The Gravitational Waveform
From Scattering Amplitudes to General Relativity

- Brandhuber, Brown, Chen, SDA, Gowdy, Travaglini
- SDA, Gonzo, Novichkov
- Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng
- Brunello, SDA

Stefano De Angelis - Amplitudes 2024, Institute for Advanced Study - June 13th, 2024
\[
\hat{h}_\mu^\nu \bigg|_{|x| \to \infty} = \eta_{\mu\nu} + \frac{\kappa^2 M}{8\pi |x|} \left[ \frac{\kappa^2 M}{\sqrt{-b^2}} \hat{h}_{\mu\nu}^{(1)} + \left( \frac{\kappa^2 M}{\sqrt{-b^2}} \right)^2 \hat{h}_{\mu\nu}^{(2)} + \ldots \right]
\]

\[
\hat{h}_+ = \frac{1}{2} \left( \hat{h}_{11} - \hat{h}_{22} \right)
\]

\[
\hat{h}_x = \hat{h}_{12}
\]

\[
u_{1,2} = \frac{1}{\sqrt{24}} (5,0,0, \pm 1)
\]

Plot authors: Aidan Herdershee, Radu Roiban, Fei Teng

- [Kovacs, Thorne] '78
- [Jakobsen, Mogull, Plefka, Steinhoff]
- [Mougiakakos, Riva, Vernizzi]
- [Jakobsen, Mogull, Plefka, Steinhoff]
- [SDA, Gonzo, Novickov]
- [Brandhuber, Brown, Chen, Gowdy, Travaglini]
- [Aoude, Haddad, Heissenberg, Helset]
Why Gravitational Waveforms?
Why Scattering?

- GW templates for matched-filtering analyses
- Analytic continuation from unbound to bound
- Scattering setups are interesting on their own
- Analytic properties of scattering amplitudes

Talk by Zvi Bern
How do we compute waveform?
KMOC as on-shell in-in formalism

• Compute expectation values from scattering amplitudes

\[ \Delta \langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{\text{out}} - \langle \mathcal{O} \rangle_{\text{in}} = \text{out} \langle \psi | \mathcal{O} | \psi \rangle_{\text{out}} - \text{in} \langle \psi | \mathcal{O} | \psi \rangle_{\text{in}} = \text{in} \langle \psi | S^\dagger [\mathcal{O}, S] | \psi \rangle_{\text{in}} \]

• The initial state of the two-body problem

\[ | \psi \rangle_{\text{in}} = \int d\Phi(p_1) d\Phi(p_2) \phi_1(p_1) \phi_2(p_2) e^{i(b_1 \cdot p_1 + b_2 \cdot p_2)} | p_1, p_2 \rangle_{\text{in}} \quad [\text{Kosower, Maybee, O'Connell}] \]

• Pick the wavefunctions

\[ 1/m_i \ll G m_i \ll b \]

  • $1/m_i$, Compton wavelengths
  • $G m_i$, Schwarzschild radii
  • $b$, impact parameter

• Pick the operator

\[ \mathcal{O} = \mathcal{W}_{GR} = \varepsilon^{\mu \nu} h_{\mu \nu} \quad g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu} \]

[Christofoli, Gonzato, Kosower, O’Connell]
[Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]
The Waveform from Scattering Amplitudes

The waveform is an in-in observable, while the amplitude is an in-out observable.

The five-point scattering amplitude

\[ \Delta \langle \mathcal{W}_h \rangle (u, \vec{n}) = \frac{1}{4\pi r} \int_0^\infty d\omega \int d\mathbf{q} \left\{ \delta^D(q_1 + q_2 - k) e^{-iou} \left[ \mathcal{A}(p_1 p_2 \rightarrow p'_1 p'_2 k^-h) - i \mathcal{A}^*(p'_1 p'_2 \rightarrow \vec{X}) \otimes \mathcal{A}(p_1 p_2 \rightarrow X k^-h) \right] + \text{c.c.} \right\} \]

Fourier transform to impact parameter space

\[ d\mu = \prod_{i=1}^2 \hat{d}^D q_i \delta(2p_i \cdot q_i + q_i^2) e^{ih_i q_i} \]
The five-point result @ NLO

