Amplitudes 2024
IAS Princeton

Recent developments
in string amplitudes

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I. Why string amplitudes?

Prominent role within string theory (starting with \[\text{Veneziano '68}\])

- already in flat spacetime: low-energy eff. actions $\sim \text{tr}(D^k F^n)$, $D^k R^n$
  $\Rightarrow$ testing / exploiting string dualities (primarily S-duality of type IIB)

- string amplitudes in AdS $\Rightarrow$ gauge/gravity duality, holography,
  bootstrap & recent crosstalk with (integrated) correlators in $\mathcal{N} = 4$

[Hansen’s talk]

Rich source of inspiration and input for other fields

- closed vs. open strings: BCJ duality & gravitational double copy
  \[\text{KLT '86, \ldots, reviews 1909.01358, 2203.13013, 2203.13017, 2204.06547, 2210.14241}\]

- function spaces for precision calculations in particle physics / gravity
  \[\text{reviews 2203.07088, 2203.09099, 2203.13014, 2203.13021, 2208.07242}\]
I. Why string amplitudes?

Numerous formalisms in amplitudes are in close contact with string theory:

- since 90’s: worldline formalisms (Bern-Kosower, . . . , WQFT)
  [review: Schubert 0101036; Uhre Jakobsen, Mogull, Plefka, Steinhoff ’20, ’21]

- since 2013: CHY formalism and ambitwistor strings
  [Cachazo, He, Yuan 1307.2199, 1309.0885; review: Mason, Geyer 2203.13017]

- tropical geometry: $\alpha' \to 0$ limit of string amplitudes
  [Tourkine 1309.3551; Lam 2405.17332]

  ... tropical moduli spaces of Feynman graphs $\leftrightarrow$ graph complexes
  [Borinsky, Brown, Munch, Tellander, Vermaseren, Vogtmann ’21 to ’24;
  Borinsky’s lecture series at amplitudes summer school next week]

- curve integral formalism
  [Arkani-Hamed, Cao, De, Dong, Figueiredo, Frost, He, Pokraka, Plamondon, Salvatori, Skowronek, Spradlin, Thomas, Volovich; Figueiredo’s & Spradlin’s talk]

- intersection theory
  [Mizera 1706.08527, 1711.00469]
Outline

I. Why string amplitudes? ✓

II. KLT and intersection theory at genus one
   [Bhardwaj’s gong-show talk; Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]
   [Stieberger 2212.06816, 2310.07755; Mazloumi, Stieberger 2403.05208]

III. Evaluating string amplitudes from convergent integrals
    [Eberhardt, Mizera 2208.12233, 2302.12733, 2403.07051]
    [Banerjee, Eberhardt, Mizera 2403.07064]

IV. Integration on higher-genus surfaces
    [D’Hoker, Hidding, OS 2306.08644 & 2308.05044; Enriquez 1112.0864]

V. Alternative double copy for single-valued periods
    [Frost, Hidding, Kamlesh, Rodriguez, OS, Verbeek 2312.00697; Dorigoni, Doroudiani, Drewitt, Hidding, Kleinschmidt, OS, Schneps, Verbeek 2403.14816, 2406.05099]
II. KLT and intersection theory at genus one

Goal: (closed strings) as (open string) \( \otimes^2 \) for integrated amplitudes

At tree level, done deal by KLT relations

\[
M_{\text{closed}}^{\text{tree}}(4 \text{ pt}; \alpha') = A_{\text{open}}^{\text{tree}}(1, 2, 3, 4; \alpha') \sin(\pi s) A_{\text{open}}^{\text{tree}}(1, 2, 4, 3; \alpha')
\]

[e.g.] \[
\int_{\mathbb{C}} \frac{d^2 z}{z(1-z)} \frac{|z|^{2s}|1-z|^{2t}}{z(1-z)} = \int_0^1 \frac{dz}{z} z^s (1-z)^t \int_0^0 \frac{d\bar{z}}{1-z} (1-\bar{z})^s (1-\bar{z})^t
\]

[Kawai, Lewellen, Tye '86]
II. 1 Tree-level KLT from intersection theory

Genus-0 integrands $\ni$ multivalued $u(z) = \prod_{i<j}(z_i-z_j)^{s_{ij}}$, their $|\cdot|$ & cc’s $\rightarrow$ use intersection theory: “dealing with multivalued integrands”

[Aomoto, Cho, Kita, Matsumoto, Mimachi, Yoshida et al: ’80s / ’90’s]

• increasingly relevant for Feynman-integral computations

[talks of Lee and Tancredi; see e.g. 2002.10476, 2203.13011 for reviews]

• open-string integrals $\leftrightarrow$ pairing twisted cycle $|\gamma \otimes u_{\gamma}|$ & rational form $\langle \varphi_L |$

$$\left\langle \frac{dz}{z} \Bigg| \{0<z<1\} \otimes z^s (1-z)^t \right\rangle = \int_0^1 z^s (1-z)^t \frac{dz}{z}$$

• closed-string integrals $\leftrightarrow$ pairing two “twisted cocycles” (with cc $|\varphi_R^\vee\rangle$)

$$\left\langle \frac{dz}{z} \left| \left( \frac{dz}{1-z} \right)^\vee \right\rangle = \int_{\mathbb{C}} \frac{d^2 z}{z} |z|^{2s} |1-z|^{2t}$$
II. 1 Tree-level KLT from intersection theory

- open-string integrals $\leftrightarrow$ pairing twisted cycle $|\gamma \otimes u_\gamma|$ & rational form $\langle \varphi_L |$  
\[ \left. \frac{dz}{z} \right|_{\{0<z<1\}} \otimes z^s(1-z)^t \right] = \int_0^1 z^s(1-z)^t \frac{dz}{z} \]

- closed-string integrals $\leftrightarrow$ pairing two “twisted cocycles” (with cc $|\varphi^\vee_R\rangle$) 
\[ \left. \frac{dz}{z} \right| \left( \frac{dz}{1-z} \right)^\vee = \int_{\mathbb{C}} \frac{d^2z |z|^{2s}|1-z|^{2t}}{z(1-\bar{z})} \]

- pairing two twisted cycles $\leftrightarrow$ regularized intersection number 
\[ \left[ \{-\infty<\bar{z}<0\} \otimes (-\bar{z})^s(1-\bar{z})^t \right|_{\{0<z<1\}} \otimes z^s(1-z)^t \right] = \frac{1}{2i \sin(\pi s)} \]

- 4pt KLT involves inverse intersection number 
\[ \int_{\mathbb{C}} \frac{d^2z |z|^{2s}|1-z|^{2t}}{z(1-\bar{z})} = \int_0^1 \frac{dz}{z} z^s(1-z)^t \sin(\pi s) \int_0^0 \frac{d\bar{z}}{1-\bar{z}} (-\bar{z})^s(1-\bar{z})^t \]
\[ \langle \varphi_L | \varphi^\vee_R \rangle = \langle \varphi_L | \gamma_L \otimes u_{\gamma_L} \rangle \left[ \gamma_L \otimes u_{\gamma_L} | \gamma_R \otimes u_{\gamma_R}^\vee \right]^{-1} \left[ \gamma_R \otimes u_{\gamma_R^\vee} | \varphi^\vee_R \right] \]

[Mizera 1706.08527, 1711.00469]
II. 1 Tree-level KLT from intersection theory

KLT at \( n \geq 5 \) points: \( \exists (n-3)! \) basis permutations \( \rho_a \in S_{n-3} \) of ...

