$W_1^{1+\infty}$ Symmetry in 4D Gravitational Scattering

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2312.08957 with Monica Pate
Review: Soft Graviton Theorems as Symmetries

Soft graviton theorems are equivalently Ward identities for symmetries of the S-matrix. Interpret soft factor as generating infinitesimal symmetry transformation on hard states. These generalized symmetries also involve the insertion of soft gravitons with null momenta parametrized by a point on the celestial sphere/plane.
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\lim_{\omega \to 0} \omega \langle \text{out} | a_+ (\omega \hat{q}) S | \text{in} \rangle = S^{-1} \langle \text{out} | S | \text{in} \rangle
\]

Leading soft graviton theorem (Weinberg 1965)

\[
S^{-1} = \sum_{k \in \text{in, out}} S_k (\hat{q}) = \sum_k \frac{\kappa \epsilon_{\mu \nu} p^\mu_k p^\nu_k}{2 \hat{q} \cdot p_k}
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Leading soft graviton theorem (Weinberg 1965):

\[\Rightarrow \langle \text{out} | [Q(\hat{q}), S] | \text{in} \rangle = 0\]

with

\[Q(\hat{q}) \equiv \lim_{\omega \to 0} \frac{1}{2} \left[ \omega a_+ (\omega \hat{q}) + \omega a_+^{\dagger} (\omega \hat{q}) \right] - \left[ \sum_{k \in \text{in}} S_k (\hat{q}) + \sum_{k \in \text{out}} S_k (\hat{q}) \right]\]
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Infinitely many (supertranslation) symmetries (graviton momentum directions)

1401.76026 He, Lysov, Mitra, Strominger
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Interpret soft factor \( S_k (z, \bar{z}) \) as generating infinitesimal symmetry transformation on hard states

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S_k (z, \bar{z}) | p_k \rangle \sim \delta (z, \bar{z}) | p_k \rangle
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- Subleading soft theorem ↔ superrotation (Virasoro) symmetry

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Extended BMS symmetry with Poincare global subgroup

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Tree-level, minimal coupling

References:
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The graviton $W^{1+\infty}$ algebra and symmetry action on hard massless particles were derived in celestial holography
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The graviton $W_{1+\infty}$ algebra and symmetry action on hard massless particles were derived in celestial holography

- Used basis of conformal primary operators that transform in highest-weight representations of $SL(2, \mathbb{C})$ global conformal (Lorentz) transformations
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This talk: derivation of $W_{1+\infty}$ symmetry action from momentum-space soft theorems for massless and massive hard particles, guided by conformal covariance
Soft Graviton Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton: Soft factors that generate the symmetry action on hard particles should also transform covariantly with this weight. To isolate the term in the soft expansion, take the limit...
Soft Graviton Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton:

\[ A(\omega \hat{q}; p_1, \cdots, p_n) = \frac{\kappa}{2} \sum_{\ell=-1}^{\infty} \omega^{\ell} A^{(\ell)}(\hat{q}; p_1, \cdots, p_n) \]
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To isolate the \( \ell \) term in the soft expansion, take the limit

\[ A^{(\ell)}(\hat{q}; \{p_i\}) \propto \lim_{\omega \to 0} \partial_{\omega}^{\ell+1} (\omega A(\omega \hat{q}; \{p_i\})) = (\ell + 1)! \lim_{\epsilon \to 0} \int_0^\infty \frac{d\omega}{\omega} \omega^{-\ell+\epsilon} A(\omega \hat{q}; \{p_i\}) \]
**Soft Graviton Expansion**

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Converges for $A(\omega) \sim \omega^{-m}$ with $m + \ell > \epsilon$
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Note definition of conformal primary basis

\[|\Delta, s, z, \bar{z}\rangle \equiv \int_0^\infty \frac{d\omega}{\omega} \omega^{\Delta} |\omega, s, z, \bar{z}\rangle \quad \quad (h, \bar{h}) = \left(\frac{\Delta + s}{2}, \frac{\Delta - s}{2}\right)\]

1705.01027 Pasterski, Shao
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\[ \implies \text{sub-}(\ell + 1) \text{ leading soft graviton is in a state with definite conformal weight} \quad (h, \bar{h}) = \left( \frac{-\ell + 2}{2}, \frac{-\ell - 2}{2} \right) \]
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Review of Lorentz $SL(2, \mathbb{C})$ Transformations
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Parametrize scattering data to facilitate analysis of global conformal transformations
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Parametrize scattering data to facilitate analysis of global conformal transformations

Massless Particles

\[ p^\mu_k = \epsilon_k \omega_k \hat{q}^\mu(z_k, \bar{z}_k) \]
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Lorentz transformations act as Mobius transformations on the celestial sphere

\[ z \rightarrow z' = \frac{az + b}{cz + d}, \quad \bar{z} \rightarrow \bar{z}' = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}, \quad ad - bc = \bar{a}\bar{d} - \bar{b}\bar{c} = 1 \]
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Momenta transform covariantly

\[ \hat{q}^\mu(z, \bar{z}) \to \hat{q}^\mu(z', \bar{z}') = \frac{1}{(cz + d)(\bar{c}\bar{z} + \bar{d})} \Lambda^\mu_\nu \hat{q}^\nu(z, \bar{z}), \quad \omega \to \omega' = (cz + d)(\bar{c}\bar{z} + \bar{d})\omega \]
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Polarization vectors require a reference point $(z_{0}, \bar{z}_{0})$ (equivalent to auxiliary spinor) to transform covariantly
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Review of Lorentz $\mathbb{SL}(2, \mathbb{C})$ Transformations, continued

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Lorentz transformations act as isometries of Massive Particles