- Heavy-mass EFT [Damgaard, Haddad, Helset], [Brandhuber, Chen, Travaglini, Wen]

Amplitudecraft:
- Generalised Unitarity
- Integrations of loops

\[
\mathcal{M}^{(1)}_{5,m_1^2,m_2^2} = \delta_{IR} + i \pi \frac{i_1 - c_{1,0}}{\sqrt{\gamma^2 - 1}} - \frac{i \pi}{16 \pi \sqrt{-q_2^2 + w_1^2}} - \frac{c_{2,0}}{16 \pi \sqrt{-q_1^2}} + I_{w_1} \log \frac{w_1^2}{\mu^2} + I_{w_2} \log \frac{w_2^2}{\mu^2} + I_q \log \frac{q_1^2}{q_2^2} + I_{\gamma} \frac{\log \left( \frac{\sqrt{\gamma^2 - 1} + \gamma}{\sqrt{\gamma^2 - 1}} \right)}{\sqrt{\gamma^2 - 1}} + \mathcal{O}(\epsilon)
\]

\[
\mathcal{M}^{(1)}_{5,m_1^2,m_2^2,\text{cut}} = \frac{d_{IR}}{\sqrt{\gamma^2 - 1}} \times \frac{1}{\epsilon} + \frac{i \pi}{16 \pi \sqrt{-q_1^2}} + \frac{c_{q_1}}{\sqrt{\gamma^2 - 1}} + \frac{c_{q_2}}{\sqrt{\gamma^2 - 1}} + \frac{\log q_1^2}{\mu^2} + \frac{\log q_2^2}{\mu^2} + \frac{\log \left( \frac{\gamma^2 - 1}{\gamma^2 - 1} \right)}{\sqrt{\gamma^2 - 1}} + \frac{\log \left( \frac{\sqrt{\gamma^2 - 1} + \gamma}{\sqrt{\gamma^2 - 1}} \right)}{\sqrt{\gamma^2 - 1}} + \frac{R}{\sqrt{\gamma^2 - 1}} + \mathcal{O}(\epsilon)
\]

- \( h \) power-counting parameter [Damour]
- Singular terms in \( 1/h \) cancel between the amplitude and the cut
- The result must be symmetrised \( m_1 \leftrightarrow m_2 \)

- The result is infrared divergent!
- Each rational coefficient has \( \sim 10^2 - 10^3 \) terms, with spurious poles [Talk by Abreu and Tancredi]
- Many terms are polynomials in \( q_1^2 \) and \( q_2^2 \), the Fourier transform will kill them [Brandhuber, Brown, Chen, SDA, Gowdy, Travaglini]
The problem of spurious poles
Impact parameter space and the small-velocity limit

\[ I_q = \frac{1}{q_1^2 q_2^2 \left[ (q_1^2)^2 - 2q_1^2 q_2^2 + (q_2^2)^2 + 4q_1^2 w_1^2 \right]^4 \left[ (q_1^2)^2 w_1^2 + (q_2^2)^2 w_2^2 - 2q_1^2 q_2^2 w_1 w_2 \gamma \right]^2} \sim \frac{1}{p_{\infty}^{12}} \]

Fake divergences! We need to expand all the variables at \( \mathcal{O}(p_{\infty}^{13}) \) to get the waveform at the first relevant order.

The problem is not just analytical: the spurious poles sit in the physical region.

\[ \gamma_1 = \frac{w_1^2 + w_2^2}{2w_1 w_2} > 1 \quad \gamma_2 = \frac{(q_1^2 w_1)^2 + (q_2^2 w_2)^2}{2q_1^2 q_2^2 w_1 w_2} > 1 \quad \Delta_1 = \frac{|q_1^2 - q_2^2|}{2\sqrt{-q_1^2}} > 0 \]

Integration-contour deformation to improve numerical convergence:

- [Herdershee, Roiban, Teng]
- [Bohnenblust, Ita, Kraus, Schlenk]

Introducing a new set of functions which are smooth on the physical sheet.

