What is this talk about?

Aim:
Discuss action of Generalized Symmetries on S-matrix and derive physical consequences (Integrable examples, but conclusions more general!)
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Generalized Symmetries

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Generalized Symmetries

Scattering Amplitudes
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Integrability

Scattering Amplitudes

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Generalized Symmetries

Integrability

Scattering Amplitudes

This Talk

Aim:
Discuss action of Generalized Symmetries on S-matrix and derive physical consequences (Integrable examples, but conclusions more general!)


Aim: Discuss action of Generalized Symmetries on S-matrix and derive physical consequences (Integrable examples, but conclusions more general!).
Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \rightarrow 2$ S-Matrix:

$$L_S = L_S(\star) = (\star \star) = (\star \star \star)$$

Imposing Symmetry ($\star$), Unitarity ($\star \star$) and YBE ($\star \star \star$) is incompatible with standard Crossing. Instead:

$$S_{ab}^d c d(\theta) = s_d a d c d S_{bc}^a d(\pi - \theta)$$

Categorical symmetries can be used efficiently in the Bootstrap program. (See Lucia's Lectures!)
Summary

- Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \rightarrow 2$ S-Matrix:
Categorical Symmetries act on (massive) kinks and lead to Ward identities for the 2 → 2 S-Matrix:

\[ L_1 S \xrightarrow{\star} L_2 S \xrightarrow{\star \star} L_3 = L_4 \xrightarrow{\star \star \star} \]

Imposing Symmetry (\( \star \)), Unitarity (\( \star \star \)) and YBE (\( \star \star \star \)) is incompatible with standard Crossing.

Instead:

\[ S_{ab\,dc}(\theta) = s_{da\,dc} d_{bd\,db} S_{bc\,ad}(i\pi - \theta) \]

Categorical symmetries can be used efficiently in the Bootstrap program. (See Lucia's Lectures!)
○ Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \to 2$ S-Matrix:

\[ \mathcal{L} \mathcal{S} = \mathcal{S}^{\star \star} = \mathcal{S}^{\star \star \star} \]

○ Imposing Symmetry ($\star$), Unitarity ($\star \star$) and YBE ($\star \star \star$) is incompatible with standard Crossing.

Instead:

\[ S_{ab}^{\prime} \equiv S_{ab} - i \pi \theta \]
Categorical Symmetries act on (massive) kinks and lead to Ward identities for the 2 → 2 S-Matrix:

\[ \mathcal{L} \mathcal{S} = \mathcal{S} \mathcal{L} = \mathcal{S} \mathcal{S} \]  

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Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \rightarrow 2$ S-Matrix:

\[
L_S = L_S(\star) = (\star \star) = (\star \star \star)
\]

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S_{ab\;dc}(\theta) = s_{da\;bc}(i\pi - \theta)
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Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \to 2$ S-Matrix:

\[ L \] \hspace{1cm} = \hspace{1cm} \star \hspace{1cm} = \hspace{1cm} \star \star \hspace{1cm} = \hspace{1cm} \star \star \star \]

Imposing Symmetry ($\star$), Unitarity ($\star \star$) and YBE ($\star \star \star$)
Categorical Symmetries act on (massive) kinks and lead to Ward identities for the 2 → 2 S-Matrix:

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Categorical Symmetries act on (massive) kinks and lead to Ward identities for the 2 \( \rightarrow \) 2 S-Matrix:

\[
L_S = L_S (\ast) = (\ast \ast) = (\ast \ast \ast)
\]

Imposing Symmetry (\ast), Unitarity (\ast \ast) and YBE (\ast \ast \ast) is incompatible with standard Crossing. Instead:

\[
S_{ab}^{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{bc}^{ad}(i\pi - \theta)
\]
Categorical Symmetries act on (massive) kinks and lead to Ward identities for the $2 \rightarrow 2$ S-Matrix:

$$L_S = L_S^*(\star) = 1^{\star\star} = 1^{\star\star\star}$$

Imposing **Symmetry** $(\star)$, **Unitarity** $(\star\star)$ and **YBE** $(\star\star\star)$ is incompatible with standard **Crossing**. Instead:

$$S_{ab}^{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{bc}^{ad}(i\pi - \theta)$$

Categorical symmetries can be used efficiently in the **Bootstrap** program. (See Lucia’s Lectures!)
Philosophy

IR $\rightarrow$ UV

IR Vacua

Symmetric TQFT

$M \rightarrow$ Massive Kinks

$K^{ab} \rightarrow$

UV CFT

Relevant Pert.