... twisted cycles \( |\gamma_a| := |\{\rho_a(0<z_1<\ldots<z_{n-3}<1)\} \otimes \prod_{i<j} \rho_a\{(z_j-z_i)^{s_{ij}}\}| \)

[Plahte '70; Bjerrum-Bohr, Damgaard, Vanhove 0907.1425; Stieberger 0907.2211]

... twisted cocycles (e.g. Parke-Taylor) \( \langle \varphi_b \rangle = \langle \prod_{j=1}^{n-3} \frac{dz_j}{z_j-z_{j+1}} \rangle \) by IBP

[Aomoto '87; Mafra, OS, Stieberger 1106.2645; Broedel, OS, Stieberger 1304.7267]

Typical open-string integrals (\( z_{pq} := z_p - z_q \))

\[
\langle \varphi_{b=1} | \gamma_a \rangle = \int_{0<z_{\rho_a(i)}<z_{\rho_a(i+1)}<1} \frac{dz_1dz_2\ldots dz_{n-3}}{z_1z_2z_3\ldots z_{n-3,n-2}} \prod_{1 \leq i < j} |z_i-z_j|^{s_{ij}}
\]

[review: Mafra, OS 2210.14241; talks of Figueiredo and Sturmfels]
II. 1 Tree-level KLT from intersection theory

KLT at \( n \geq 5 \) points: \( \exists (n-3)! \) basis permutations \( \rho_a \in S_{n-3} \) of ...

... twisted cycles \( |\gamma_a| := \left\{ \rho_a(0<z_1< \ldots <z_{n-3}<1) \right\} \otimes \prod_{i<j} \rho_a \{ (z_j-z_i)^{s_{ij}} \} \]

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\((n-3)! \times (n-3)! \) intersection matrix \( H_{ab} = [\gamma_a | \gamma_b^\vee] \sim \sin^{3-n} (\pi \sum_{a,b} s_{ab}) \)

\( \Rightarrow \) KLT formula looks like resolution of identity \( 1 = \sum_{c,d} |\gamma_c| H^{-1}_{cd} [\gamma_d^\vee] \)

[Mizera 1706.08527, 1711.00469]

\[ M_{\text{tree}}^{\text{closed}}(n \text{ pt}; \alpha') = \sum_{1 \leq c,d \leq (n-3)!} A_{\text{open}}(\rho_c; \alpha') H_{cd}^{-1} \tilde{A}_{\text{open}}(\rho_d; \alpha') \]

\[ \langle \varphi_a | \varphi_b^\vee \rangle = \sum_{1 \leq c,d \leq (n-3)!} \langle \varphi_a | \gamma_c \rangle H_{cd}^{-1} [\gamma_d^\vee | \varphi_b^\vee \rangle \]
II. 1 Tree-level KLT from intersection theory

KLT formula looks like resolution of identity $1 = \sum_{c,d} |\gamma_c| H^{-1}_{cd} [\gamma^\vee_d]$

$$\langle \varphi_a | \varphi_b^\vee \rangle = \sum_{c,d=1}^{\text{dim}} \langle \varphi_a | \gamma_c \rangle H^{-1}_{cd} [\gamma^\vee_d | \varphi_b^\vee \rangle$$

... and generalizes to sphere integrals with unintegrated punctures $x_i$, e.g.

$$\int_{\mathbb{C}} d^2 z \, |z|^{2s_0}|1-z|^{2s_1}|x-z|^{2s_x} \varphi_L(z, x) \overline{\varphi_R(z, x)} = \left( \begin{array}{c} \int_0^x dz \, (x-z)^{s_x} \\ \int_x^1 dz \, (z-x)^{s_x} \end{array} \right) z^{s_0} (1-z)^{s_1} \varphi_L(z, x)$$

$$\times \left( \begin{array}{cc} \sin(\pi s_0) & \sin(\pi(s_0+s_x)) \\ \sin(\pi s_x) & 0 \end{array} \right) \left( \begin{array}{c} \int_0^0 d\bar{z} \, (-\bar{z})^{s_0} \\ \int_x^1 d\bar{z} \, \bar{z}^{s_0} \end{array} \right) (1-\bar{z})^{s_1} (\bar{x}-\bar{z})^{s_x} \overline{\varphi_R(z, x)}$$

[Vanhover, Zerbini 1812.03018; Britto, Mizera, Rodriguez, OS 2102.06206]

Contain single-valued hypergeometric / Lauricella functions.

[Brown, Dupont 1907.06603; Duhr, Porkert 2309.12772]
II. 2 Genus-one double copy for Riemann Wirtinger integrals

→ “warm-up” integrals towards 1-loop string amplitudes: meromorphic
Riemann Wirtinger (RW) integrals on (univ. cover of) torus $T^2(\tau) = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$

$$\langle \varphi_a | \gamma_b \rangle = \int_{0}^{z_b} dz \ e^{2\pi i s_A z} \left( \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right)^{s_0} F(z-z_a, \eta|\tau)$$

with $z_a, z_b \in \{1, x\}$ and constant $s_B := \tau s_A - x s_x - \eta$ and twisted cycles

$$|\gamma_b| = |\{0 < z < z_b\} \otimes u_{RW}\rangle$$ where $u_{RW}(z|\tau) = e^{2\pi i s_A z} \left( \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right)^{s_0}$

[Mano '08, 09; Mano, Watanabe ’12; Ghazouani, Pirio 1605.02356; Goto 2206.03177]

eliminated $B$-cycle : $(1-e^{2\pi i s_A})[\gamma_B] = (1-e^{2\pi i s_B})[\gamma_A] - (1-e^{-2\pi i s_0})[\gamma_x]$
II. 2 Genus-one double copy for Riemann Wirtinger integrals

→ “warm-up” integrals towards 1-loop string amplitudes: meromorphic Riemann Wirtinger (RW) integrals on (univ. cover of) torus $T^2(\tau) = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}

\langle \varphi_a | \gamma_b \rangle = \int_0^{\hat{z}_b} \mathrm{d}z \, e^{2\pi i s_A z} \left( \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right)^{s_0} F(z-z_a, \eta|\tau)

[Mano '08, 09; Mano, Watanabe '12; Ghazouani, Pirio 1605.02356; Goto 2206.03177]

Twisted cocycle $\langle \varphi_a | \rangle$: Kronecker-Eisenstein series [talks of Porkert, Tancredi]

$F(z, \eta|\tau) = \frac{\theta'_1(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} g^{(k)}(z|\tau) = F(z+1, \eta|\tau)$

$\theta_1(z+\tau|\tau) = -e^{-2\pi i z - i\pi \tau} \theta_1(z|\tau) \quad \Rightarrow \quad F(z+\tau, \eta|\tau) = e^{-2\pi i \eta} F(z, \eta|\tau)$

2dim twisted cohomology $\varphi_a \in \mathrm{d}z \{ F(z, \eta|\tau), F(z-x, \eta|\tau) \}$; note that expansion variable $\eta$ is constrained to yield constant $\tau s_A - x s_x - \eta = s_B$
II. 2 Genus-one double copy for Riemann Wirtinger integrals

Complex RW integral [Ghazouani, Pirio 1906.11857] obeys genus-one KLT

\[
\langle \varphi_a | \varphi_b^\vee \rangle = \int_{T^2(\tau)} d^2z \, e^{2\pi is_A(z-\bar{z})} \left| \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right|^{2s_0} F(z-z_a, \eta|\tau) \overline{F(z-z_b, \eta|\tau)}
\]

\[
= \frac{i}{2} \sin(\pi s_0) \left( \begin{array}{c} \langle \varphi_a | \gamma A \rangle \\ \langle \varphi_a | \gamma x \rangle \end{array} \right) \left( \begin{array}{cc} 0 & e^{i\pi(s_0-s_A)} \\ -e^{i\pi(s_A-s_0)} & 2i \sin(\pi(s_A-s_0)) \end{array} \right) \left( \begin{array}{c} [\gamma_A^\vee | \varphi_b^\vee \rangle \\ [\gamma_x^\vee | \varphi_b^\vee \rangle \end{array} \right)
\]