\[ L_q = \frac{1}{\Delta_2^2} \left\{ 2 \log \frac{q_2^2}{q_1^2} + 2 \log \frac{w_2}{w_1} - \frac{(q_2^2 w_2)^2 - (q_1^2 w_1)^2}{q_1^2 q_2^2 w_1 w_2} \left[ \left( 1 + \frac{\gamma \Delta_2}{\gamma^2 - 1} \right) \frac{\text{arccosh} \gamma}{\sqrt{\gamma^2 - 1}} - \frac{\Delta_2}{\gamma^2 - 1} \right] \right\} \]

- [Bern, Dixon, Kosower]
- [Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]
The waveform in General Relativity

The Multipolar-Post-Minkowskian formalism

- The MPM formalism computes the time-domain waveform as a sum over irreducible multipolar contributions, keyed by their multipole order and their spatial parity:

\[ W_{\text{MPM}}(T_r, \theta, \phi) = U_2 + \frac{1}{c} (V_2 + U_3) + \frac{1}{c^2} (V_3 + U_4) + \frac{1}{c^3} (V_4 + U_5) + \frac{1}{c^4} (V_5 + U_6) + \frac{1}{c^5} (V_6 + U_7) + \cdots \]

- It computes each radiative multipole moment in terms of the stress-energy tensor of the material source. Each radiative multipole is given by a sum of contributions involving both source variables at a time and hereditary integrals over the past behaviour of the source:

\[ U_{ij}(t) = \frac{d^2I_{ij}(t)}{dt^2} + \frac{2GM}{c^3} \left[ \int_0^\infty d\tau I_{ij}^{(4)}(t-\tau) \left( \log \left( \frac{\tau}{2b_0} \right) + \frac{11}{12} \right) + \frac{G}{c^5} \left( \frac{1}{7} I_{a(i}^{(5)} I_{j)a} - \frac{5}{7} I_{a(i}^{(4)} I_{j)a} - \frac{2}{7} I_{a(i}^{(3)} I_{j)a} \right) + \cdots \right] \]

\[ I_{ij}^{2.5PN}(t) = \nu M \left( 1 + \frac{a_2}{c^2} + \frac{a_4}{c^4} - \frac{24}{7} \frac{\nu}{c^5} \frac{G^2M^2}{r^2} r \right) x^{(i} x^{j)} + \cdots \]
A Tale of Two Formalisms
Matching KMOC and MPM

- The two computations are set up in “different frames” [Bini, Damour, Geralico]

- The rotation $\phi \rightarrow \phi + \Delta \chi^{1\text{PM}}/2$ is equivalent to the KMOC cut terms [Georgoudis, Heissenberg, Russo]

- In conventional (or ’t Hooft) dimreg scheme, we need to take into account $\epsilon/\epsilon$ contributions (not in FDH) [Bini, Damour, SDA, Geralico, Herdershee, Roiban, Teng]
A Tale of Two Formalisms
Backreaction on the Static Background

- In MPM, there is a constant-in-time leading contribution. We match it by introducing one-particle graviton emission at zero energy:

\[
\mathcal{W}_{\text{const}}(k) \propto \frac{\hat{\delta}(\omega)}{r} \left[ \frac{m_1 (\varepsilon \cdot u_1)^2}{u_1 \cdot n} + \frac{m_2 (\varepsilon \cdot u_2)^2}{u_2 \cdot n} \right]
\]

- We can divide the momentum-space matrix element into three categories:

\[
\mathcal{W}(k, q_1, q_2) = \mathcal{W}_{\text{const}}(k, q_1, q_2) + \mathcal{W}_{\text{conn}}(k, q_1, q_2) + \mathcal{M}_{\text{disc}}(k, q_1, q_2)
\]

- The waveform's constant part depends on the choice of the BMS frame.
A Tale of Two Formalisms
Backreaction on the Static Background

• If zero-energy gravitons are kept in the spectrum, the three-point amplitude generates time-dependent terms through the cut term.

• Since the zero energy graviton is supported only at the origin of phase space, we must regularize it so that this point, $|\ell| = 0$, remains in the integration domain.