$\phi$

Action on vacua of $M$

Action on Kinks

Preserved by $\phi$

Symmetry $C$ is present at all steps.
IR $\longrightarrow$ UV
IR $\rightarrow$ UV

Symmetric TQFT $\mathcal{M}$

IR Vacua

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Action on vacua of $\mathcal{M}$

Action on Kinks $K_{ab}$

Preserved by Symmetry $C$ is present at all steps.
IR \rightarrow UV

IR Vacua
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Massive Kinks
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IR $\rightarrow$ UV

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Massive Kinks $K_{ab}$

UV CFT
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Action on vacua of $\mathcal{M}$
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Symmetry $C$ is present at all steps.
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Symmetry $C$ is present at all steps.
IR $\rightarrow$ UV

IR Vacua $\rightarrow$ Massive Kinks $\rightarrow$ UV CFT

Symmetric TQFT $\mathcal{M}$

$K_{ab}$

Relevant Pert. $\phi$

Action on vacua of $\mathcal{M}$
Action on Kinks

Preserved by $\phi$

Symmetry $\mathcal{C}$ is present at all steps.
Categorical symmetries (Review)

Implemented by topological lines:

\[ \text{[Petkova, Zuber '02; Gaiotto, Kapustin, Seiberg, Willett '14; Chang, Lin, Shao, Wang, Yin '18; ...]} \]

Associativity (F-symbols):

\[ L \cdot L' \cdot L'' = X L v L' L'' L'' L'' L' L'' L'' \]

Fusion structure:

\[ L \cdot L' = X L 3 N L' L'' L' L'' L'' N L' L'' N L' \in N \]
Categorical symmetries (Review)

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\[
\mathcal{L} = \mathcal{L} \quad \rightarrow \quad \mathcal{L}
\]
Categorical symmetries (Review)

Implemented by topological lines:

\[ \text{[Petkova, Zuber '02; Gaiotto, Kapustin, Seiberg, Willett '14; Chang, Lin, Shao, Wang, Yin '18; ...]} \]

\[ \mathcal{L} \mathcal{L} \mathcal{L}' = \mathcal{L} \mathcal{L}' \mathcal{L}'' \]

Fusion structure

\[ \sum_{\mathcal{L}_3} N_{\mathcal{L} \mathcal{L} \mathcal{L}'}^{\mathcal{L}'} = \sum_{\mathcal{L}_3} N_{\mathcal{L} \mathcal{L} \mathcal{L}'}^{\mathcal{L}''} \in \mathbb{N} \]
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\[ L = L' = L'' = L''' \]

Fusion structure:

\[ = \sum_{L_3} N^{L''}_{L L'} \]

\[ N^{L''}_{L L'} \in \mathbb{N} \]

Topological junctions (vector space):

\[ x \in V^{L''}_{L L'} : \]
Categorical symmetries (Review)

Implemented by topological lines:

\[ \text{[Petkova, Zuber '02; Gaiotto, Kapustin, Seiberg, Willett '14; Chang, Lin, Shao, Wang, Yin '18; ...]} \]

Fusion structure

\[ = \sum_{L_3} N_{L L L'} \]

\[ N_{L L L'} \in \mathbb{N} \]

Associativity (F-symbols):

\[ = \sum_{L_3} N_{L L L'} \]

Topological junctions (vector space):

\[ x \in \mathcal{V}_{L L L'} : \]
Categorical symmetries (Review)

Implemented by topological lines:

\[ \text{[Petkova, Zuber '02; Gaiotto, Kapustin, Seiberg, Willett '14; Chang, Lin, Shao, Wang, Yin '18; ...]} \]

\[ \mathcal{L} = \mathcal{L} \]

Fusion structure

\[ \mathcal{L} \mathcal{L}' = \sum_{\mathcal{L}_3} N_{\mathcal{L}, \mathcal{L}, \mathcal{L}'} \]

\[ N_{\mathcal{L}, \mathcal{L}, \mathcal{L}'} \in \mathbb{N} \]