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\[ p^\mu_k = \epsilon_k m_k \hat{p}^\mu_k, \quad \hat{p}^\mu_k = \frac{1}{2y_k} \left( y_k^2 n^\mu + \hat{q}^\mu(w_k, \bar{w}_k) \right), \quad \hat{p}_k^2 = -1, \quad n^\mu \equiv \partial_z \partial_{\bar{z}} \hat{q}^\mu(z, \bar{z}) \]
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Lorentz transformations act as isometries of $\text{AdS}_3$

\[ ds^2 = \frac{dy^2 + dwd\bar{w}}{y^2} \]

\[ w_k \rightarrow w'_k = \frac{(aw_k + b)(\bar{c}w_k + \bar{d}) + acy_k^2}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2} \]

\[ \bar{w}_k \rightarrow \bar{w}'_k = \frac{(\bar{a}w_k + \bar{b})(cw_k + d) + \bar{a}c\bar{y}_k^2}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2} \]

\[ y_k \rightarrow y'_k = \frac{y_k}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2} \]
Review of Lorentz $\mathbb{SL}(2, \mathbb{C})$ Transformations, continued

Parametrize scattering data to facilitate analysis of global conformal transformations

Massive Particles

$$p_k^\mu = \epsilon_k m_k \hat{p}_k^\mu,$$
$$\hat{p}_k^\mu = \frac{1}{2y_k} \left(y_k^2 n^\mu + \hat{q}^\mu(w_k, \bar{w}_k)\right),$$
$$\hat{p}_k^2 = -1,$$
$$n^\mu \equiv \partial_z \bar{\partial}_{\bar{z}} \hat{q}^\mu(z, \bar{z})$$

Lorentz transformations act as isometries of $\text{AdS}_3$

$$ds^2 = \frac{dy^2 + dw d\bar{w}}{y^2}$$

$$w_k \rightarrow w'_k = \frac{(aw_k + b)(\bar{c}w_k + \bar{d}) + a\bar{c}y_k^2}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2}$$
$$\bar{w}_k \rightarrow \bar{w}'_k = \frac{(\bar{a}w_k + \bar{b})(cw_k + d) + \bar{a}c\bar{y}_k^2}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2}$$
$$y_k \rightarrow y'_k = \frac{y_k}{(cw_k + d)(\bar{c}w_k + \bar{d}) + c\bar{c}y_k^2}$$

$$p_k^\mu(y_k, w_k, \bar{w}_k) \rightarrow p_k'^\mu(y'_k, w'_k, \bar{w}'_k) = \Lambda^\nu_\mu p_k^\nu(y_k, w_k, \bar{w}_k)$$
Subleading Soft Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton: The leading and subleading terms take the familiar form. Beyond subleading order, soft expansion was found to take the form. "Universal" term from gauge invariance, "non-universal" subleading in (1801.05528 Hamada, Shiu; 1802.03148 Li, Lin, Zhang) Tree-level, minimal coupling.
Subleading Soft Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton:

\[ A(\omega \hat{q}; p_1, \ldots, p_n) = \frac{\kappa}{2} \sum_{\ell=-1}^{\infty} \omega^\ell A^{(\ell)}(\hat{q}; p_1, \ldots, p_n) \]
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The leading and subleading terms take the familiar form

\[ A^{(-1)}(\hat{q}, p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(-1)} A(p_1, \cdots, p_n), \quad S_k^{(-1)} = \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k} \]

Weinberg 1965
Subleading Soft Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton:

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Weinberg 1965

$$A^{(0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(0)} A(p_1, \cdots, p_n), \quad S_k^{(0)} = \frac{\varepsilon_{\mu\nu\rho\sigma} p_k^\mu (i \hat{q} \sigma \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k}$$

1404.4091 Cachazo, Strominger
Subleading Soft Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton:

\[ \mathcal{A}(\omega \hat{q}; p_1, \cdots, p_n) = \frac{\kappa}{2} \sum_{\ell = -1}^{\infty} \omega^\ell \mathcal{A}^{(\ell)}(\hat{q}; p_1, \cdots, p_n) \]

The leading and subleading terms take the familiar form

\[
\mathcal{A}^{(-1)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(-1)} \mathcal{A}(p_1, \cdots, p_n), \quad S_k^{(-1)} = \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k}
\]

\[
\mathcal{A}^{(0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(0)} \mathcal{A}(p_1, \cdots, p_n), \quad S_k^{(0)} = \frac{\varepsilon_{\mu\nu} p_k^\mu (i\hat{q}_\sigma \mathcal{L}^{\nu\sigma}_k)}{\hat{q} \cdot p_k}
\]

Weinberg 1965 \quad \mathcal{L}_{k\mu\nu} = -i \left( p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right)

1404.4091 Cachazo, Strominger
Consider the low-energy expansion of an amplitude with an outgoing graviton:

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Weinberg 1965

\[ \mathcal{L}_{k\mu\nu} = -i \left( p_k^\mu \frac{\partial}{\partial p_k^\nu} - p_k^\nu \frac{\partial}{\partial p_k^\mu} \right) \]

1404.4091 Cachazo, Strominger

Beyond subleading order, \( \ell \geq 1 \) soft expansion was found to take the form

\[ \mathcal{A}^{(\ell > 0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(\ell)} A(p_1, \cdots, p_n) + \varepsilon_{\mu\nu} B_\ell^{\mu\nu}(\hat{q}; p_1, \cdots, p_n), \quad S_k^{(\ell)} = \frac{1}{(\ell + 1)!} \frac{i q_\mu \mathcal{L}_k^{\mu\rho}(i q_\sigma \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k} (\hat{q} \cdot \frac{\partial}{\partial p_k})^{\ell - 1} \]
Subleading Soft Expansion

Consider the low-energy expansion of an amplitude with an outgoing graviton:

\[ \mathcal{A}(\omega \hat{q}; p_1, \cdots, p_n) = \frac{k}{2} \sum_{\ell=-1}^{\infty} \omega^\ell \mathcal{A}^{(\ell)}(\hat{q}; p_1, \cdots, p_n) \]

The leading and subleading terms take the familiar form

\[ A^{(-1)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(-1)} A(p_1, \cdots, p_n), \quad S_k^{(-1)} = \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k} \]

\[ A^{(0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(0)} A(p_1, \cdots, p_n), \quad S_k^{(0)} = \frac{\varepsilon_{\mu\nu} p_k^\mu (i \dot{q}_\sigma \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k} \]