[Bhardwaj’s gong-show talk; Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]

with \( z_a, z_b \in \{0, x\} \) & reality condition \( \text{Im} \, \eta = s_A \text{Im} \, \tau - s_x \text{Im} \, x \)

KLT formula amounts to \( 1 = \sum_{c,d} |\gamma_c| H^{-1}_{cd} [\gamma_d^\vee] \) with intersection matrix

\[
H = \left( \begin{array}{cc} [\gamma_A | \gamma_A^\vee] & [\gamma_A | \gamma_x^\vee] \\ [\gamma_x | \gamma_A^\vee] & [\gamma_x | \gamma_x^\vee] \end{array} \right) = \frac{\sin(\pi s_A)}{\sin(\pi s_0)} \left( \begin{array}{cc} 2i \sin(\pi(s_A-s_0)) & -e^{i\pi(s_0-s_A)} \\ e^{i\pi(s_A-s_0)} & 0 \end{array} \right)
\]

Generalizes to any \#(unintegrated \( x_1, x_2, \ldots \)), but only for 1 integrated \( z \).
II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int d^D \ell$

$$M_{\text{closed}}^{1\text{-loop}}(n \text{ pt}) = \int d^D \ell \int_{\mathcal{M}_1} d^2\tau \left( \prod_{j=2}^{n} \int_{T^2(\tau)} d^2z_j \right) \mathcal{F}_n(\epsilon, k, \ell|z, \tau) \mathcal{F}_n(\bar{\epsilon}, k, \ell|z, \tau)$$

$\mathcal{F}_n(\epsilon, k, \ell|z, \tau)$ meromorphic in $z_i, \tau$

$\longrightarrow$ “chiral splitting”  

[Verlinde, Verlinde '87; D'Hoker, Phong '88, '89]

Loop momenta $\ell_I$ in string theory = zero modes w.r.t. $A_I$ cycles

$\ell^m_I = \frac{1}{2\pi} \oint_{A_I} \partial_z X^m = \frac{1}{2\pi} \oint_{A_I} \partial_{\bar{z}} X^m$, shared between L & R
II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int d^D \ell$

$$M_{\text{closed}}^{1\text{-loop}}(n \text{ pt}) = \int d^D \ell \int_{\mathbb{M}_1} d^2 \tau \left( \prod_{j=2}^{n} \int_{T^2(\tau)} d^2 z_j \right) \mathcal{F}_n(\epsilon, k, \ell | z, \tau)$$

$\longrightarrow$ “chiral splitting” [Verlinde, Verlinde ’87; D’Hoker, Phong ’88, ’89]

Loop momenta $\ell_I$ in string theory = zero modes w.r.t. $A_I$ cycles

$$\ell^m_I = \frac{1}{2\pi} \oint_{A_I} \partial z X^m = \frac{1}{2\pi} \oint_{A_I} \partial \bar{z} X^m,$$  

shared between L & R

Loop momentum jumps when transporting punctures around $B_I$ cycles

$$z_j \rightarrow z_j + B_1 \quad \Rightarrow \quad A_1 \rightarrow A_1 + \begin{array}{c} \implies \\ \circ \end{array} \quad \Rightarrow \quad \ell_1 \rightarrow \ell_1 + k_j$$
II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before $\int d^D \ell$

$$M^{1\text{-loop}}_{\text{closed}}(n \text{ pt}) = \int d^D \ell \int_{\mathfrak{m}_1} d^2 \tau \left( \prod_{j=2}^{n} \int_{T^2(\tau)} d^2 z_j \right) F_n(\epsilon, k, \ell|z, \tau)\quad \text{chiral amplitude}$$

$\rightarrow$ “chiral splitting” [Verlinde, Verlinde ’87; D’Hoker, Phong ’88, ’89]

Chiral amplitude $F_n \ni$ universal Koba-Nielsen factor $u_{\text{ST}}$

$$u_{\text{ST}}(z|\tau) = \exp \left( \frac{i\pi \alpha'}{2} \tau \ell^2 - i\pi \alpha' \sum_{j=2}^{n} (\ell \cdot k_j) z_j \right) \prod_{1 \leq i < j}^{n} \theta_1(z_i - z_j|\tau)^{s_{ij}}$$

$B$-monodromy compensated by loop mom. shift $z_j \rightarrow z_j + \tau \Rightarrow \ell \rightarrow \ell + k_j$
II. 3 Crashcourse in chiral splitting

Closed-string loop integrands factorize before \( \int d^D \ell \) chiral amplitude

\[
M_{\text{closed}}^{1\text{-loop}}(n \text{ pt}) = \int d^D \ell \int_{\mathfrak{M}_1} d^2 \tau \left( \prod_{j=2}^{n} \int_{T^2(\tau)} d^2 z_j \right) \mathcal{F}_n(\epsilon, k, \ell|z, \tau) \mathcal{F}_n(\tilde{\epsilon}, k, \ell|z, \tau)
\]

\( \rightarrow \) “chiral splitting” [Verlinde, Verlinde '87; D’Hoker, Phong '88, '89]

Chiral amplitude \( \mathcal{F}_n \) \( \ni \) universal Koba-Nielsen factor \( u_{\text{ST}} \)

\[
u_{\text{ST}}(z|\tau) = \exp \left( \frac{i \pi \alpha'}{2} \tau \ell^2 - i \pi \alpha' \sum_{j=2}^{n} (\ell \cdot k_j) z_j \right) \prod_{1 \leq i < j}^{n} \theta_1(z_i - z_j|\tau)^{s_{ij}}
\]

\( B \)-monodromy compensated by loop mom. shift \( z_j \rightarrow z_j + \tau \Rightarrow \ell \rightarrow \ell + k_j \)

Compare with Riemann-Wirtinger integral at \( s_A = -\frac{\alpha'}{2} \ell \cdot k \)

\[
u_{\text{RW}}(z|\tau) = e^{2\pi i s_A z} \theta_1(z|\tau)^{s_0} \prod_{j \geq 1} \theta_1(z - x_j|\tau)^{s_j}, \quad s_0 + \sum_{j \geq 1} s_j = 0
\]

overall \( B \)-monodromies \( F(z+\tau, \eta)u_{\text{RW}}(z+\tau) = e^{2\pi i s_B} F(z, \eta)u_{\text{RW}}(z) \).
II. 4 Genus-one double copy for string amplitudes

For rectangular tori $\text{Re} (\tau) = 0$, contour def’s $\Rightarrow$ factorize $\int d^2 z = \int d\xi d\chi$

$$M_{n \text{pt}}^{\text{1-loop closed}} \mid_{\text{Re}(\tau)=0} = \int d^D \ell \sum_{\rho,\sigma \in S_{n-1}} S_{\alpha'}(\rho|\sigma) \int_{0<\xi_\sigma<\xi_{\sigma+1}<1} d\xi_2 \ldots d\xi_n \mathcal{F}^\text{op}_n(\epsilon, k, \ell | \xi, \tau) \times \int_{0<\chi_\rho<\chi_{\rho+1}<1} d\chi_2 \ldots d\chi_n \mathcal{F}^\text{op}_n(\bar{\epsilon}, k, \ell | \chi, \tau) \prod_{j=2}^n \Psi(\xi_j, \chi_j, \ell \cdot k_j)$$

- get exactly the open-string incarnation of chiral amplitudes,

$$\mathcal{F}^\text{op}_n(\epsilon, k, \ell | \xi, \tau) = \exp \left( \frac{i\pi \alpha'}{2} \tau \ell^2 - i\pi \alpha' \sum_{j=2}^n (\ell \cdot k_j) \xi_j \right) \prod_{1 \leq i < j} |\theta_1(\xi_i-\xi_j | \tau)|^{s_{ij}} \frac{Q_n(\epsilon, k, \ell | \xi, \tau)}{1 - e^{-i\pi \alpha' \ell \cdot k_j}}$$