Associativity (F-symbols):

\[ \mathcal{L}_{x'} \mathcal{L}_v \mathcal{L}_u = \sum_{\mathcal{L}_v} \left[ \mathcal{L}_{x'} \mathcal{L}_v \mathcal{L}_u \right]_{x'yv} \]

Quantum dimension:

\[ d_{\mathcal{L}} = d_{\mathcal{L}} \geq 1 \]
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:

\[ Z^2 \text{ ordered} \]
\[ Z^2 \text{ disordered} \]

Ising CFT

KW duality exchanges high and low T

\[ \text{KW} \iff \]

At critical point this becomes a symmetry \( N \).

Ising Symmetry:

\[ \text{Ising} = \{ 1, \eta, N \} \]

Fusion algebra:

\[ \eta^2 = 1, \quad \eta^N = N \eta = N \]
\[ N^2 = 1 + \eta^d, \quad \eta^d = 1, \quad \eta^N = \sqrt{2} \]

The KW defect line \( N \) is non-invertible!
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:
Consider the 1+1d Ising phase diagram:

\begin{align*}
\mathbb{Z}_2 \text{ ordered} & \quad \mathbb{Z}_2 \text{ disordered} \\
\text{Ising} & \quad \text{CFT} \\
T & \quad
\end{align*}
Consider the 1+1d Ising phase diagram:

\[ \mathbb{Z}_2 \text{ ordered} \quad \mathbb{Z}_2 \text{ disordered} \]

KW duality exchanges high and low T.

Ising Symmetry:

\[ \text{Ising} = \{1, \eta, N\} \]

Fusion algebra:

\[ \eta^2 = 1, \eta N = N \eta, N^2 = 1 + \eta, d = \sqrt{2} \]

The KW defect line \(N\) is non-invertible!
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:

\[ \mathbb{Z}_2 \] ordered \quad \Rightarrow \quad \mathbb{Z}_2 \] disordered

KW duality exchanges high and low T

KW

√2

The KW defect line is non-invertible!
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:

\[ \mathbb{Z}_2 \text{ ordered} \quad \mathbb{Z}_2 \text{ disordered} \]

KW duality exchanges high and low T

At critical point \( \circ \) this becomes a symmetry \( \mathcal{N} \).
Consider the 1+1d Ising phase diagram: Ising Symmetry:

```
\[\mathbb{Z}_2\] ordered \quad \mathbb{Z}_2 \quad \text{disordered}
```

KW duality exchanges high and low T

At critical point $\circ$ this becomes a symmetry $\mathcal{N}$. 

**Ising Symmetry:**

\[
\text{Ising} = \{1, \eta, N\}
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**Fusion algebra:**

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\eta^2 = 1, \quad \eta N = N \eta = N
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\[
N^2 = 1 + \eta d, \quad d N = \sqrt{2}
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**Ising CFT**

**Ising Symmetry:**

\[ \text{Ising} = \{1, \eta, \mathcal{N}\} \]

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\[ KW \leftrightarrow \]

At critical point \( \bigcirc \) this becomes a symmetry \( \mathcal{N} \).
Consider the 1+1d Ising phase diagram:

Ising CFT

$\mathbb{Z}_2$ ordered $\mathbb{Z}_2$ disordered

$T$

KW duality exchanges high and low $T$

Ising Symmetry:

$\text{Ising} = \{1, \eta, \mathcal{N}\}$

Fusion algebra:

$\eta^2 = 1$, $\eta\mathcal{N} = \mathcal{N}\eta = \mathcal{N}$

$\mathcal{N}^2 = 1 + \eta$

At critical point $\circ$ this becomes a symmetry $\mathcal{N}$. 

Example: Ising Symmetry
Consider the 1+1d Ising phase diagram:

\[ Z_2 \text{ ordered} \quad \xrightarrow{T} \quad Z_2 \text{ disordered} \]

Ising CFT

KW duality exchanges high and low T

At critical point \( \bigcirc \) this becomes a symmetry \( \mathcal{N} \).