Beyond subleading order, \( \ell \geq 1 \) soft expansion was found to take the form

\[ A^{(\ell>0)}(\hat{q}; p_1, \cdots, p_n) = \sum_{k=1}^{n} S_k^{(\ell)} A(p_1, \cdots, p_n) + \varepsilon_{\mu\nu} B_{\ell}^{\mu\nu}(\hat{q}; p_1, \cdots, p_n), \quad S_k^{(\ell)} = \frac{1}{(\ell + 1)!} \frac{\varepsilon_{\mu\nu} (i \dot{q}_\rho \mathcal{L}_k^{\mu\rho})(i \dot{q}_\sigma \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k} \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1} \]

"Universal" term \( S_k^{(\ell)} \) from gauge invariance, "non-universal" \( B_{\ell}^{\mu\nu} \) subleading in \( \hat{q} \cdot p_k \)

Tree-level, minimal coupling

1801.05528 Hamada, Shiu; 1802.03148 Li, Lin, Zhang

Weinberg 1965

\[ \mathcal{L}_{k\mu\nu} = -i \left( p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right) \]
Covariant Massless Soft Factors

The soft factors do not transform as primaries.

Natural proposal: complete them to angular momentum generators.

Resulting modified soft factors transform covariantly with identical weight to soft gravitons:

\[ 1812.06895 \text{ Guevara, Ochirov, Vines} \]

Note: division by implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant.

However, the gauge and conformal non-invariance will drop out at the level of symmetry action.
Covariant Massless Soft Factors

The soft factors $S^{\mu(\ell)}_k$ do not transform as primaries.

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$$S^{\mu(\ell)}_k = \frac{1}{(\ell + 1)!} \frac{\varepsilon_{\mu\nu} (i\hat{q}_\rho \mathcal{L}_k^{\mu\rho}) (i\hat{q}_\sigma \mathcal{L}_k^{\nu\sigma})}{\hat{q} \cdot p_k} \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1}$$

Note: division by $\hat{q} \cdot p_k$ implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant. However, the gauge and conformal non-invariance will drop out at the level of symmetry action.
Covariant Massless Soft Factors

The soft factors $S_k^{\mu(\ell)}$ do not transform as primaries

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\[
S_k^{\mu(\ell)} = \frac{1}{(\ell + 1)!} \varepsilon_{\mu\nu} (i \hat{q}_\rho \mathcal{L}_k^{\mu\rho})(i \hat{q}_\sigma \mathcal{L}_k^{\nu\sigma}) \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1}
\]

Note: division by implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant. However, the gauge and conformal non-invariance will drop out at the level of symmetry action.
Covariant Massless Soft Factors

The soft factors $S_k^{n(\ell)}$ do not transform as primaries

Natural proposal: complete them to angular momentum generators

$$S_k^{n(\ell)} = \frac{1}{(\ell + 1)!} \frac{\varepsilon_{\mu\nu}(i \hat q_\rho \mathcal{L}_k^{\mu\rho})(i \hat q_\sigma \mathcal{L}_k^{\nu\sigma})}{\hat q \cdot p_k} \left( \hat q \cdot \frac{\partial}{\partial p_k} \right)^{\ell - 1}$$

$\hat q^\mu \frac{\partial}{\partial p_k^\mu} \to \frac{1}{\varepsilon_+ \cdot p_k} \left( (\varepsilon_+ \cdot p_k) \hat q^\mu \frac{\partial}{\partial p_k^\mu} - (\hat q \cdot p_k) \varepsilon_+^\mu \frac{\partial}{\partial p_k^\mu} \right)$
Covariant Massless Soft Factors

The soft factors $S_k^{\mu(\ell)}$ do not transform as primaries

$$S_k^{\mu(\ell)} = \frac{1}{(\ell + 1)!} \frac{\varepsilon_{\mu\nu} (\hat{q} \hat{q}_\rho \mathcal{L}_{k}^{\mu\rho}) (i \hat{q}_\sigma \mathcal{L}_{k}^{\nu\sigma})}{\hat{q} \cdot p_k} \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1}$$

Natural proposal: complete them to angular momentum generators

$$\hat{q}^\mu \frac{\partial}{\partial p_k^\mu} \rightarrow \frac{1}{\varepsilon_+ \cdot p_k} \left( (\varepsilon_+ \cdot p_k) \hat{q}^\mu \frac{\partial}{\partial p_k^\mu} - (\hat{q} \cdot p_k) \varepsilon_+^\mu \frac{\partial}{\partial p_k^\mu} \right) = \frac{1}{\varepsilon_+ \cdot p_k} F_{+\mu}^{\nu} \mathcal{L}_{k\mu\nu}$$

$$F_{\pm}^{\mu\nu} = i \varepsilon_{\pm}^{[\mu} \hat{q}^{\nu]}$$

$$\mathcal{L}_{k\mu\nu} = -i \left( p_{k\mu} \frac{\partial}{\partial p_k^\nu} - p_{k\nu} \frac{\partial}{\partial p_k^\mu} \right)$$

Note: division by $\varepsilon_+$ implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant. However, the gauge and conformal non-invariance will drop out at the level of symmetry action.
Covariant Massless Soft Factors

The soft factors $S'^{n(\ell)}_k$ do not transform as primaries

$$S'^{n(\ell)}_k = \frac{1}{(\ell + 1)!} \frac{\varepsilon_{\mu\nu}(i\hat{q}_\rho \mathcal{L}^{\mu\rho}_k)(i\hat{q}_\sigma \mathcal{L}^{\nu\sigma}_k)}{\hat{q} \cdot p_k} \left(\hat{q} \cdot \frac{\partial}{\partial p_k}\right)^{\ell - 1}$$

Natural proposal: complete them to angular momentum generators

$$\hat{q}^\mu \frac{\partial}{\partial p_k^\mu} \rightarrow \frac{1}{\varepsilon \cdot p_k} \left( (\varepsilon \cdot p_k) \hat{q}^\mu \frac{\partial}{\partial p_k^\mu} - (\hat{q} \cdot p_k) \varepsilon^\mu \frac{\partial}{\partial p_k^\mu} \right) = \frac{1}{\varepsilon \cdot p_k} F^{\mu\nu}_+ \mathcal{L}_{k\mu\nu}$$

Resulting modified soft factors transform covariantly with identical weight to soft gravitons:

$$S'^{(\ell)}_k (z, \bar{z}) = \frac{\varepsilon^\mu \mu\nu p_k^\mu p_k^\nu}{\hat{q} \cdot p_k} \frac{1}{(\ell + 1)!} \left(\frac{F_+ \cdot \mathcal{L}_k}{\varepsilon \cdot p_k}\right)^{\ell + 1}$$

Note: division by $\epsilon$ implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant. However, the gauge and conformal non-invariance will drop out at the level of symmetry action.