- splitting fct. obstructs (open string)$\otimes 2$ factorization of $\xi_j$ and $\chi_j$-integrals

$$\Psi(\xi_j, \chi_j, \ell \cdot k_j) = \frac{1 + e^{-i\pi \alpha' \ell \cdot k_j}}{1 - e^{-i\pi \alpha' \ell \cdot k_j}} e^{i\pi \alpha' \ell \cdot k_j \Theta[\chi_j - \xi_j]}, \quad (\text{Heaviside } \Theta)$$

- KLT kernel $S_{\alpha'}(\rho|\sigma)$ is inverse of twisted intersection matrix à la Goto

[Stieberger 2212.06816, 2310.07755]

[Mazloumi, Stieberger 2403.05208]
II. 4 Genus-one double copy for string amplitudes

For rectangular tori $\text{Re}(\tau) = 0$, contour def's $\Rightarrow$ factorize $\int d^2 z = \int d\xi d\chi$

$$M^{1\text{-loop}}_{\text{closed}}(n \text{ pt}) \bigg|_{\text{Re}(\tau)=0} = \int d^D \ell \sum_{\rho, \sigma \in S_{n-1}} [\gamma_\sigma | \gamma_\rho^\vee]^{-1} \int \limits_{0<\xi_{\sigma_i}<\xi_{\sigma_{i+1}}<1} d\xi_2 \ldots d\xi_n \mathcal{F}_n^{\text{op}}(\epsilon, k, \ell | \xi, \tau)$$

$$\times \int \limits_{0<\chi_{\rho_i}<\chi_{\rho_{i+1}}<1} d\chi_2 \ldots d\chi_n \mathcal{F}_n^{\text{op}}(\tilde{\epsilon}, k, \ell | \chi, \tau) \prod_{j=2}^{n} \Psi(\xi_j, \chi_j, \ell \cdot k_j)$$

[Stieberger 2212.06816, 2310.07755]

splitting fct. obstructs (open string) $\otimes^2$ factorization of $\xi_j$ and $\chi_j$-integrals

$$\Psi(\xi_j, \chi_j, \ell \cdot k_j) = \frac{1 + e^{-i\pi\alpha' \ell \cdot k_j}}{1 - e^{-i\pi\alpha' \ell \cdot k_j}} e^{i\pi\alpha' \ell \cdot k_j \Theta[\chi_j - \xi_j]}$$

... but imposes level matching

... & admits interpretation as non-planar cylinder with closed-string bulk insertion

[figure taken from Stieberger 2212.06816]
II. 4 Genus-one double copy for string amplitudes

For rectangular tori \( \text{Re}(\tau) = 0 \), contour def's \( \Rightarrow \) factorize \( \int d^2 z = \int d\xi d\chi \)

\[
M_{\text{closed}}^{1\text{-loop}}(n \text{ pt})\big|_{\text{Re}(\tau)=0} = \int d^D \ell \sum_{\rho,\sigma \in S_{n-1}} [\gamma_\sigma | \gamma_\rho^\vee]^{-1} \int_0^{\xi_{\sigma_i}} \cdots \int_0^{\xi_{\sigma_{i+1}}} d\xi_2 \cdots d\xi_n \mathcal{F}_n^{\text{op}}(\epsilon, k, \ell | \xi, \tau) \\
\times \int_0^{\chi_{\rho_i}} \cdots \int_0^{\chi_{\rho_{i+1}}} d\chi_2 \cdots d\chi_n \mathcal{F}_n^{\text{op}}(\tilde{\epsilon}, k, \ell | \chi, \tau) \prod_{j=2}^n \Psi(\xi_j, \chi_j, \ell \cdot k_j)
\]

[Stieberger 2212.06816, 2310.07755]

From the degeneration limit \( \tau \to i\infty \), recover ...

... via \( \alpha' \to 0 \), the one-loop KLT formula for supergravity \( \ell \)-integrands

[He, OS 1612.00417; He, OS, Zhang 1706.00640]

... upon \( \alpha' \)-expansion, the KLT formula for \( D^{2k} R^n \) 1-loop matrix elements

[Edison, Guillen, Johansson, OS, Teng 2107.08009]

\( \rightarrow \) maybe find a way around the linearized Feynman propagators

\( (\ell + K)^2 \to 2\ell \cdot K + K^2 \) in the (effective) field-theory KLTs from '16–'21?
II. 5 Discussion of genus-one double copies

Have seen two flavors of one-loop double copy formulae

- cplx. Riemann Wirtinger integral with double copy “\(1 = \sum_{c,d} |\gamma_c| H_{cd}^{-1}[\gamma_d^\vee]|\)”

\[
\int_{T^2(\tau)} d^2 z \ e^{2\pi i s_A(z-\bar{z})} \left| \frac{\theta_1(z|\tau)}{\theta_1(z-x|\tau)} \right|^{2s_0} \ F(z-z_a, \eta|\tau) \ F(z-z_b, \eta|\tau)
\]

[Bhardwaj, Pokraka, Ren, Rodriguez 2312.02148]

* so far, only for one integrated puncture \(z\)

* reality constraint \(\text{Im} \ \eta = s_A \text{Im} \ \tau - s_x \text{Im} \ x\)

- closed-string \(n\)-point one-loop amplitudes (integrand w.r.t. \(\ell\) and \(\tau\))

[Stieberger 2212.06816, 2310.07755; Mazloumi, Stieberger 2403.05208]

* so far, only for rectangular tori \(\text{Re} (\tau) = 0\)

* splitting fct’s \(\Psi(\xi_j, \chi_j, \ell \cdot k_j)\) interlocking \(\int\)’s over open-string \(\xi_j, \chi_j\)’s

Rewarding to study both approaches in tandem & combine their strengths!
III. Evaluating string amplitudes from convergent integrals

Recent progress in overcoming the following concerns on traditional integration contours over moduli space $\mathcal{M}_{g,n}$ in string amplitudes

- Eberhardt, Mizera 2208.12233, 2302.12733, 2403.07051

- tension between Lorentzian spacetime and Euclidean worldsheet
  [Witten 1307.5124]

- integrals don’t converge for phys. kinematics (e.g. $\int_0^1 \frac{dz}{z} |z|^s \mathcal{I} \mathcal{P} \mathcal{I} \mathcal{R} s \leq 0$)

- traditional formulae for loop amplitudes are manifestly real whereas optical theorem requires imaginary part for the discontinuities in $s$

Why did we rarely hear about these concerns?

- marked points $z_i$ more forgiving than cplx. structure moduli $\tau_j$

- no problem in $\alpha'$-expansion, only finite $\alpha'$ requires new contours
III. 1 New amplitude prescription at one loop

Consider planar 1-loop 4pt amplitude of open superstring: gauge grp. $SO(32)$

$\implies$ cylinder ($\tau \in i\mathbb{R}^+$) & Möbius strip ($\tau \in \frac{1}{2} + i\mathbb{R}^+$) @ relative factor $-1$

$\Rightarrow \text{contour in } \text{[textbooks]} \implies \text{Im} A_{\text{open}}^{1\text{-loop}} = 0$

\textbf{NOT} related by contour deformation!

[Eberhardt, Mizera 2302.12733; figures taken from the reference]
III. 1 New amplitude prescription at one loop

New prescription yields non-zero $\text{Im } A_{\text{open}}^{1}\text{-loop}$ and $\text{Im } M_{\text{closed}}^{1}\text{-loop}$ localizing on

- open string: cylinder and Möbius strip contribution
- closed string: contour for $\text{Re } \tau$ and $\text{Im } \tau$

consistent with unitarity cuts!