Ising Symmetry:

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Fusion algebra:

\[ \eta^2 = 1, \quad \eta \mathcal{N} = \mathcal{N} \eta = \mathcal{N} \]

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\[ d_\eta = 1, \quad d_\mathcal{N} = \sqrt{2}. \]
Example: Ising Symmetry

Consider the 1+1d Ising phase diagram:

\[ \mathbb{Z}_2 \text{ ordered } \rightarrow T \rightarrow \mathbb{Z}_2 \text{ disordered} \]

KW duality exchanges high and low T

At critical point this becomes a symmetry \( \mathcal{N} \).

Ising Symmetry:

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Fusion algebra:

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The KW defect line \( \mathcal{N} \) is non-invertible!
$C$-symmetric TQFTs

We describe 1+1d TQFT $M$ via a collection of boundary conditions (states) $a, b, c, ...$

The symmetry action is described by topological junctions.

Which satisfy associativity conditions

\[
\phi_{a \; b \; c} = \tau_{L \; L'} \phi_{a \; c \; L} \phi_{L \; L' \; b}
\]

Parallel fusion is described by an integer-valued matrix:

\[
\begin{pmatrix}
 n_L \\
 b \\
 a
\end{pmatrix}
\]

Satisfying the algebra:

\[
\phi_{b \; \left( n_{L'} \right) \; c} = \phi_{n_{L''} \; L \; L'} \phi_{a \; \left( n_{L'} \right) \; c}
\]

This endows $M$ with the mathematical structure of a module category over $C$. 
We describe 1+1d TQFT $\mathcal{M}$ via a collection of boundary conditions (states) $a, b, c, \ldots$ [Huang, Lin, Seifnashri '21, ...]
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$$L a c L' = \sum \varphi^{a b c}_{L L' L''} L'' a c$$

This endows $\mathcal{M}$ with the mathematical structure of a module category over $\mathcal{C}$.
We describe 1+1d TQFT $\mathcal{M}$ via a collection of boundary conditions (states) $a, b, c, \ldots$ [Huang, Lin, Seifnasri '21, ...]

Parallel fusion is described by an integer-valued matrix:

$$a \xrightarrow{\mathcal{L}} b = (n_{\mathcal{L}})_a^b$$

The symmetry action is described by topological junctions. Which satisfy associativity conditions

$$\mathcal{L} \xrightarrow{\mathcal{L}'} = \sum_{\mathcal{L}''} \varphi_{\mathcal{L} \mathcal{L}'}^{a \ b \ c} \mathcal{L} \mathcal{L}'' \mathcal{L}''$$
$\mathcal{C}$-symmetric TQFTs

We describe 1+1d TQFT $\mathcal{M}$ via a collection of boundary conditions (states) $a, b, c, \ldots$ [Huang, Lin, Seifnashri '21, ...]

The symmetry action is described by topological junctions. Which satisfy associativity conditions

Parallel fusion is described by an integer-valued matrix:

$$a \quad \quad \quad = \quad (n_{\mathcal{L}})_a^b \quad b$$

Satisfying the algebra:

$$\sum_b (n_{\mathcal{L}})_a^b (n_{\mathcal{L}'}_b^c) = \sum_{\mathcal{L}''} N_{\mathcal{L} \mathcal{L}'} (n_{\mathcal{L}''})_a^c,$$

The symmetry action is described by topological junctions.
\(C\)-symmetric TQFTs

We describe 1+1d TQFT \( \mathcal{M} \) via a collection of boundary conditions (states) \( a, b, c, \ldots \) [Huang, Lin, Seifnashri '21, ...]

The symmetry action is described by topological junctions. Which satisfy associativity conditions

\[
\phi^{ab}_{L \cdot L'} = \sum_{L''} \phi^{abc}_{L \cdot L' \cdot L''} = \sum_{L''} \phi^{abc}_{L \cdot L' \cdot L''}
\]

Parallel fusion is described by an integer-valued matrix:

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
= \begin{array}{c}
\sum_{L''} (n_{L})^{b}_{a} (n_{L'})^{c}_{b} \\
\sum_{L''} N_{L \cdot L' \cdot L''}^{c} (n_{L'})^{c}_{a}
\end{array}
\]

