, 161901 (2023)
Chen et al. PLB 833, 137391 (2022)
Whyte, Wilson, Thomas.

$m_\pi \approx 280$ MeV

168 MeV

$T^+_{cc}$ $D^*D$ $DD\pi$
The left-hand cut problem

Role of the left-hand cut contributions on pole extractions

Presence of the left-hand cut:
a) invalidates the Lüscher

Incorporating DDπ effects and left-hand cuts in lattice QCD studies of $T_{cc}$

\[ s_{lhc} = s_{\text{thr}} - m_\pi^2 + (m_{D^*} - m_D)^2 \]

\[ \sqrt{s_{lhc}} \approx 3966 \text{ MeV} \quad \sqrt{s_{\text{thr}}} \approx 3975 \text{ MeV} \]

\[ s_{lhc,2} = s_{\text{thr}} - 4m_\pi^2 + (m_{D^*} - m_D)^2 \]

\[ \sqrt{s_{lhc,2}} \approx 3937 \text{ MeV} \quad \sqrt{s_{\text{thr}}} \approx 3975 \text{ MeV} \]
Single-channel approximation

\[ \mathcal{K}_3^E = 0 \]

\[ \mathcal{K}_3^E = -1.9 \cdot 10^6 \]

\[ [\mathcal{M}_3(3S_1|3S_1)] \quad (\kappa = m_{\pi}/m_D = 0.145) \]

\[ J^P = 1^+ \]

Pole position

Preliminary

Lattice: PRL 129, 032002

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