$\text{Im } \begin{array}{c} 1 \times \ \ \ \ \ \ \ \ 3 \\ 2 \times \ \ \ \ \ 4 \end{array} = \sum_{(n_1,n_2) \atop \sqrt{s} \geq \sqrt{n_1} + \sqrt{n_2}} \text{polarizations}$

[\text{Eberhardt, Mizera 2208.12233; figures taken from the reference}]
III. 1 New amplitude prescription at one loop

For open strings, full $A_{\text{open}}^{1\text{-loop}}$ (both Re & Im) most conveniently evaluated on “Rademacher contour” $\Gamma_{\infty}$ ($\infty$ collection of circles at $\mathbb{Q} + i\mathbb{Q}$ centers)

→ analytical checks & numerical control at finite $\alpha'$, say $(k_1 + k_2)^2 \sim \frac{10}{\alpha'}$

[Eberhardt, Mizera 2302.12733; figures taken from the reference]
III. 2 Checks and applications

Simplified formulae for imaginary parts of loop amplitudes at all energies

\[
\text{Im } \begin{array}{c}
\times 1 \\
\times 2 \\
\times 3 \\
\times 4
\end{array}
= \sum_{(n_1, n_2) \geq \sqrt{n_1 + n_2}} \text{polarizations}
\begin{array}{c}
\times 1 \\
\times 2 \\
\times 3 \\
\times 4
\end{array}
\]

[Eberhardt, Mizera 2208.12233; figure taken from the reference]

- low-energy expansions of \(\text{Im } M_{1\text{-loop}}^{\text{closed}}\) \& \(\text{Im } A_{1\text{-loop}}^{\text{open}}\) match “log(s)-part” of
  [D’Hoker, Green 1906.01652; Edison, Guillen, Johansson, OS, Teng 2107.08009]

- at high-energies, Regge limit sometimes dominated by imaginary part
  [Banerjee, Eberhardt, Mizera 2403.07064]

Similarly: new integration contours identified for \(n\)-point tree amplitudes

\[\implies \text{convergent integral representations, numerical control at finite } \alpha'\]

[Eberhardt, Mizera 2403.07051]
IV. Integration on higher-genus surfaces

Strong symbiosis between two questions:

**Q1:** What is a good set of integration kernels on Riemann surfaces such that their iterated integrals close under taking primitives?

**Q2:** What is an integration-friendly function space for integrands of multiloop string amplitudes, universal to type I & II / het / bos theories?

Example that string-theoretic objectives / techniques are useful for other fields

<table>
<thead>
<tr>
<th>worldsheet-based string amplitudeology</th>
<th>higher-genus polylog’s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>algebraic geometry</td>
</tr>
<tr>
<td></td>
<td>particle physics</td>
</tr>
<tr>
<td></td>
<td>stringy bootstrap</td>
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<tr>
<td></td>
<td>gravity / cosmology</td>
</tr>
<tr>
<td></td>
<td>number theory</td>
</tr>
</tbody>
</table>

[e.g. talks of Bern, Hansen, McLeod, Lee, Porkert, Sturmfels, Tancredi]
IV. Integration on higher-genus surfaces

Strong symbiosis between two questions:

**Q1:** What is a good set of integration kernels on Riemann surfaces such that their iterated integrals close under taking primitives?

**Q2:** What is an integration-friendly function space for integrands of multiloop string amplitudes, universal to type I & II / het / bos theories?

→ genus zero: $d \log(z-a)$ kernels of multiple polylogarithms resonate with Parke-Taylor basis for string tree amplitudes in arbitrary theories

→ genus one: Kronecker-Eisenstein kernels $g^{(k)}(z|\tau)$ or $f^{(k)}(z|\tau)$ are unified language for elliptic polylogs, modular forms, 1-loop string amp’s

→ now: higher-genus generalization of $f^{(k)}$ [D’Hoker, Hidding, OS 2306.08644]
IV. 1 Double-life of Kronecker-Eisenstein kernel

2× periodic but non-mero’ kernels \( f^{(k)}(z|\tau) = f^{(k)}(z+1|\tau) = f^{(k)}(z+\tau|\tau) \)

instead of meromorphic / multivalued \( g^{(k)}(z|\tau) \) generated by

\[
\exp \left( 2\pi i \eta \frac{\text{Im } z}{\text{Im } \tau} \right) \frac{\theta'_1(0|\tau)\theta_1(z+\eta|\tau)}{\theta_1(z|\tau)\theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} f^{(k)}(z|\tau)
\]

- backbone of elliptic polylogs in formulation of \[\text{[Brown, Levin 1110.6917]}\]

- function space for 1-loop string integrands (or \( g^{(k)}(z|\tau) \) before \( \int d^D \ell \)) \[\text{[Broedel, Mafra, OS 1412.5535; Gerken, Kleinschmidt, OS 1811.02548]}\]

- \( f^{(k)}(z|\tau) \) at rational pt’s \( z \in \mathbb{Q} + \tau\mathbb{Q} \Rightarrow \) modular forms of congruence subgroups \( \Gamma(N) \Rightarrow \) symbol alphabet for elliptic polylogs at rational pt’s \[\text{[Broedel, Duhr, Dulat, Penante, Tancredi 1803.10256]}\]

- convolutions of \( f^{(k)} \)’s \( \Rightarrow \) modular graph forms & sv elliptic polylog’s \[\text{[Gerken, Kleinschmidt, OS 1911.03476; D’Hoker, Kleinschmidt, OS 2012.09198]}\]
IV. 1 Double-life of Kronecker-Eisenstein kernel

\[ 2 \times \text{periodic but non-mero}' \text{ kernels } f^{(k)}(z|\tau) = f^{(k)}(z+1|\tau) = f^{(k)}(z+\tau|\tau) \]

instead of meromorphic / multivalued \( g^{(k)}(z|\tau) \) generated by

\[ \exp \left( 2\pi i \eta \frac{\text{Im } z}{\text{Im } \tau} \right) \frac{\theta'_1(0|\tau) \theta_1(z+\eta|\tau)}{\theta_1(z|\tau) \theta_1(\eta|\tau)} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} f^{(k)}(z|\tau) \]

Alternatively construction via bosonic (Arakelov) Green function on \( T^2(\tau) \)

\[ G(z|\tau) = -\log \left| \frac{\theta_1(z|\tau)}{\eta(\tau)} \right|^2 + 2\pi \frac{(\text{Im } z)^2}{\text{Im } \tau} \]

- base case is derivative: \( f^{(1)}(z|\tau) = -\partial_z G(z|\tau) = \partial_z \log \theta_1(z|\tau) + 2\pi i \frac{\text{Im } z}{\text{Im } \tau} \)

- higher \( k \geq 2 \) kernels recursively obtained from convolutions with \( G \)

\[ f^{(k)}(x|\tau) = \int_{T^2(\tau)} \frac{d^2 z}{\text{Im } \tau} \partial_x G(x-z|\tau) f^{(k-1)}(z|\tau) \]
IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog’s

Instead of $\theta_1$-representation of $f^{(k)}$, generalize their construction from $G$:

Arakelov Green function $G(x, y)$ on higher-genus surface $\Sigma$ depending on 2 pt’s $x, y \in \Sigma$ is uniquely defined by symmetry $G(x, y) = G(y, x)$ and

- Laplace eq: $\partial_x \partial_{\bar{x}} G(x, y) = \pi \kappa(x) - \pi \delta^2(x, y)$ “locally behaves like log”
- absence of zero mode $\int_{\Sigma} d^2x \kappa(x) G(x, y) = 0$

with $\kappa(x)$ the Kähler form on $\Sigma$ with unit normalization $\int_{\Sigma} d^2x \kappa(x) = 1$.