• We find that the cut gives a finite contribution:

$$\mathcal{M}_{\text{disc}} = \delta^{1 \text{ loop disc}} - i\omega GE \left[ \frac{1}{\epsilon} - \log \frac{\beta^2}{\pi} \right] \mathcal{M}_{\text{tree}}$$

$$\delta^{1 \text{ loop disc}} = iG \left[ m_1 w_1 \log \frac{w_1^2}{\omega^2} + m_2 w_2 \log \frac{w_2^2}{\omega^2} \right] \mathcal{M}_{\text{tree}}$$

• The second term can be removed by a finite time shift. The first term $\delta^{1 \text{ loop disc}}$ coincides with a BMS supertranslation. [Veneziano, Vilkovisky], [Georgoudis, Heissenberg, Russo]
An Improved Framework for Waveforms [Brunello, SDA]

• From the analytic properties of scattering amplitudes, we isolate long-range interactions.

• We avoid spurious poles in the $q_i^2$ integrals by performing tensor-to-scalar decomposition at the level of Fourier + loop integrals. [Anastasiou, Karler, Vicini]

• We introduce IBP relations for Fourier + loop integrals. The final waveform is a sum of Master Integrals — the Fourier transforms of loop MIs and their derivatives w.r.t. the impact parameter.

$$\mathcal{J}_{a_1,a_2,a_3,a_4,a_5,a_6} = \int \frac{Dq}{q^2} \prod_{i=1}^{11} \left( \frac{1}{D_1 q_i} \right) a_{11} - \int \frac{Dq}{q^2} \frac{\partial}{\partial q_i} a_{11} \left( \prod_{i=1}^{11} D_1 q_i \right) = 0$$

We can use packages developed for loops:
• LiteRed [Lee]
• LiteIBP [Peraro]

At tree level, the $q_i^2$-integration is trivialised (factorisation in the complex $q_i^2$ channels):
• [SDA, Gonzo, Novichkov]
The Fourier integrals

\[ \mathcal{F} \left[ (-q^2)^a \right] \propto K_{\alpha + \frac{1}{2} - 1} \left( \sqrt{-b^2 \hat{w}_2} \right) \]

\[ \frac{i}{16 \pi \sqrt{-b^2 p_\infty}} \left\{ \int_0^\infty dx \left[ e^{-z \cosh x} H_{-1} \left( z \sqrt{p_\infty \sinh x} \right) \right] - i \frac{e^{-\sqrt{1 + p_\infty}}}{\sqrt{p_\infty}} \right\} \]

Struve-H function

\[ \frac{1}{8 \sqrt{(-q_1^2)}} + \mathcal{O}(\epsilon^1), \]

\[ \frac{i}{8 \pi (-q_2^2) \sqrt{\gamma^2 - 1}} \left( \frac{-q_2^2}{w_{11} \mu_{11}} \right)^2 \left[ \frac{1}{\epsilon} - \log (\gamma^2 - 1) - 2 \log \left( \gamma + \sqrt{\gamma^2 - 1} \right) + i \pi \right] + \mathcal{O}(\epsilon^1) \]

\[ \frac{i}{8 \pi (-q_2^2) \sqrt{\gamma^2 - 1}} \left( \frac{-q_1^2}{w_{22} \mu_{22}} \right)^2 \left[ \frac{1}{\epsilon} - \log (\gamma^2 - 1) + i \pi \right] + \mathcal{O}(\epsilon^1) \]

- Can we write this Fourier transform as iterated integrals?
- Can we compute these integrals using differential equations? [Henn]
Summary and future directions

- The scattering waveform at NLO in momentum space
- Comparison to MPM results and interesting connections to BMS
- The analytic waveform in impact parameter space

- NNLO waveform
  - What is the interesting physics we are going to learn?
  - Challenge for the QCD community
- Peeling? The $1/r$ expansion of the waveform
- Resummation in $G$ and comparisons with Numerical Relativity

“...The fact that the road leading to the present successful EFT/MPM comparison had some bumps, which taught us interesting lessons, is another example of the useful synergy between amplitude-based, and classical perturbation-theory-based, approaches to gravitational physics.”

Continuous-spin particles from the on-shell approach

[Brando Bellazzini, SDA, Marcello Romano]