The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities

Lauren K. Williams, Harvard

Based on: arXiv:2404.03026,
joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler
Outline

- Tricolored subdivisions and partial cyclic orders
- Applications to Parke-Taylor identities and Parke-Taylor polytopes
- What is the amplituhedron?
- Magic number conjecture for the amplituhedron
- Proof of Magic number conjecture when \( m = 2 \)
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A tricolored subdivision $\tau$ of an $n$-gon is a subdivision of the polygon into smaller polygons (black, grey, or white) in which every edge connects two vertices of the $n$-gon.

From each $\tau$, can read off a cyclic order $C_\tau$ (is a cyclic analogue of partial order). To get $C_\tau$ from $\tau$, read vertices of white (resp black) polygons clockwise (resp counterclockwise), and ignore the grey.

The $C_\tau$ from our example requires that $(2, 5, 7)$, $(5, 7, 6)$, and $(1, 8, 7, 2)$ are circularly ordered.

A circular extension of $C_\tau$ is a total circular order compatible with $C_\tau$. E.g. one circular extension of our example is: $(2, 5, 1, 8, 7, 6, 3, 4)$. 

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\[
\begin{array}{c}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} \\
\end{array}
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The Grassmannian \( Gr_{k,n}(\mathbb{C}) := \{ V \mid V \subset \mathbb{C}^n, \dim V = k \} \)

Represent an element of \( Gr_{k,n} \) by a full-rank \( k \times n \) matrix \( C \).

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\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}
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Given \( I \in \binom{[n]}{k} \), the Plücker coordinate \( p_I(C) \) is the minor of the \( k \times k \) submatrix of \( C \) in column set \( I \).
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- Given a permutation $w = w_1 \ldots w_n$, define the *Parke-Taylor function*

$$\text{PT}(w) := \frac{1}{P_{w_1w_2}P_{w_2w_3} \ldots P_{w_nw_1}},$$

where the $P_{ij}$ are Plücker coordinates on the Grassmannian $\text{Gr}_{2,n}$.

We get the following identity.

**Theorem (Parisi–Sherman–Bennett–Tessler–W)**

Let $\tau$ be a tricolored subdivision with at least one grey polygon, and let $C_\tau$ be the cyclic partial order. Then

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Example:

The circular extensions of $C_\tau$ are $(1234), (1243), (1423)$, so Thm says $\frac{1}{P_{12}P_{23}P_{34}P_{41}} + \frac{1}{P_{12}P_{24}P_{43}P_{31}} + \frac{1}{P_{14}P_{42}P_{23}P_{31}} = 0$.

(Rk: 3-term Plücker relation)
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Let \( \tau \) be a tricolored subdivision with at least one grey polygon, and let \( C_{\tau} \) be the cyclic partial order. Then

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We can associate a Parke-Taylor polytope $\Gamma_\tau \subset \mathbb{R}^{n-1}$ to each tricolored subdivision on $[n]$: for any compatible arc $i \to j$ with $i < j$,

$\text{area}(i \to j) \leq x_i + x_{i+1} + \cdots + x_{j-1} \leq \text{area}(i \to j) + \text{gr-area}(i \to j) + 1$.

A compatible arc is an edge of a polygon or lies entirely inside a black or white polygon.

$\text{area}(i \to j)$ (resp. gr-area($i \to j$)) is the “black area” (resp. “grey area”) to the left of the arc.

Above, $2 \to 7$ is a compatible arc. Gives inequality:

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Above, $2 \to 7$ is a compatible arc. Gives inequality:
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We’ve seen how each tricolored subdivision $\tau$ gives rise to:

- a partial cyclic order $C_\tau$ and a Parke-Taylor polytope $\Gamma_\tau$.

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Let $\tau$ be a tricolored subdivision. Then the Parke-Taylor polytope $\Gamma_\tau$ has a triangulation

$$\Gamma_\tau = \bigcup \Delta(w)$$

into unit simplices $\Delta(w)$, where $w$ ranges over all circular extensions of the partial cyclic order $C_\tau$. In particular, the normalized volume of $\Gamma_\tau$ equals the number of circular extensions of $C_\tau$. 

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The Grassmannian and the matroid stratification

Recall: the **Grassmannian** $Gr_{k,n}(\mathbb{C}) := \{V \mid V \subset \mathbb{C}^n, \dim V = k\}$

Represent an element of $Gr_{k,n}$ by a full-rank $k \times n$ matrix $C$.