1812.06895 Guevara, Ochirov, Vines
1405.1410 He, Huang, Wen
1504.01364 Lipstein
Covariant Massless Soft Factors

The soft factors $S_k^{\mu(\ell)}$ do not transform as primaries

$$S_k^{\mu(\ell)} = \frac{1}{(\ell + 1)!} \varepsilon_{\mu\nu} \left( i\hat{q}_\mu \mathcal{L}_{\nu k}^\mu \right) \left( \hat{q} \cdot p_k \right) \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1}$$

Natural proposal: complete them to angular momentum generators

$$\hat{q}_\mu \frac{\partial}{\partial p_k^\mu} \rightarrow \frac{1}{\varepsilon_+ \cdot p_k} \left( (\varepsilon_+ \cdot p_k) \hat{q}_\mu \frac{\partial}{\partial p_k^\mu} - (\hat{q} \cdot p_k) \varepsilon_+ \frac{\partial}{\partial p_k^\mu} \right) = \frac{1}{\varepsilon_+ \cdot p_k} F_{+\mu\nu} \mathcal{L}_{k\mu\nu}$$

Resulting modified soft factors transform covariantly with identical weight to soft gravitons:

$$S_k^{\mu(\ell)}(z, \bar{z}) = \frac{\varepsilon_+ \mu \nu \rho \sigma \eta \kappa \lambda}{\hat{q} \cdot p_k} \left( \frac{1}{(\ell + 1)!} \left( \frac{F_+ \cdot \mathcal{L}_k}{\varepsilon_+ \cdot p_k} \right)^{\ell+1} \right)$$

$$S_k^{\mu(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (c \bar{z} + d)^{-\ell-2} S_k^{\mu(\ell)}(z, \bar{z})$$

1812.06895 Guevara, Ochirov, Vines
1405.1410 He, Huang, Wen
1504.01364 Lipstein
**Covariant Massless Soft Factors**

The soft factors $s_k^{m(\ell)}$ do not transform as primaries

$$ s_k^{m(\ell)} = \frac{1}{(\ell + 1)!} \varepsilon_{\mu\nu}(i\hat{q}_\rho \mathcal{L}^\rho_{\mu\nu})(i\hat{q}_\sigma \mathcal{L}^\sigma_{\nu\kappa}) \left( \hat{q} \cdot \frac{\partial}{\partial p_k} \right)^{\ell-1} $$

Natural proposal: complete them to angular momentum generators

$$ \hat{q}^\mu \frac{\partial}{\partial p_k^\mu} \rightarrow \frac{1}{\varepsilon_+ \cdot p_k} \left( (\varepsilon_+ \cdot p_k) \hat{q}^\mu \frac{\partial}{\partial p_k^\mu} - (\hat{q} \cdot p_k) \varepsilon_+ \frac{\partial}{\partial p_k^\mu} \right) = \frac{1}{\varepsilon_+ \cdot p_k} F^{\mu\nu}_{\ell+1} \mathcal{L}_{\mu\nu} $$

Resulting modified soft factors transform covariantly with identical weight to soft gravitons:

$$ s_k^{(\ell)}(z, \bar{z}) = \varepsilon_+ \mu \nu p_k^\mu p_k^\nu \frac{1}{\hat{q} \cdot p_k} \left( \frac{F_+ \cdot \mathcal{L}_k}{\varepsilon_+ \cdot p_k} \right)^{\ell+1} $$

$$ s_k^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (\bar{c}z + \bar{d})^{-\ell-2} s_k^{(\ell)}(z, \bar{z}) $$

Note: division by $\varepsilon_+ \cdot p_k$ implies this is not gauge invariant, and the dependence on the reference point means the "universal" partition is not strictly conformally invariant.

However, the gauge and conformal non-invariance will drop out at the level of symmetry action.
$W_{1+\infty}$ Action on Massless Particles
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To construct symmetry action, consider a primary descendant of the modified massless soft factor:

$$\partial_{\bar{z}}^{\ell+3} S^\ell_k(z, \bar{z})$$
**W\textsubscript{1+∞} Action on Massless Particles**

To construct symmetry action, consider a primary descendant of the modified massless soft factor:

\[
\partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (\bar{c} \bar{z} + \bar{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)}(z, \bar{z})
\]

It transforms with weight \(\left(-\frac{\ell+2}{2}, \frac{\ell+4}{2}\right)\)
$W_{1+\infty}$ Action on Massless Particles

To construct symmetry action, consider a primary descendant of the modified massless soft factor:

$$\partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)} (z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (\bar{c} \bar{z} + \bar{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)} (z, \bar{z})$$

It transforms with weight $\left( -\frac{\ell+2}{2}, \frac{\ell+4}{2} \right)$ and does not depend on $z_{0}$: fully covariant and invariantly partitioned.
To construct symmetry action, consider a primary descendant of the modified massless soft factor:

\[
\partial_\zeta^{\ell+3} S_k^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2}(\bar{c}\zeta + \bar{d})^{\ell+4} \partial_\zeta^{\ell+3} S_k^{(\ell)}(z, \bar{z})
\]

It transforms with weight \((\frac{-\ell+2}{2}, \frac{\ell+4}{2})\) and does not depend on \(z_0\): fully covariant and invariantly partitioned