[Faltings ’84; Alvarez-Gaumé, Moore, Nelson, Vafa, Bost ’86]

- also $\exists$ representation in terms of the “prime form” (higher-genus $\theta$-fct’s)
- separating and non-separating degenerations of $G(x, y)$ well studied

[D’Hoker, Green, Pioline 1712.06135]
IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog’s

Instead of $\theta_1$-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

Convolute with Abelian differential $\omega_{I=1,2,\ldots,h}(x)$ on genus-$h$ surface $\Sigma$

\[
\oint_{A_I} \omega_J(z) dz = \delta_{IJ}
\]

period matrix

\[
\oint_{B_I} \omega_J(z) dz = \Omega_{IJ}
\]

normalization

with cplx. conjugates $\bar{\omega}^I(x) = [(\text{Im } \Omega)^{-1}]^{IJ} \bar{\omega}_J(x) @ I, J = 1, 2, \ldots, h$

Even though $\mathcal{G}(x, z)$ integrates to zero against $\kappa(z) = \frac{1}{h} \bar{\omega}^I(z) \omega_I(z)$ obtain tensorial $f^{(1)}$ kernel from remaining “traceless” $h^2 - 1$ vol. forms $\bar{\omega}^J(z) \omega_I(z)$

\[
f^{IJ}(x, y) = \int_{\Sigma} d^2 z \partial_x \mathcal{G}(x, z) \bar{\omega}^J(z) \omega_I(z) - \delta_{IJ} \partial_x \mathcal{G}(x, y)
\]
IV. 2 Higher-genus generalization of \( f^{(k)} \) and elliptic polylog’s

Instead of \( \theta_1 \)-representation of \( f^{(k)} \), generalize their construction from \( \mathcal{G} \):  

**Tensorial** \( f^{(1)} \) kernel from remaining “traceless” \( h^2 - 1 \) vol. forms \( \bar{\omega}^J(z) \omega_I(z) \)

\[
f^{I \; J}(x, y) = \int_\Sigma d^2 z \partial_x \mathcal{G}(x, z) \bar{\omega}^J(z) \omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x, y)
\]

Higher kernels \( f^{(k \geq 2)} \) with \( k+1 \) free indices mimic recursion of \( h = 1 \) case

\[
f^{I_1 \ldots I_k \; J}(x, y) = \int_\Sigma d^2 z \partial_x \mathcal{G}(x, z) \bar{\omega}^{I_1}(z) f^{I_2 \ldots I_k \; J}(z, y)
\]

Kernels \( f^{I_1 \ldots I_k \; J}(x, y) \) at rank \( k \geq 2 \) are regular throughout \( \Sigma \times \Sigma \),

only \( k = 1 \) case has simple pole \( f^{I \; J}(x, y) = \frac{\delta^I_J}{x-y} + \mathcal{O}((x-y)^0) \)
IV. 2 Higher-genus generalization of $f^{(k)}$ and elliptic polylog’s

Instead of $\theta_1$-representation of $f^{(k)}$, generalize their construction from $\mathcal{G}$:

tensorial $f^{(1)}$ kernel from remaining “traceless” $h^2 - 1$ vol. forms $\bar{\omega}^J(z)\omega_I(z)$

$$f^I_J(x, y) = \int \sum d^2 z \partial_x \mathcal{G}(x, z) \bar{\omega}^J(z)\omega_I(z) - \delta^I_J \partial_x \mathcal{G}(x, y)$$

Higher kernels $f^{(k \geq 2)}$ with $k+1$ free indices mimic recursion of $h = 1$ case

$$f^{I_1 \ldots I_k}_J(x, y) = \int \sum d^2 z \partial_x \mathcal{G}(x, z) \bar{\omega}^{I_1}(z)f^{I_2 \ldots I_k}_J(z, y)$$

Assembly line for higher-genus polylogarithms [D’Hoker, Hidding, OS 2306.08644]

- combine $f$’s to flat connection $\mathcal{J}(z, y) = -\pi d\bar{z} \bar{\omega}^I(z)b_I + dz \Psi_J(z, y)a^J$
  
  where $\Psi_J(z, y) = \omega_J(z) + \text{ad}_{b_I} f^I_J(z, y) + \text{ad}_{b_{I_1}} \text{ad}_{b_{I_2}} f^{I_1 I_2}_J(z, y) + \ldots$

- expand homotopy-inv. $\text{Pexp}(\int_y^x \mathcal{J}(z, y))$ in words in non-comm. $a^J, b_I$
IV. 3 Applications to string-amplitude computations

Bottleneck in $h \geq 2$ loop amplitudes of RNS superstring: simplify $\prod$ of

$$S_\delta(x, y) = \frac{\theta[\delta](\int_y^x \omega_I)}{\theta[\delta](0) E(x, y)}$$

fermion Green fct’s or “Szegö kernel”

and their summation over “spin structures $\delta$”

$\rightarrow 2^{2h}$ configurations of $\pm$ that 2dim fermions pick up under $A_I, B_J$ shifts

Higher-genus $f^{I_1 \cdots I_k} J(x, y)$-kernels completely disentangle $z_i$-dependence from $\delta$-dependence in cyclic products $S_\delta(z_1, z_2)S_\delta(z_2, z_3) \cdots S_\delta(z_n, z_1)$

- $\sum_\delta (S_\delta$-cycles) are essential parts of chiral amplitudes at $h = 1, 2$ loops
  [D’Hoker, Phong 0501197; D’Hoker, OS 2108.01104]

- part of recent proposal for 4pt chiral amplitude at $h = 3$ loops
  [Geyer, Monteiro, Stark-Muchão 2106.03968]
IV. 3 Applications to string-amplitude computations

Bottleneck in \( h \geq 2 \) loop amplitudes of RNS superstring: simplify \( \prod \) of

\[
S_\delta(x, y) = \frac{\theta[\delta]\left( \int_y^x \omega_I \right)}{\theta[\delta](0) E(x, y)} \quad \text{fermion Green fct’s or “Szegö kernel”}
\]

Higher-genus \( f^I_1 \cdots I_k J(x, y) \)-kernels completely disentangle \( z_i \)-dependence from \( \delta \)-dependence in cyclic products \( S_\delta(z_1, z_2)S_\delta(z_2, z_3) \ldots S_\delta(z_n, z_1) \)

\[
S_\delta(z_1, z_2)S_\delta(z_2, z_3)S_\delta(z_3, z_1) = F^{(3)}_{IJK}(\vec{z}) C^{IJK}_\delta + F^{(2)}_{JK}(\vec{z}) C^{JK}_\delta + F^{(0)}(\vec{z})
\]

with \( F^{(3)}_{IJK}(\vec{z}) = \omega_I(1)\omega_J(2)\omega_K(3) \) and \( z_i \)-independent, govern SUSY decomposition

\[
F^{(2)}_{JK}(\vec{z}) = \omega_I(1)f^I f_{J}(2, 3)\omega_K(3) + \text{cycl}(1, 2, 3)
\]

\[
F^{(0)}(\vec{z}) = \left( \partial_1 \mathcal{G}(1, 3) - \partial_1 \mathcal{G}(1, 2) \right) \partial_2 \partial_3 \mathcal{G}(2, 3) - \frac{1}{h} \omega_I(1) \partial_3 f^{IK} K(2, 3)
\]

[D’Hoker, Hidding, OS 2308.05044]
IV. 3 Applications to string-amplitude computations

Bottleneck in $h \geq 2$ loop amplitudes of RNS superstring: simplify $\prod$ of

$$S_\delta(x, y) = \frac{\theta[\delta](\int_x^y \omega_I)}{\theta[\delta](0) E(x, y)} \text{ fermion Green fct’s or “Szegö kernel”}$$

Higher-genus $f^{I_1 \ldots I_k} J(x, y)$-kernels completely disentangle $z_i$-dependence from $\delta$-dependence in cyclic products $S_\delta(z_1, z_2)S_\delta(z_2, z_3) \ldots S_\delta(z_n, z_1)$

$$S_\delta(z_1, z_2)S_\delta(z_2, z_3) \ldots S_\delta(z_n, z_1) = F^{(0)}(\vec{z}) + \sum_{r=2}^n F^{(r)}_{I_1 \ldots I_r}(\vec{z}) C^{I_1 \ldots I_r}_\delta$$

with $F^{(r)}_{I_1 \ldots I_r}(\vec{z})$ indep. on $\delta$ & modular tensors $C^{I_1 \ldots I_r}_\delta$ indep. on $z_i$