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Let $Gr_{k,n}^\geq$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all $I$.

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Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_\mathcal{M} := \{ C \in Gr_{k,n}^0 | p_I(C) > 0 \text{ iff } I \in \mathcal{M} \}$.

In contrast to terrible topology of matroid strata ...

(Postnikov, see also Rietsch) If $S_\mathcal{M}$ is non-empty it is a (positroid) cell, i.e. homeomorphic to an open ball. So we have *positroid cell decomposition*

$$Gr_{k,n}^0 = \sqcup S_\mathcal{M}.$$

Can classify the (nonempty) cells ...
What is the positive Grassmannian?

Background: 1994 Lusztig total positivity for $G/P$, 1997 Rietsch, 2006 Postnikov preprint on \textit{totally non-negative} (TNN) or “positive” Grassmannian.

Let $Gr_{k,n}^{\geq 0}$ be subset of $Gr_{k,n}(\mathbb{R})$ where Plucker coords $p_I \geq 0$ for all $I$.

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What is the amplituhedron?

The amplituhedron $\mathcal{A}_{n,k,m}(Z)$, Arkani-Hamed–Trnka (2013).

Fix $n, k, m$ with $k + m \leq n$.

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Motivation for the amplituhedron ($\mathcal{N} = 4$ SYM):


- Hodges (2009) observed that in some cases, the amplitude is the volume of a polytope, with spurious poles arising from internal boundaries of a triangulation of the polytope. Asked if in general each amplitude is the volume of some geometric object.

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- If $k = 1$ and $m = 2$, $\mathcal{A}_{n,k,m} \subset Gr_{1,3}$ is equivalent to an $n$-gon in $\mathbb{RP}^2$.
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Lauren K. Williams (Harvard) 2024
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Fix \( n, k, m \) with \( k + m \leq n \), let \( Z \in \text{Mat}^{>0}_{n,k+m} \) (max minors > 0). Let \( \tilde{Z} \) be map \( \text{Gr}_{k,n}^{>0} \rightarrow \text{Gr}_{k,k+m} \) sending a \( k \times n \) matrix \( C \) to \( CZ \).

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We’d like to “triangulate” or “tile” the amplituhedron

Have $Gr_{k,n}^{\geq 0} = \bigsqcup \pi S_\pi$ cell complex, and $\tilde{Z} : Gr_{k,n}^{\geq 0} \to A_{n,k,m}(Z)$ a continuous surjective map onto $km$-dim'l amplituhedron $A_{n,k,m}(Z)$.

A tiling of $A_{n,k,m}(Z)$ is a collection $\{\tilde{Z}(S_\pi) | \pi \in C\}$ of closures of images of $km$-dimensional cells, such that:

- $\tilde{Z}$ is injective on each $S_\pi$ for $\pi \in C$ ($\tilde{Z}(S_\pi)$ a tile)
- their union equals $A_{n,k,m}(Z)$
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We will work with all-$Z$ tilings, coming from collections of cells that give tilings for all $Z$.

Motivation:
the “volume” of the amplituhedron computes scattering amplitudes;
AH-T conjectured that certain “BCFW cells” give a tiling of $A_{n,k,4}(Z)$;
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Let \( M(a, b, c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2} \).

All known tilings of \( A_{n,k,m} \) for even \( m \) have cardinality \( M(k, n - k - m, \frac{m}{2}) \). Call this prediction the **Magic Number Conjecture**.

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$k = 1$: Thm says that all triangulations of an $n$-gon have size $n - 2$.

Ideas of the proof:

- There is a classification of tiles for the $m = 2$ amplituhedron using bicolored subdivisions (P–SB–W).
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- Use above decompositions to define the P-T function of $A_{n,k,2}(Z)$ and each tile, and show that this function is the same for ALL tiles.
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- There is a classification of tiles for the $m = 2$ amplituhedron using bicolored subdivisions (P–SB–W).
- Just as each Parke-Taylor polytope has a decomposition into \( w \)-simplices where \( w \) ranges over certain circular extensions, each tile has a decomposition into “\( w \)-chambers” where \( w \) ranges over certain circular extensions.
- Use above decompositions to define the P-T function of $\mathcal{A}_{n,k,2}(Z)$ and each tile, and show that this function is the same for ALL tiles.
- Therefore each tiling of $\mathcal{A}_{n,k,2}(Z)$ has the same size.
- Rk: total number of \( w \)-chambers of $\mathcal{A}_{n,k,2}(Z)$ is the Eulerian number.
The magic number theorem for the $m = 2$ amplituhedron

**Magic Number Theorem (P–SB–T–W)**

All tilings of ampl. $A_{n,k,2}(Z)$ have size $M(k, n - k - 2, 1) = \binom{n-2}{k}$.