Define the right-weight of the primary descendant as \(p = \bar{h} = \frac{\ell+4}{2}\)
To construct symmetry action, consider a primary descendant of the modified massless soft factor:

\[ \partial_{\bar{z}}^{\ell+3} S_k^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (\bar{c} \bar{z} + \bar{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_k^{(\ell)}(z, \bar{z}) \]

It transforms with weight \( \left( -\frac{\ell+2}{2}, \frac{\ell+4}{2} \right) \) and does not depend on \( z_0 \): fully covariant and invariantly partitioned

Define the right-weight of the primary descendant as \( p = \bar{h} = \frac{\ell+4}{2} \)

Primaries of weight \( p \) have a finite set of modes closed under \( SL(2, \mathbb{R}) \): \( 1 - p \leq m \leq p - 1 \)
To construct symmetry action, consider a primary descendant of the modified massless soft factor:

\[ \partial_{\bar{z}}^{\ell+3} S_{\ell}^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (c\bar{z} + d)^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_{\ell}^{(\ell)}(z, \bar{z}) \]

It transforms with weight \((\frac{-\ell+2}{2}, \frac{\ell+4}{2})\) and does not depend on \(z_0\): fully covariant and invariantly partitioned.

Define the right-weight of the primary descendant as \(p = \bar{h} = \frac{\ell+4}{2}\).

Primaries of weight \(p\) have a finite set of modes closed under \(\text{SL}(2, \mathbb{R})\): \(1 - p \leq m \leq p - 1\).

Consider the action of

\[ \delta_m^{(p)} |p_k\rangle = -\frac{1}{2} \int \frac{d^2 z}{2\pi} \bar{z}^{p+m-1} \partial_{\bar{z}}^{2p-1} S_{\ell}^{(2p-4)}(z, \bar{z}) |p_k\rangle \]
**$W_{1+\infty}$ Action on Massless Particles**

To construct symmetry action, consider a primary descendant of the modified massless soft factor:

$$\partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)}(z, \bar{z}) \rightarrow (c z + d)^{-\ell+2} (\overline{c \bar{z}} + \overline{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_{k}^{(\ell)}(z, \bar{z})$$

It transforms with weight $(\frac{-\ell+2}{2}, \frac{\ell+4}{2})$ and does not depend on $z_0$: fully covariant and invariantly partitioned.

Define the right-weight of the primary descendant as $p = \bar{h} = \frac{\ell+4}{2}$.

Primaries of weight $p$ have a finite set of modes closed under $\text{SL}(2, \mathbb{R})$: $1 - p \leq m \leq p - 1$.

Consider the action of

$$\delta^{p}_{m}|p_{k}\rangle \equiv -\frac{1}{2} \int \frac{d^2 z}{2\pi} z^{p+m-1} \partial_{z}^{2p-1} S_{k}^{(2p-4)}(z, \bar{z})|p_{k}\rangle$$

Examples:

$p = \frac{3}{2} \rightarrow \ell = -1$ leading soft theorem, closed set of 2 modes: (chiral) translations.
W_{1+\infty} Action on Massless Particles

To construct symmetry action, consider a primary descendant of the modified massless soft factor:

$$\partial_{\bar{z}}^{\ell+3} S_{k}^{|(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2}(\bar{c}\bar{z} + \bar{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_{k}^{|(\ell)}(z, \bar{z})$$

It transforms with weight \((\frac{-\ell+2}{2}, \frac{\ell+4}{2})\) and does not depend on \(z_0\): fully covariant and invariantly partitioned

Define the right-weight of the primary descendant as \(p = \tilde{h} = \frac{\ell+4}{2}\)

Primaries of weight \(p\) have a finite set of modes closed under \(\text{SL}(2, \mathbb{R})\): \(1 - p \leq m \leq p - 1\)

Consider the action of

$$\delta_{m}^{p}|p_{k}\rangle \equiv -\frac{1}{2} \int \frac{d^{2}z}{2\pi} z^{p+m-1} \partial_{z}^{2p-1} S_{k}^{(2p-4)}(z, \bar{z})|p_{k}\rangle$$

Examples:

\(p = \frac{3}{2} \rightarrow \ell = -1\) leading soft theorem, closed set of 2 modes: (chiral) translations

\(p = 2 \rightarrow \ell = 0\) subleading soft theorem, closed set of 3 modes: \(\text{SL}(2, \mathbb{R})\) chiral half of Lorentz
**W_{1+\infty} Action on Massless Particles**

To construct symmetry action, consider a **primary descendant** of the modified massless soft factor:

\[
\partial_z^{\ell+3} S_k^{(\ell)}(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2} (\bar{c} \bar{z} + \bar{d})^{\ell+4} \partial_{\bar{z}}^{\ell+3} S_k^{(\ell)}(z, \bar{z})
\]

It transforms with weight \((\frac{-\ell+2}{2}, \frac{\ell+4}{2})\) and does not depend on \(z_0\): **fully covariant and invariantly partitioned**

Define the right-weight of the primary descendant as \(p = \bar{h} = \frac{\ell+4}{2}\)

Primaries of weight \(p\) have a **finite set of modes closed under** \(\text{SL}(2, \mathbb{R})\): \(1 - p \leq m \leq p - 1\)

Consider the action of

\[
\delta_m^p |p_k\rangle \equiv -\frac{1}{2} \int \frac{d^2 z}{2\pi} z^{p+m-1} \partial_{\bar{z}}^{2p-1} S_k^{(2p-4)}(z, \bar{z}) |p_k\rangle
\]

**Examples:**

\(p = \frac{3}{2} \rightarrow \ell = -1\) leading soft theorem, closed set of 2 modes: (chiral) translations

\(p = 2 \rightarrow \ell = 0\) subleading soft theorem, closed set of 3 modes: \(\text{SL}(2, \mathbb{R})\) chiral half of Lorentz