Satisfying the algebra:

\[
\sum_{b} (n_{L})^{b}_{a} (n_{L'})^{c}_{b} = \sum_{L''} N_{L \cdot L' \cdot L''}^{c} (n_{L'})^{c}_{a},
\]

This endows \( \mathcal{M} \) with the mathematical structure of a module category over \( \mathcal{C} \).
Example: Ising TQFT

As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with $\mathbb{Z}_2$ symmetry:

$$|+\rangle, |-\rangle, |0\rangle$$

The KW symmetry interchanges the two sets in a $\mathbb{Z}_2$-neutral way:

$$N|0\rangle = |+\rangle + |-\rangle, N|\pm\rangle = |0\rangle.$$

One can check that there are no consistent TQFTs with 1 or 2 vacua.

$$|0\rangle = |N\rangle, |+\rangle = |1\rangle, |-\rangle = |\eta\rangle.$$

This is a special case of the regular representation. One identifies

$$\{a, b, c, ...\} = \{L, L', L'' , ...\}$$

and:

$$L'L''L = 1,$$

This enforces:

$$\phi_{L_1 L_2 L_3 L L'} = h F_{L_1 L L_3 L'} i L_2 L'.$$

And describes the complete SSB of the symmetry $C$. 

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\phi_{L_1 L_2 L_3} = h_{FL_1 L_3} L_2 L_4.
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**Example: Ising TQFT**

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\end{align*}

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As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with \( \mathbb{Z}_2 \) symmetry:

\[
|+\rangle | -\rangle |0\rangle
\]

The KW symmetry interchanges the two sets in a \( \mathbb{Z}_2 \)-neutral way:

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Example: Ising TQFT

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The KW symmetry interchanges the two sets in a $\mathbb{Z}_2$-neutral way:

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As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with \( \mathbb{Z}_2 \) symmetry:

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\[
\mathcal{N}|0\rangle = |+\rangle + |\pm\rangle, \quad \mathcal{N}|\pm\rangle = |0\rangle.
\]

This is a special case of the Regular representation. One identifies \( \{a, b, c...\} = \{\mathcal{L}, \mathcal{L}', \mathcal{L}'' ...\} \) and:

One can check that there are no consistent TQFTs with 1 or 2 vacua.
As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with \( \mathbb{Z}_2 \) symmetry:

\[
|+\rangle, \quad |−\rangle, \quad |0\rangle
\]

The KW symmetry interchanges the two sets in a \( \mathbb{Z}_2 \)-neutral way:

\[
\mathcal{N}|0\rangle = |+\rangle + |−\rangle, \quad \mathcal{N}|±\rangle = |0\rangle.
\]

One can check that there are no consistent TQFTs with 1 or 2 vacua.

This is a special case of the Regular representation. One identifies \( \{a, b, c, \ldots\} = \{L, L', L''\ldots\} \) and:

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L' L'' = 1
\]

And describes the complete SSB of the symmetry \( C \).
Example: Ising TQFT

As an example let us study a TQFT with Ising symmetry.

We are familiar with the TQFTs with $\mathbb{Z}_2$ symmetry:

$|+\rangle$, $|-\rangle$, $|0\rangle$

The KW symmetry interchanges the two sets in a $\mathbb{Z}_2$-neutral way:

$\mathcal{N}|0\rangle = |+\rangle + |-\rangle$, $\mathcal{N}|\pm\rangle = |0\rangle$.

One can check that there are no consistent TQFTs with 1 or 2 vacua.

This is a special case of the **Regular representation**. One identifies \{a, b, c...\} = \{L, L', L'' ...\} and:

$\mathcal{N} |0\rangle = |L\rangle$, $|+\rangle = |L'\rangle$, $|-\rangle = |L''\rangle$.

This enforces:

$\mathcal{N} |0\rangle = |L\rangle$, $|+\rangle = |L'\rangle$, $|-\rangle = |L''\rangle$.

This enforces:

$\mathcal{F}_{L_1 L_2 L_3} \equiv \left[ F_{L_1 L_2 L_3} \right]_{L_2 L''}$.
Example: Ising TQFT

As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with $\mathbb{Z}_2$ symmetry:

This is a special case of the **Regular** representation. One identifies $\{a, b, c...\} = \{L, L', L''...\}$ and:

The KW symmetry interchanges the two sets in a $\mathbb{Z}_2$-neutral way:

This enforces:

One can check that there are no consistent TQFTs with 1 or 2 vacua.