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Ideas of the proof:

- There is a classification of tiles for the $m = 2$ amplituhedron using \textit{bicolored subdivisions} (P–SB–W).
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Tiles of the amplituhedron

Recall: $\tilde{Z}(S_\pi)$ is a tile for $\tilde{Z}: Gr_{k,n}^{\geq 0} \to A_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on $km$-dim'l cell $S_\pi$. Lukowski–Parisi–Spradlin–Volovich conjectured:

**Theorem (Parisi–Sherman-Bennett–W)**

The tiles for $A_{n,k,2}(Z) \leftrightarrow$ collections of bicolored subdivisions of an $n$-gon with total “area” $k$. To construct the cell $S_\pi$:

- Choose triangulation of black polygons into $k$ black triangles.
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\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
* & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
0 & * & * & 0 & 0 & 0 & * & 0 & 0 \\
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Let $Z \in \text{Mat}_{n,k+2}^{>0}$. Let $\tilde{Z}$ be map $\text{Gr}_{k,n}^{>0} \to \text{Gr}_{k,k+2}$ sending $C \mapsto CZ$. Let $Z_1, \ldots, Z_n$ be rows of $Z$. Let $Y \in \text{Gr}_{k,k+2}$ (viewed as matrix).

Given $I = \{i_1 < i_2\} \subset [n]$, define the twistor coordinate as

$$\langle YZ_I \rangle = \langle YZ_{i_1} Z_{i_2} \rangle := \det \begin{bmatrix} - & Y & - \\ - & Z_{i_1} & - \\ - & Z_{i_2} & - \end{bmatrix}$$

Inspired by matroid stratification, we define the amplituhedron sign stratification – decompose $A_{n,k,2}(Z)$ into pieces based on the signs of twistor coordinates. (Parisi–Sherman-Bennett–W.; Karp-W.)

Call the top-dimensional pieces chambers.

Thm: (P-SB-W) The number of nonempty chambers of $A_{n,k,2}(Z)$ is the Eulerian number.
Chambers of the amplituhedron $\mathcal{A}_{n,k,2}(Z)$

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The Magic Number Theorem for \( A_{n,k,2}(\mathbb{Z}) \)

- Given any region \( R \) of \( A_{n,k,2}(\mathbb{Z}) \) that admits a tiling, we define its weight function
  \[
  \Omega(R) := \sum \text{PT}(\Delta^Z(w)),
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  where the sum is over all \( w \)-chambers \( \Delta^Z_w \subset R \).

- We prove that for any tile \( Z_\tau \) of \( A_{n,k,2}(\mathbb{Z}) \),
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  \Omega(Z_\tau) = (-1)^k \text{PT}(I_n),
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- It is known that there is a tiling of \( A_{n,k,2}(\mathbb{Z}) \) consisting of \( \binom{n-2}{k} \) tiles, so \( \Omega(A_{n,k,2}(\mathbb{Z})) = (-1)^k \binom{n-2}{k} \text{PT}(I_n) \).

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Given any region $R$ of $A_{n,k,2}(Z)$ that admits a tiling, we define its \textit{weight function}

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- Given any region $R$ of $\mathcal{A}_{n,k,2}(\mathbb{Z})$ that admits a tiling, we define its weight function
  \[ \Omega(R) := \sum \text{PT}(\Delta^Z_w), \]
  where the sum is over all $w$-chambers $\Delta^Z_w \subset R$.
- We prove that for any tile $Z_\tau$ of $\mathcal{A}_{n,k,2}(\mathbb{Z})$,
  \[ \Omega(Z_\tau) = (-1)^k \text{PT}(I_n), \]
  where $I_n$ is the identity permutation.
- It is known that there is a tiling of $\mathcal{A}_{n,k,2}(\mathbb{Z})$ consisting of $\binom{n-2}{k}$ tiles,
  so $\Omega(\mathcal{A}_{n,k,2}(\mathbb{Z})) = (-1)^k \binom{n-2}{k} \text{PT}(I_n)$.
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The magic number conjecture for the $m = 2$ amplituhedron and Parke-Taylor identities [arXiv:2404.03026](http://arxiv.org/abs/2404.03026), joint with Matteo Parisi, Melissa Sherman-Bennett, and Ran Tessler.