[D’Hoker, Hidding, OS 2308.05044]

Next steps:

• simplify integral representations of $C^{I_1 \ldots I_r}_\delta$ & rewrite via $\theta$-fct’s

• extend to open chains $S_\delta(x, z_1)S_\delta(z_1, z_2) \ldots S_\delta(z_n, y)$ at $x \neq y$
IV. 4 Fay identities

Closure of polylogs under $\int dz$ requires bilinear identities among kernels

- genus zero: partial fraction $\frac{1}{(y-z)(z-x)} + \text{cycl}(x, y, z) = 0$

$$\int_0^u dz \frac{G(a_1, \ldots, a_n; z)}{(y-z)(z-x)} = \frac{1}{x-y} \int_0^u dz \left[ \frac{1}{z-x} - \frac{1}{z-y} \right] G(a_1, \ldots, a_n; z)$$

$$= \frac{1}{x-y} \left[ G(x, a_1, \ldots, a_n; u) - G(y, a_1, \ldots, a_n; u) \right]$$

- genus one: Fay identities among Kronecker-Eisenstein kernels

$$f^{(s)}(x-z)f^{(r)}(y-z) = -(-1)^s f^{(r+s)}(y-x)$$

$$+ \sum_{\ell=0}^s \binom{\ell+r-1}{\ell} f^{(s-\ell)}(x-y)f^{(r+\ell)}(y-z)$$

$$+ \sum_{\ell=0}^r \binom{\ell+s-1}{\ell} f^{(r-\ell)}(y-x)f^{(s+\ell)}(x-z)$$

no repeated appearance

of $z$ on right-hand side!

$\Rightarrow$ friendly to $\int dz$

[Brown, Levin 1110.6917; Broedel, Mafra, Matthes, OS 1412.5535]
IV. 4 Fay identities

Closure of polylogs under $\int dz$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1 \ldots I_k} J(x, y)$ obey tensorial Fay identities such as

$$f^{I} J(x, y) f^{J} K(y, z) + f^{I} J(y, x) f^{J} K(x, z) - f^{I} J(x, z) f^{J} K(y, z)$$

$$+ \omega J(x) f^{IJ} K(y, x) + \omega J(y) f^{JI} K(x, z) + \omega J(x) f^{JI} K(y, z) = 0$$

• trace w.r.t. $I, K$ yields higher-genus uplift of partial-fraction identity

$$\frac{1}{(x-y)(y-z)} + \frac{1}{(z-x)(x-y)} + \frac{1}{(y-z)(z-x)} + \text{non-singular} = 0$$

• at genus one, translation invariance yields cyclic form

$$f^{(1)}(x-y) f^{(1)}(y-z) + f^{(2)}(x-z) + \text{cycl}(x, y, z) = 0$$
IV. 4 Fay identities

Closure of polylogs under $\int dz$ requires bilinear identities among kernels

Higher-genus kernels $f^{I_1 \ldots I_k} J(x, y)$ obey tensorial Fay identities

$$f^{I_1 \ldots I_r} J(z, x) f^{P_1 \ldots P_s J} K(y, z) = f^{I_1 \ldots I_r} J(z, x) f^{P_1 \ldots P_s J} K(y, x)$$

$$+ \sum_{m=0}^{s} (-1)^{m-s-1} \sum_{\ell=0}^{r} f^{(P_s \ldots P_{m+1} \uplus I_1 \ldots I_\ell)} J(z, y) f^{P_1 \ldots P_m J I_{\ell+1} \ldots I_r} K(y, x)$$

$$+ \sum_{m=0}^{s} (-1)^{m-s-1} f^{P_1 \ldots P_m} J(y, x) [f^{(P_s \ldots P_{m+1} J \uplus I_1 \ldots I_{r-1}) I_r} K(z, x)$$

no repeated $z$ on RHS!

$$+ f^{(P_s \ldots P_{m+1} \uplus I_1 \ldots I_r) J} K(z, y)]$$

with shuffles such as $f^{\cdot \cdot \cdot (P \uplus I) \cdot \cdot \cdot} J(x, y) = f^{\cdot \cdot \cdot PI \cdot \cdot \cdot} J(x, y) + f^{\cdot \cdot \cdot IP \cdot \cdot \cdot} J(x, y)$

[D’Hoker, OS 2406.abcde]
IV. 5 Meromorphic kernels

How do mero’ Kronecker-Eisenstein kernels $g^{(k)}$ generalize beyond genus 1?

→ Enriquez implicitly defined meromorphic but multi-valued connection ...

... with mero’ coefficients $\omega^{I_1\ldots I_k J}(x, y)$ multiplying $\text{ad}_{b_{I_1}} \ldots \text{ad}_{b_{I_k}} a^J$

... with monodromies $\omega^{I_1\ldots I_k J}(x + B_L, y) = \sum_{\ell=0}^{k} \frac{1}{\ell!} \delta_{I_1}^{I_1} \ldots \delta_{I_k}^{I_k} \omega^{I_1\ldots I_k J}(x, y)$

generalizing $g^{(k)}(x + \tau) = \sum_{\ell=0}^{k} \frac{1}{\ell!} (-2\pi i)^{\ell} g^{(k-\ell)}(x)$ to arbitrary genus

... including $\omega_J(x) = \omega^0 J(x, y)$ as $k = 0$ instance  

[Enriquez 1112.0864]

• in chiral splitting / before $\prod_{J=1}^{h} \int d^D \ell_J$, expect $\omega^{I_1\ldots I_k J}(x, y)$ to be suitable function space for chiral amplitudes $\mathcal{F}_n(\epsilon, k, \ell| z, \Omega)$

• expressing $\omega^{I_1\ldots I_k J}(x, y)$ in terms of $f^{I_1\ldots I_k J}(x, y)$: under investigation  

[D’Hoker, Enriquez, OS, Zerbini: work in progress]
Conjecture: Fay id’s of $f^{I_1\ldots I_k} J(x, y)$ hold in identical form for $\omega^{I_1\ldots I_k} J(x, y)$

$$f^{I} J(x, y)f^{J} K(y, z) + f^{I} J(y, x)f^{J} K(x, z) - f^{I} J(x, z)f^{J} K(y, z)$$

$$+ \omega J(x)f^{I} J K(y, x) + \omega J(y)f^{J} I K(x, z) + \omega J(x)f^{J} I K(y, z) = 0$$

$$\omega^{I} J(x, y)\omega^{J} K(y, z) + \omega^{I} J(y, x)\omega^{J} K(x, z) - \omega^{I} J(x, z)\omega^{J} K(y, z)$$

$$+ \omega J(x)\omega^{I} J K(y, x) + \omega J(y)\omega^{J} I K(x, z) + \omega J(x)\omega^{J} I K(y, z) = 0$$

[D’Hoker, OS 2406.abcde; proof under discussion with Enriquez, Zerbini]
IV. 5 Meromorphic kernels

Conjecture: Fay id’s of $f^{I_1 \ldots I_k} J(x, y)$ hold in identical form for $\omega^{I_1 \ldots I_k} J(x, y)$

$$\omega^{I_1 \ldots I_r} J(z, x) \omega^{P_1 \ldots P_s J} K(y, z) = \omega^{I_1 \ldots I_r} J(z, x) \omega^{P_1 \ldots P_s J} K(y, x)$$

$$+ \sum_{m=0}^{s} (-1)^{m-s-1} \sum_{\ell=0}^{r} \omega^{(P_s \ldots P_{m+1} \sqcup I_1 \ldots I_\ell)} J(z, y) \omega^{P_1 \ldots P_m J \sqcup I_{\ell+1} \ldots I_r} K(y, x)$$

$$+ \sum_{m=0}^{s} (-1)^{m-s-1} \omega^{P_1 \ldots P_m J(y, x)} [\omega^{(P_s \ldots P_{m+1} J \sqcup I_1 \ldots I_{r-1}) I_r} K(z, x)$$

$$+ \omega^{(P_s \ldots P_{m+1} \sqcup I_1 \ldots I_r) J} K(z, y)]$$

no repeated $z$ on RHS!