**Can prove that this action satisfies the** \(W_{1+\infty}\) **algebra**

\[
[\delta_m^p, \delta_n^q] = [m(q - 1) - n(p - 1)] \delta_{m+n}^{p+q-2}
\]
The prescription for massless particles (taking modes of a primary descendant of the soft factor) can be generalized, provided that the relevant primary descendant is partitioned invariantly. However, for $l = -1, 0$, the primary descendant depends on $\mu$, so to find the correct symmetry action, we need to revisit the soft factor itself. Soft factor $\mathcal{S}$ can be split into two separately covariant pieces, one of which is independent of $\mu$. For $l = -1, 0$, the primary descendant is independent of $\mu$ and takes the simple form. Covariant Massive Soft Factors. Caution in massless limit.
Covariant Massive Soft Factors

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For $\ell = -1, 0, 1$, the primary descendant is independent of $z_0$ and takes the simple form

$$\partial_{\bar{z}}^{\ell+3} S_k^{\mu(\ell)}(z, \bar{z}) = \mathcal{N}_\ell \frac{p_k^4}{(\hat{q} \cdot p_k)^{\ell+4}} (F_- \cdot \mathcal{L}_k)^{\ell+1}$$

$$\mathcal{N}_\ell = (-\sqrt{2})^{\ell-1}(\ell + 3)(\ell + 2)$$
Covariant Massive Soft Factors

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However, for $\ell > 1$, its primary descendant depends on $z_0$, so to find the correct symmetry action, we need to revisit the soft factor itself.
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For $\ell = -1, 0, 1$, the primary descendant is independent of $z_0$ and takes the simple form:

$$\partial_{z \bar{z}}^{\ell+3} S_k^{(\ell)} (z, \bar{z}) = N_{\ell} \frac{p_k^4}{(\hat{q} \cdot p_k)^{\ell+4}} (F_- \cdot L_k)^{\ell+1}$$

$$N_{\ell} = (-\sqrt{2})^{\ell-1}(\ell + 3)(\ell + 2)$$

Recall:

$$S_k^{(\ell)} (z, \bar{z}) = \frac{\epsilon_{+ \mu \nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k} \frac{1}{(\ell + 1)!} \left( \frac{F_+ \cdot L_k}{\epsilon_+ \cdot p_k} \right)^{\ell+1}$$
Covariant Massive Soft Factors

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However, for \( \ell > 1 \), its primary descendant depends on \( z_0 \), so to find the correct symmetry action, we need to revisit the soft factor itself.

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\[
S_k^{(\ell)}(z, \bar{z}) = \frac{\varepsilon_{+\mu \nu} p_k^\mu p_k^\nu}{\hat{q} \cdot p_k} \frac{1}{(\ell + 1)!} \left( \frac{F_+ \cdot \mathcal{L}_k}{\varepsilon_+ \cdot p_k} \right)^{\ell+1}
\]

Soft factor \( S_k^{(\ell)}(z, \bar{z}) \) can be split into two separately covariant pieces, one of which is independent of \( z_0 \)

\[
S_k^{(\ell)}(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} f^{2\ell+2} \frac{1}{p_k^{2\ell-2}} \partial_{\bar{z}}^{\ell-1} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f^{\ell+4}} \right] - \frac{\mathcal{N}_\ell}{(\ell + 3)!(\ell - 2)!} \frac{1}{p_k^{2\ell-2}} \sum_{j=0}^{\ell-2} \binom{\ell - 2}{j} (-1)^j f^{\ell - 2 - j} f_0^{\ell + 4 + j} \partial_{\bar{z}}^{\ell - 1} \left[ \frac{f^j (F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f_0^{\ell + 4 + j}} \right]
\]

\[
f \equiv \hat{q} \cdot p_k,
\]

\[
f_0 \equiv \hat{q}_0 \cdot p_k
\]

\[
\hat{q}_0 \equiv \hat{q}(z_0, \bar{z})
\]
The prescription for massless particles (taking modes of a primary descendant of the soft factor) can be generalized, provided that the relevant primary descendant is partitioned invariantly.

For $\ell = -1, 0, 1$, the primary descendant is independent of $z_0$ and takes the simple form

$$\partial_{\bar{z}}^{\ell+3} S^{(\ell)}_k(z, \bar{z}) = \mathcal{N}_\ell \frac{p_k^4}{(\hat{q} \cdot p_k)^{\ell+4}} (F_+ \cdot \mathcal{L}_k)^{\ell+1} \quad \mathcal{N}_\ell = (-\sqrt{2})^{\ell-1}(\ell + 3)(\ell + 2)$$

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$$S^{(\ell)}_k(z, \bar{z}) = \frac{\varepsilon_{+\mu\nu} p_k^{\mu} p_k^{\nu}}{\hat{q} \cdot p_k} \frac{1}{(\ell + 1)!} \left( \frac{F_+ \cdot \mathcal{L}_k}{\varepsilon_+ \cdot p_k} \right)^{\ell+1}$$

Soft factor $S^{(\ell)}_k(z, \bar{z})$ can be split into two separately covariant pieces, one of which is independent of $z_0$.

Caution in massless limit:

$$S^{(\ell)}_k(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} \frac{1}{p_k^{2\ell-2}} \partial^{\ell-1}_{\bar{z}} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f_{\ell+4}} \right]$$

$$- \frac{\mathcal{N}_\ell}{(\ell + 3)!(\ell - 2)!} \frac{1}{p_k^{2\ell-2}} \sum_{j=0}^{\ell-2} \binom{\ell - 2}{j} (-1)^j f_{\ell-2-j} f_0^{\ell+4+j} \partial^{\ell-1}_{\bar{z}} \frac{f_j (F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f_0^{\ell+4+j}}$$

Caution in massless limit:

$$f \equiv \hat{q} \cdot p_k,$$

$$f_0 \equiv \hat{q}_0 \cdot p_k,$$

$$\hat{q}_0 \equiv \hat{q}(z_0, \bar{z})$$
$w_{1+\infty}$ Action on Massive Particles
$W_{1+\infty}$ Action on Massive Particles

Natural proposal for the "universal" soft factor for massive external particles at order $\ell > 1$

$$S_k^{(\ell)}(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} \frac{q \cdot p_k}{p_k^{2\ell-2}} \partial_{\bar{z}}^{\ell-1} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{(q \cdot p_k)^{\ell+4}} \right]$$

$$\mathcal{N}_\ell = (-\sqrt{2})^{\ell-1}(\ell + 3)(\ell + 2)$$
Natural proposal for the "universal" soft factor for massive external particles at order $\ell > 1$.