This enforces:

And describes the **complete** SSB of the symmetry $C$. 

$$\Phi_{L_1 L_2 L_3} = \left[ F_{L_1 L_2 L_3} \right]_{L_2 L_1 L_3}.$$
As an example let us study a TQFT with Ising symmetry. We are familiar with the TQFTs with $\mathbb{Z}_2$ symmetry:

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One can check that there are no consistent TQFTs with 1 or 2 vacua.

$$|0\rangle = |\mathcal{N}\rangle, \quad |+\rangle = |1\rangle, \quad |-\rangle = |\eta\rangle$$

This is a special case of the Regular representation. One identifies \{a, b, c...\} = \{L, L', L''...\} and:

This enforces:

$$\mathcal{F}_{L_1L_2L_3} = \left[ F_{L'_1L_1L_3} \right]_{L_2L''}.$$ 

And describes the complete SSB of the symmetry $C$. 

\[\text{Example: Ising TQFT}\]
Kink Multiplets ...

To understand symmetry action on kinks we describe their Hilbert space $H_{ab}$ as the strip Hilbert space with $L \gg 1/M_{\text{kink}}$ and TQFT b.c. [Cordova, Garcia-Sepulveda, Holfester '24]:

$H_{a}, b \cong b \cdot a \cdot L \Rightarrow H_{ab} \Rightarrow H_{cd}$

Composition of two lines $L, L' \Rightarrow \Rightarrow \Rightarrow$

Gives the algebra:

$L'_{ef} \cdot [L_{cd}]_{ab} = X_{L''_{\varphi ace LL'_{L''_{\varphi cdf}}}}$. 
To understand symmetry action on kinks we describe their Hilbert space $\mathcal{H}_{ab}$ as the strip Hilbert space with $L \gg 1/M_{\text{kin}}$ and TQFT b.c. [Cordova, Garcia-Sepulveda, Holfester '24]:
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$$\mathcal{H}_{a,b} \simeq \begin{array}{c}
\begin{array}{c}
\vdots \\
a \\
\end{array} \\
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\vdots \\
b \\
\end{array}
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\quad & \quad \\
\hline
a & b
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$$\begin{array}{c|c|c|c}
\quad & \quad & \quad & \\
\hline
\quad & \quad & \quad & \\
\hline
a & b & c & d
\end{array} = \sqrt{d} \mathcal{L} [\mathcal{L}]_{ab}^{cd} \begin{array}{c|c}
\quad & \quad \\
\hline
\quad & \quad \\
\quad & \quad
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$$\mathcal{L} \mathcal{L}' \mathcal{L} = \sqrt{d_L} \mathcal{L} \left[ \mathcal{L} \right]_{ab}^{cd} \mathcal{L}' \mathcal{L} \mathcal{L}'$$
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$$L \cdot \mathcal{H}_{ab} \Rightarrow \mathcal{H}_{cd}$$

Composition of two lines $L$, $L'$

Gives the algebra:

$$[L'][ef]_{cd} \cdot [L]_{ab} = \sum_{L''} \varphi_{L'\ell L'' \ell_1 \ell_2} \varphi_{L' \ell L' \ell_1 \ell_2} \mathcal{L}'' \cdot [L''][ef]_{ab}.$$
The irreducible representations of this algebra are labelled by lines $v \in \mathbb{C}^\ast$. For $M = \text{Reg } \mathbb{C}^\ast = \mathbb{C}$, in this case the kink creation operator descends from the $v$-twisted sector in the UV CFT. We call $K_{ab}$ the kink multiplet. The fusion algebra $v \times v' = P v'' e_{N v v' v'}$ encodes the tensor product decomposition of irreps $\rightarrow$ kink bound states!
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We call $K^\nu_{ab}(x)$ the **Kink multiplet**.
The irreducible representations of this algebra are labelled by lines $\nu \in \mathbb{C}^*_{\mathcal{M}}$. For $\mathcal{M} = \text{Reg } \mathbb{C}^*_{\mathcal{M}} = \mathbb{C}$. In this case the kink creation operator descends from the $\nu$-twisted sector in the UV CFT.