[D’Hoker, OS 2406.abcde; proof under discussion with Enriquez, Zerbini]

Alternative to meromorphic & multivalued connection of [Enriquez 1112.0864]:

meromorphic and single-valued connection with higher poles $(x-y) \leq -2$

[Enriquez, Zerbini 2110.09341, 2212.03119]
V. Alternative double copy for single-valued periods

This section: no \( \sin(\pi s) \) or related trigonometric intersection numbers

- genus-0 target: single-valued polylog’s \( \ni \) multi-Regge kinematics of SYM
  
  [Dixon, Duhr, Penington 1207.0186; Broedel, Sprenger, Torres Orjuela 1606.08411,
  Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek ’16-19]

- genus-1 target: non-holo “modular graph forms” \( \ni \) closed-strings @1loop
  
  [D’Hoker, Green, Gürdogan, Vanhove 1512.06779; D’Hoker, Green 1603.00839]

Both are double copies of meromorphic quantities (genus-0 polylog’s

or iterated Eisenstein integrals) \( \times \) their complex conjugates \( \times \) MZVs

Devil in the detail: the MZV part is surprisingly hard!

\[
e.g. \quad G^{sv}(0, 0, 1, 1; z) = G(0, 0, 1, 1; z) + \overline{G(1; z)}G(0, 0, 1; z) + \overline{G(1, 1; z)}G(0, 0; z) \\
+ \overline{G(1, 1, 0; z)}G(0; z) + \overline{G(1, 1, 0, 0; z)} + 2\zeta_3 \overline{G(1; z)}
\]
V. Alternative double copy for single-valued periods

This section: no \(\sin(\pi s)\) or related trigonometric intersection numbers

- genus-0 target: \textit{single-valued polylog’s} \(\ni\) multi-Regge kinematics of SYM
  \[\text{[Dixon, Duhr, Penington 1207.0186; Broedel, Sprenger, Torres Orjuela 1606.08411, Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek ’16-19]}\]
  - genus-1 target: \textit{non-holo “modular graph forms”} \(\ni\) closed-strings @1loop
    \[\text{[D’Hoker, Green, Gürdogan, Vanhove 1512.06779; D’Hoker, Green 1603.00839]}\]

Both are double copies of meromorphic quantities (genus-0 polylog’s or iterated Eisenstein integrals) \(\times\) their complex conjugates \(\times\) MZVs

Devil in the detail: the MZV part is surprisingly hard!

\(\text{e.g. generating series } G(e_0, e_1; z) \& G^{sv}(e_0, e_1; z)\) of mero’ / sv polylogs

\[G^{sv}(e_0, e_1; z) = \frac{G(e_0, \hat{e}_1; z)^t G(e_0, e_1; z)}{\Phi^{sv}(e_0, e_1)e_1\Phi^{sv}(e_0, e_1)^{-1}}\text{ sv Drinfeld associator}\]
V. 1 Zeta generators

Reformulated construction of sv polylogs in [Brown '04] (and multi-variable generalizations [1606.08807]) via “zeta generators” $\sigma_{2k+1}$ with Lie brackets

$$[\sigma_{2k+1}, e_0] = 0, \quad [\sigma_3, e_1] = [[[e_1, e_0], e_0+e_1], e_1], \quad \text{etc.}$$

[Ihara '92; Furusho 0011261]

[Frost, Hidding, Kamlesh, Rodriguez, OS, Verbeek 2312.00697]

Clue: Reformulated construction smoothly extends beyond genus zero!

Above $\sigma_{2k+1} = \sigma_{2k+1}^{(g=0)}$ acting on braid operators $e_0, e_1$ have organic uplift to zeta generators $\sigma_{2k+1}^{(g=1)}$ at genus one acting on non-comm. variables $\epsilon_k$

dual to holomorphic Eisenstein series $G_k(\tau)$ at $k = 0, 2, 4, \ldots$

[Tsuongai '95; Enriquez 1003.1012; Brown 1504.04737; Schneps 1506.09050; Hain-Matsumoto 1512.03975; Dorigoni, Doroudiani, Drewitt, Hidding, Kleinschmidt, OS, Schneps, Verbeek (DDDHKSSV) 2406.05099]
V. 1 Zeta generators

Combine $\sigma^{(g=0)}_{2k+1}$ and $\sigma^{(g=1)}_{2k+1}$ into genus-agnostic generating series

$$M^{sv}(\sigma^{(g)}_{2k+1}) = 1 + 2 \sum_{k=1}^{\infty} \zeta_{2k+1} \sigma^{(g)}_{2k+1} + 2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{2k+1} \zeta_{2\ell+1} \sigma^{(g)}_{2k+1} \sigma^{(g)}_{2\ell+1} + \text{higher depth}$$

Then, obtain universal form for generating series of

• single-valued polylogs: zeta generators $\sigma^{(g=0)}_{2k+1}$ acting on $e_0, e_1$

$$G^{sv}(e_0, e_1; z) = M^{sv}(\sigma^{(0)}_{2k+1})^{-1} \overline{G(e_0, e_1; z)}^t M^{sv}(\sigma^{(0)}_{2k+1}) G(e_0, e_1; z)$$

• single-valued iterated Eisenstein integrals / modular graph forms:

zeta generators $\sigma^{(g=1)}_{2k+1}$ acting on $\epsilon_k \leftrightarrow \int G_k(\tau)$ in mero’ series $I(\epsilon_k; \tau)$

$$I^{sv}(\epsilon_k; \tau) = M^{sv}(\sigma^{(1)}_{2k+1})^{-1} \overline{I(\epsilon_k; \tau)}^t M^{sv}(\sigma^{(1)}_{2k+1}) I(\epsilon_k; \tau)$$

[DDDHKSSV 2403.14816]
V. 1 Zeta generators

- single-valued iterated Eisenstein integrals / modular graph forms:

  \[ \int G_k(\tau) \text{ in mero' series } \mathbb{I}(\epsilon_k; \tau) \]
  \[ \mathbb{I}^{\text{sv}}(\epsilon_k; \tau) = \mathbb{M}^{\text{sv}}(\sigma_{2k+1}^{(1)})^{-1} \mathbb{I}(\epsilon_k; \tau)^t \mathbb{M}^{\text{sv}}(\sigma_{2k+1}^{(1)}) \mathbb{I}(\epsilon_k; \tau) \]

  [DDDHKSSV 2403.14816]

- concrete genus-one realization of general theory of single-valued periods
  
  [Brown, Dupont 1810.07682]

- makes Brown’s equivariant iterated Eisenstein integrals fully explicit
  
  [Brown 1707.01230, 1708.03354]

- inspires \( \sigma_{2k+1}^{(1)} \)-based proposal for motivic coaction of elliptic MZVs
  
  [Kleinschmidt, Porkert, OS: in progress]
Conclusion & Outlook

• 2 flavors of one-loop double copy formulae à la KLT from intersection theory with complementary strengths and (? temporary ?) limitations

• new $\int$ contours for string amplitudes $\Rightarrow$ unprecedented control @ finite $\alpha'$

• progress on construction & properties of integration kernels for higher-genus polylogarithms; $\exists$ first links with Enriquez’ meromorphic kernels

• zeta generators $\Rightarrow$ generating series for sv periods in genus-agnostic form

Thank you for your attention!