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S^{(\ell)}_k(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} \frac{(\hat{q} \cdot p_k)^{2\ell+2}}{p_k^{2\ell-2}} \partial_{\bar{z}}^{\ell-1} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{(\hat{q} \cdot p_k)^{\ell+4}} \right]
\]

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\mathcal{N}_\ell = (-\sqrt{2})^{\ell-1} (\ell + 3)(\ell + 2)
\]

\[
S^{(\ell)}_k(z, \bar{z}) \rightarrow (cz + d)^{-\ell+2}(\bar{c}\bar{z} + \bar{d})^{-\ell-2} S^{(\ell)}_k(z, \bar{z})
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\]

As in the massless case, define $p = \bar{h} = \frac{\ell+4}{2}$, $1 - p \leq m \leq p - 1$, and consider the action

\[
\delta_{m|p_k}^p \equiv -\frac{1}{2} \int \frac{d^2z}{2\pi} z^{p+m-1} \partial_{\bar{z}}^{2p-1} \left\{ \begin{array}{ll}
S_k^{(2p-4)}(z, \bar{z})|p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\
S_k^{(2p-4)}(z, \bar{z})|p_k\rangle, & p > \frac{5}{2}
\end{array} \right. 
\]
**$W_{1+\infty}$ Action on Massive Particles**

Natural proposal for the "universal" soft factor for massive external particles at order $\ell > 1$

$$S^{(\ell)}_k(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} \frac{(\hat{q} \cdot p_k)^{2\ell+2}}{p_{k\ell-2}^2} \partial_{\bar{z}}^{\ell-1} \left[ \frac{(F_+ \cdot L_k)^{\ell+1}}{(\hat{q} \cdot p_k)^{\ell+4}} \right]$$

$$\mathcal{N}_\ell = (-\sqrt{2})^{\ell-1} (\ell + 3)(\ell + 2)$$

Primary descendant generalizes the form already found for $\ell = -1, 0, 1$

$$\partial_{\bar{z}}^{\ell+3} S^{(\ell)}_k(z, \bar{z}) = \mathcal{N}_\ell \frac{p_k^4}{(\hat{q} \cdot p_k)^{\ell+4}} (F_- \cdot L_k)^{\ell+1}$$

As in the massless case, define $p = \frac{\bar{h} = \frac{\ell+4}{2}}{2} \quad 1 - p \leq m \leq p - 1$, and consider the action

$$\delta^p_m|p_k\rangle = -\frac{1}{2} \int \frac{d^2z}{2\pi} \bar{z}^{p+m-1} \partial_{\bar{z}}^{2p-1} \left\{ \begin{array}{ll}
S^{(2p-4)}_k(z, \bar{z})|p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\
S^{(2p-4)}_k(z, \bar{z})|p_k\rangle, & p > \frac{5}{2}
\end{array} \right.$$
Natural proposal for the "universal" soft factor for massive external particles at order $\ell > 1$

$$S_k^{(\ell)}(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} \frac{(\hat{q} \cdot p_k)^{2\ell+2}}{p_k^{2\ell-2}} \partial_{\bar{z}}^{\ell-1} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{(\hat{q} \cdot p_k)^{\ell+4}} \right]$$

$\mathcal{N}_\ell = (-\sqrt{2})^{\ell-1}(\ell + 3)(\ell + 2)$

Primary descendant generalizes the form already found for $\ell = -1, 0, 1$

$$\partial_{\bar{z}}^{\ell+3} S_k^{(\ell)}(z, \bar{z}) = \mathcal{N}_\ell \frac{p_k^4}{(\hat{q} \cdot p_k)^{\ell+4}} (F_- \cdot \mathcal{L}_k)^{\ell+1}$$

As in the massless case, define $p = \frac{\ell+4}{2}, 1 - p \leq m \leq p - 1$, and consider the action

$$\delta^p_m|p_k\rangle = -\frac{1}{2} \int \frac{d^2 z}{2\pi} \bar{z}^{p+m-1} \partial_{\bar{z}}^{2p-1} \begin{cases} 
S_k^{(2p-4)}(z, \bar{z})|p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\
S_k^{(2p-4)}(z, \bar{z})|p_k\rangle, & p > \frac{5}{2} 
\end{cases}$$

$$= -\frac{\mathcal{N}_{2p-4}}{2} \int \frac{d^2 z}{2\pi} \bar{z}^{p+m-1} \frac{p_k^4}{(\hat{q} \cdot p_k)^{2p}} (F_- \cdot \mathcal{L}_k)^{2p-3}|p_k\rangle.$$ 

Can again prove that this action respects the $W_{1+\infty}$ algebra:

$$[\delta^p_m, \delta^q_n] = [m(q - 1) - n(p - 1)] \delta_{m+n}^{p+q-2}$$
Summary

• Clarified origin of $W_{1+\infty}$ symmetry action from tower of momentum-space soft factors
• Proposed new "universal" massive soft factors motivated by $SL(2, \mathbb{C})$ covariance
• Discovered nontrivial symmetry action on massive particles
• Proved that symmetry action satisfies $W_{1+\infty}$ acting on massive particles

Future Directions

• How is the symmetry algebra deformed by loops and higher-dimension operators?
• What theories solve $W_{1+\infty}$ constraints?
• Organization of "non-universal" terms in soft expansion from symmetry principles?
• Simplification of symmetry action in integer massive conformal primary basis?
• Kinematic algebra description of symmetry action on massive particles?
More details
Massless $W_{1+\infty}$ Action

\[ \delta^p_m |p_k\rangle = -\frac{1}{2} \int \frac{d^2 z}{2\pi} z^{p+m-1} \partial_z^{2p-1} S_k^{(2p-4)}(z, \bar{z}) |p_k\rangle \]

\[ p = \bar{h} = \frac{\ell+4}{2} \]

