Welfare egalitarianism in non-rival environments

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Abstract

We study equity in economies where a set of agents commonly own a technology producing a non-rival good from their private contributions. A social ordering function associates to each economy a complete ranking of the allocations. We build social ordering functions satisfying the property that individual welfare levels exceeding a legitimate upper bound should be reduced. Combining that property with efficiency and robustness properties with respect to changes in the set of agents, we obtain a kind of welfare egalitarianism based on a constructed numerical representation of individual preferences.
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1 Introduction

We consider an economy where a non-rival good can be produced from a private good. A group of agents own the production technology in common and each of them is endowed with a (relatively large) quantity of the private

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good. Exclusion, perhaps partial, is possible: different agents may consume different quantities of the non-rival good. If we are interested in efficiency and equity, how should we choose the production level of the non-rival good, the consumption levels of the agents, and their private good contributions?

To answer this question, one may construct allocation rules. An allocation rule specifies which feasible allocations are the most desirable as a function of the parameters of the problem, namely, the production technology and the agents' preferences over bundles of non-rival and private goods. Examples of this approach include Foley (1970), Mas-Colell (1980), Moulin (1987, 1992), and Sprumont (1998). Quite naturally, all these papers impose Pareto efficiency, which immediately rules out exclusion.

Alternatively, one may look for social ordering functions specifying a complete ranking of all allocations, feasible or not, as a function of the parameters of the problem. That approach is adopted in Maniquet and Sprumont (2002), and we follow it here too. Because of a variety of information and incentive constraints, or because of other political or legal restrictions, the set of achievable allocations in often very uncertain and may have almost any shape. If a full ordering of the conceivable allocations is available, a best allocation may be selected from virtually any set. In particular, it is possible to choose among allocations where agents are partially excluded from consuming the non-rival good. This is useful because partial exclusion helps alleviate the free-rider problem, as Moulin (1994) demonstrates.

The literature on allocation rules has identified a number of interesting normative principles that can be adapted to our approach. Particularly relevant to our analysis is the "Unanimity Upper Bounds" axiom proposed by Moulin (1992) and used by Sprumont (1998) to characterize a particular welfare-egalitarian rule. Let us define the unanimity (welfare) level of an agent to be the level she would enjoy at an efficient allocation if the others had the same preferences as hers and everyone was treated equally. In the non-rival environment we are considering, disagreeing constitutes a social burden in the sense that the unanimity welfare levels are not jointly feasible when preferences differ. The Unanimity Upper Bounds axiom requires that everyone take up a share of this burden: no agent's welfare should exceed her unanimity level.

In this paper, we extend the Unanimity Upper Bounds principle to social ordering functions. We propose the following "Excessive Welfare Transfer Principle". Suppose that an agent enjoys an "excessive" welfare level, that is, a level exceeding her unanimity level. Then, a private good transfer from her to an agent whose welfare is lower than his unanimity level should be viewed as a social improvement. This principle expresses a form of welfare inequality aversion, but a rather limited one. We study social ordering functions which satisfy that property and other more standard efficiency and robustness conditions.

Three conclusions may be drawn from our work. First, the Excessive Welfare Transfer Principle is compatible with a large range of properties. It can be combined with efficiency properties and with properties of robustness to changes in the preference profile or in the set of agents. On the other hand, it is incompatible with another equity property, called Free Lunch Aversion, which was the central axiom in Maniquet and Sprumont (2002).

Second, combining the Excessive Welfare Transfer Principle with properties of efficiency and robustness with respect to changes in the agent set precipitates an infinite aversion towards excessive welfare. Removing an arbitrarily large quantity of private good from the bundle allocated to an "excessive welfare" agent, and transferring an arbitrarily small fraction of it to an agent whose welfare is not excessive, must be regarded as a social improvement. It is worth noting that related derivations of other extreme forms of welfare inequality aversion were obtained by Fleurbaey (2001), Maniquet and Sprumont (2002) and Maniquet (2002).