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The irreducible representations of this algebra are labelled by lines $\nu \in C^*_.\text{ For } M = \text{Reg } C^*_M = C. \text{ In this case the kink creation operator descends from the } \nu\text{-twisted sector in the UV CFT.}

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The fusion algebra $\nu \times \nu' = \sum_{\nu''} N^\nu_{\nu\nu'} \nu''$ encodes the tensor product decomposition of irreps $\rightarrow$ kink bound states!
Example: Tricritical Ising $\rightarrow$ Ising TQFT

The classical example is to study the $\phi^4$, $\phi^3$ deformation of the $M_{4,3}$ minimal model. [Zamolodchikov '89, ...]

This preserves an Ising symmetry and flows to three degenerate vacua $|0\rangle$, $|\pm\rangle$.

The kink multiplet is $K_{N\pm0}$, $K_{N0\pm}$.

The literature proposes the following integrable S-matrix: [Bernard, Leclair '90; Zamolodchikov '91; Fendley, Saleur, Zamolodchikov '93]

$$S_{abdc}(\theta) = \frac{1}{Z(\theta)} \left[ s_{da} s_{dc} s_{db} d_{d} \right] \frac{i\theta}{2\pi} \sinh \frac{\theta}{4} \delta_{bd} + \sinh \frac{i\pi}{2} - \frac{\theta}{4} \delta_{ac} #$$

The green factor enforces crossing symmetry. Otherwise $S_{abdc}(\theta) = s_{da} s_{dc} s_{db} d_{d} S_{bcad}(i\pi - \theta)$.

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Ward identities and the S-matrix

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\[ A.C. \sim S \]
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Ward identities and the S-matrix

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\[
\begin{array}{c}
\text{A.C.} \\
\begin{array}{c}
\text{v} \\
\text{v} \\
\text{v} \\
\text{v}
\end{array}
\end{array}
\] 

\[
\begin{array}{c}
\text{S} \\
\begin{array}{c}
\text{v} \\
\text{v} \\
\text{v} \\
\text{v}
\end{array}
\end{array}
\] 

We insert a symmetry line on the disk and deform it either upwards or downwards:
Ward identities and the S-matrix

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\[ \approx \]

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![Diagram showing symmetry action](image)

We insert a symmetry line on the disk and deform it either upwards or downwards:
Ward identities and the S-matrix

To derive the symmetry action we construct the S-matrix by analytic continuation from a large disk:

\[
\begin{align*}
S & \simeq \begin{pmatrix}
 L & v \\
 v & v \\
 v & v \\
 v & v
\end{pmatrix} \\
& \quad \simeq \begin{pmatrix}
 a & \ast \\
 \ast & b \\
 c & \ast \\
 \ast & d
\end{pmatrix}
\end{align*}
\]

The symmetry action is given by:

\[
\left[ \mathcal{L} ; v \right]_{c b}^{e} \left[ \mathcal{L} ; v \right]_{b a}^{e a'} \sqrt{\frac{d_a}{d_c}} S_{a' c}^{b'} (\theta) = \sum_{e'} \left[ \mathcal{L} ; v \right]_{b' c'}^{e' c} \left[ \mathcal{L} ; v \right]_{a' b'}^{3 e'} \sqrt{\frac{d_{a'}}{d_{c'}}} S_{a b}^{c e'} (\theta).
\]
To derive the symmetry action we construct the S-matrix by analytic continuation from a large disk:

\[ \mathcal{M}_{4,3} \rightarrow \text{Ising:} \]

\[ \mathcal{N} : \quad S_{0+}^{0+}(\theta) = S_{0+}^{+0}(\theta) + S_{0-}^{+0}(\theta). \]
To analyze crossing symmetry we assume that the Disk 4pf is crossing invariant. We then introduce the in and out states:

\[ |\psi\rangle_{\text{in}} = d_b c v v \langle \psi | \rangle_{\text{out}} = d_b a v v \]

Their norms are schematically:

\[ \langle \psi | \psi \rangle_{\text{in}} = v v v v = d_v \sqrt{d_b d \cdot d_a c} \]