1 - $p \leq m \leq p - 1$

Examples:

\[ p = \frac{3}{2} \rightarrow \ell = -1 \] leading soft theorem, closed set of 2 modes: (chiral) translations

\[ \delta^{\frac{3}{2}} |p_k\rangle = \frac{\omega_k}{2} |p_k\rangle \quad \delta^{\frac{3}{2}} |p_k\rangle = \frac{\omega_k \bar{z}_k}{2} |p_k\rangle \]

\[ p = 2 \rightarrow \ell = 0 \] subleading soft theorem, closed set of 3 modes: $SL(2, \mathbb{R})$ chiral half of Lorentz

Acting on scalars

\[ \delta^2_m |p_k\rangle = \bar{z}_k^{m-1} \left( \bar{z}_k^2 \partial_{\bar{z}_k} - \frac{m+1}{2} \omega_k \partial_{\omega_k} \right) |p_k\rangle \]

Action in the conformal primary basis follows directly:

\[ \delta^n_{\phi \Delta_k} (z_k, \bar{z}_k) = \int_0^\infty \frac{d\omega_k}{\omega_k} \omega_k^{\Delta_k} \delta^n_{\phi \Delta_k} |p_k\rangle \]

Can prove by induction that this action satisfies the $W_{1+\infty}$ algebra

\[ [\delta^p_m, \delta^q_n] = [m(q - 1) - n(p - 1)] \delta^{p+q-2}_{m+n} \]

Base cases:

\[ p = \frac{3}{2}, \frac{5}{2} \]

\[ \delta^2_m : \delta^q_k \rightarrow \delta^q_{k+m} \]

\[ \delta^2_m : \delta^q_k \rightarrow \delta^q_{k+m} \]
Massive $W_{1+\infty}$ Action

\[ p = \bar{h} = \frac{\ell+4}{2} \]

\[ 1 - p \leq m \leq p - 1 \]

---

Leading: (chiral) translations

\[ \delta^p_m |p_k\rangle \equiv -\frac{1}{2} \int \frac{d^2z}{2\pi} z^{p+m-1} \partial_z^{2p-1} \begin{cases} S_k^{(2p-4)}(z, \bar{z}) |p_k\rangle, & p = \frac{3}{2}, 2, \frac{5}{2} \\ S_k^{(2p-4)}(z, \bar{z}) |p_k\rangle, & p > \frac{5}{2} \end{cases} \]

\[ = -\frac{\mathcal{N}_{2p-4}}{2} \int \frac{d^2z}{2\pi} z^{p+m-1} \frac{p_k^4}{(q \cdot p_k)^{2p-1}} (F_- \cdot \mathcal{L}_k)^{2p-3} |p_k\rangle. \]

---

\[ \delta_{-\frac{1}{2}}^{\frac{3}{2}} |p_k\rangle = \frac{\epsilon_k m_k}{4y_k} |p_k\rangle \]

\[ \delta_{\frac{3}{2}}^{\frac{3}{2}} |p_k\rangle = \frac{\epsilon_k m_k \bar{z}_k}{4y_k} |p_k\rangle \]

---

Subleading: $SL(2, \mathbb{R})$ Killing vectors on $(y_k, w_k, \bar{w}_k)$ hyperboloid

\[ \delta^2_m |p_k\rangle = \frac{1}{2} \bar{w}_k^{m-1} \left( 2w_k^2 \partial_{\bar{w}_k} + (m+1)\bar{w}_k y_k \partial_{y_k} - m(m+1)y_k^2 \partial_{w_k} \right) |p_k\rangle \]
Splitting the Soft Factor

To separate the soft factor and isolate the covariant piece independent of $z_0$, use completeness:

$$1 = \frac{p_k^2}{\hat{p}_k^2} = \frac{(\partial_z \hat{q} \cdot p_k)(\partial_z \hat{q} \cdot p_k) - (\hat{q} \cdot p_k)(n \cdot p_k)}{p_k^2}$$

For example, when $\ell = 2$

$$S_k^{(2)}(z, \bar{z}) = \frac{\mathcal{N}_2}{6!} \left[ \frac{1}{p_k^2} (\hat{q} \cdot p_k)^6 \partial_z \left( \frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q} \cdot p_k)^6} \right) \right] - \frac{1}{\hat{p}_k^2} (\hat{q}_0 \cdot p_k)^6 \partial_z \left( \frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q}_0 \cdot p_k)^6} \right)$$

Follows primary descendant pattern

$$\partial_z^5 \left[ \frac{\mathcal{N}_2}{6!} \frac{1}{p_k^2} (\hat{q} \cdot p_k)^6 \partial_z \left( \frac{(F_+ \cdot \mathcal{L}_k)^3}{(\hat{q} \cdot p_k)^6} \right) \right] = \mathcal{N}_2 \frac{p_k^4}{(\hat{q} \cdot p_k)^6} (F_- \cdot \mathcal{L}_k)^3$$

There is an analogous separation of terms at every order:

$$S_k^{(\ell)}(z, \bar{z}) = \frac{\mathcal{N}_\ell}{(2\ell + 2)!} f^{2\ell+2} \frac{1}{p_k^{2\ell-2}} \partial_z^{\ell-1} \left[ \frac{(F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f^{\ell+4}} \right]$$

$$- \frac{\mathcal{N}_\ell}{(\ell + 3)! (\ell - 2)!} \frac{1}{p_k^{2\ell-2}} \sum_{j=0}^{\ell-2} \binom{\ell - 2}{j} \frac{(-1)^j}{j + \ell + 4} f^{\ell-2-j} f_0^{\ell+4+j} \partial_z^{\ell-1} \left[ \frac{f^j (F_+ \cdot \mathcal{L}_k)^{\ell+1}}{f_0^{\ell+4+j}} \right]$$

$f \equiv \hat{q} \cdot p_k$, $f_0 \equiv \hat{q}_0 \cdot p_k$,