Using the normalized states in both channels the Disk crossing is continued to:

\[ S_{ab} d c (\theta) = S_{bc} a d (\iota \pi - \theta) \]
Modified Crossing

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  d \\
  c \\
  v \\
  v
\end{array} \quad \begin{array}{c}
  b \\
  v
\end{array} \]

\[ \langle \psi |_{\text{out}} = \begin{array}{c}
  a \\
  d \\
  b \\
  v \\
  v
\end{array} \]

Their norms are schematically:

\[ \langle \psi |_{\text{in}} \langle \psi | = \langle d | v \rangle^2 \langle b | v \rangle^2 = d_b \]

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\[ b = d_v \sqrt{d_b d_d} \]
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\[
|\psi\rangle_{\text{in}} = d_b \begin{array}{c} \text{a} \\ \text{c} \end{array} v \\
\langle\psi|_{\text{out}} = d_b \begin{array}{c} \text{a} \\ \text{c} \end{array} v
\]

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\[
\text{in}\langle\psi|\psi\rangle_{\text{in}} = d_v \sqrt{d_b d_d}
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To analyze crossing symmetry we assume that the Disk 4pf is crossing invariant. We then introduce the in and out states:

\[ |\psi\rangle_{in} = |d \ b \ c \ v \rangle \quad \text{and} \quad \langle \psi|_{out} = \langle d \ b | a \ v \ v \rangle \]

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To analyze crossing symmetry we assume that the Disk 4pf is crossing invariant. We then introduce the in and out states:

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\nu \\
d \quad b \\
c \\
\nu
\end{array} \]

\[ \langle \psi |_{out} = \begin{array}{c}
\nu \\
d \quad b \\
x \\
\nu
\end{array} \]

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\[ S_{ab}^{dc}(\theta) = \sqrt{d_{a}d_{c}} \cdot S_{bc}^{ab}(i\pi - \theta) \]
Bootstrap for Fibonacci symmetry

To conclude let me flash some results coming from the S-matrix Bootstrap.

Unitarity + Analiticity + Modified Crossing + Symmetry

$C = \text{Fibonacci (two vacua)}$

$W_2 = 1 + W$.

$g^2$: cubic coupling $K \bar{K} \rightarrow B$.

Fat dot: integrable flow $M_4, \phi_2, \phi_1$ [Smirnov '91; Colomo, Koubek, Mussardo '92; ...]

Other integrable point: cusp at $g = 0$, Potts $S + S^*$-deformation

Symmetry $Z_2 \times \text{Fib}$, \{1, $W, W'$ $\equiv \eta W, \eta$\}. Kink is in $W'$ multiplet. But now: $W' \times W' = 1 + W = B W, W \not\in K W, 1 \times K W, W$. 

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Fat dot: integrable flow

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\[
W' \times W' = 1 + W \implies B_{W',W} \notin K_{W,1} \times K_{1,W} .
\]
Future Prospects

There are many avenues yet to pursue. For example:

▶ Physical observables related to modified crossing. Promising: TBA for twisted sector data along RG flow (WIP).

▶ Reconstructing UV CFT data from integrable IR. Haagerup - Double Haagerup symmetric CFTs? [Huang, Lin, Ohmori, Tachikawa, Tezuka '21; Van Hoove, Lootens, Van Damme, Wolf, Osborne '21]

▶ Applications to higher dimensions. (3d results [Mehta, Minwalla, Patel, Prakash, Sharma '22]) Monopole scattering in 4d [Van Beest, Boyle Smith, Delmastro, Komargodski, Tong '23, ...]

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▶ Reconstructing UV CFT data from integrable IR. Haagerup - Double Haagerup symmetric CFTs? [Huang, Lin, Ohmori, Tachikawa, Tezuka '21; Van Hoove, Lootens, Van Damme, Wolf, Osborne '21]

▶ Applications to higher dimensions. (3d results [Mehta, Minwalla, Patel, Prakash, Sharma '22])

Monopole scattering in 4d [Van Beest, Boyle Smith, Delmastro, Komargodski, Tong '23, ...].

▶ Relationship between modified crossing and 't Hooft anomalies.