Third, our axioms pin down a specific class of egalitarian, or maximin, social ordering functions. Those social ordering functions maximize the welfare of the worst-off agent calibrated along multiples of the cost function. Maximizing any of those social orderings subject to the technological feasibility constraints yields the allocation rule characterized by Sprumont (1998). Maniquet and Sprumont (2002) identified a different class of egalitarian social ordering functions where welfare is measured in terms of the equivalent free consumption of the non-rival good; maximizing those orderings yields the rule characterized by Moulin (1987).

The rest of the paper is organized as follows. We define the model in the following section and introduce the axioms in section 3. Some important implications of the axioms are proved in section 4. Section 5 is devoted to the presentation and discussion of our main result. All proofs are gathered in section 6.

2 Setup

One non-rival good can be produced from one private good. The *cost function* $C : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly convex, and $C(0) = 0$. The set of all cost functions is denoted by $\mathcal{C}$. The *agent set* N is a finite nonempty set of integers. The set of all agent sets is denoted by $\mathcal{N}$. A *preference* for agent $i \in N$ is a binary relation R_i over bundles $z_i = (x_i, y_i) \in \mathbb{R} \times \mathbb{R}_+$ which is complete, transitive, continuous, strictly increasing in the non-rival good consumption level y_i, strictly decreasing in the private good contribution level x_i, and convex. We let P_i and I_i denote the corresponding strict preference and indifference relations, respectively. We often write i instead of $\{i\}$ and ij instead of $\{i, j\}$. The set of all preferences is denoted by $\mathcal{R}$.

An *economy* is a list $E = (N, R, C)$, where $N \in \mathcal{N}$, $R \in \mathcal{R}^N$, and $C \in \mathcal{C}$. We call R a *preference profile.* Formally, R is a mapping from N to $\mathcal{R}$. Thus, for any $R^0 \in \mathcal{R}$, $R^{-1}(R^0) := \{i \in N : R(i) = R^0\}$ denotes the set of agents whose preference is R^0. For every $S \subset N$, R_S is the restriction of R to S. We often abuse our notation, however, and let R_S denote also the list of preferences assigned to S by R, i.e., we write R_S instead of $R(S)$. Similarly, we use both R_i and $R(i)$ to denote agent i's preference. We refer to a pair (N, R) as a *society* and let $n = |N|$.

An *allocation* (for the economy E) is a mapping $z \in (\mathbb{R} \times \mathbb{R}_+)^N$ specifying a *bundle* $z(i)$ for each agent $i \in N$. We often write $z_i = (x_i, y_i)$ instead of $z(i)$ and $z_S = (x_S, y_S)$ instead of $z(S)$. An allocation z is *admissible* (for the society (N, R)) if there exists a production level $\overline{y}$ such that, for each $i \in N$,

$$(0, \overline{y})\ R_i\ z_i\ R_i\ (0, 0).$$

The first part of this condition says that no agent prefers her bundle to the opportunity of consuming any quantity of the non-rival good for free. The second part means that everyone gets a non-negative share of the surplus generated by the production of the non-rival good. We denote the sets of admissible allocations by $Z(N, R)$. We also write $Z(R_i)$ instead of $Z(\{i\}, R_{\{i\}})$. Our formulation allows different agents to consume different quantities of the non-rival good at an admissible allocation: exclusion, complete or partial, is possible. Furthermore, an admissible allocation $z = (x, y)$ need not be *feasible* for E, that is, it may violate the constraint $\sum_{i \in N} x_i \geq C(\max_{i \in N} y_i)$.

A *social ordering* (*for* E) is a complete and transitive binary relation defined over the set of all admissible allocations, $Z(N, R)$. A *social ordering function* $\mathbf{R}$ assigns to each economy E a social ordering $\mathbf{R}(E)$. Thus, $z\ \mathbf{R}(E)$

z' means that the allocation z is at least as good as z' in the economy E. Similarly, we use $\mathbf{P}(E)$ and $\mathbf{I}(E)$ to denote strict social preference and social indifference.

3 Axioms

This section defines the four properties that we impose on social ordering functions. An arbitrary economy $E = (N, R, C)$ is fixed throughout. We begin with a basic efficiency condition requiring that if all agents find an allocation at least as good as another, so does society. This property is weaker than the traditional strong version of the Pareto principle because it never forces strict social preference.

Unanimity *Let $z, z' \in Z(N, R)$. Then, $\{z_i R_i z'_i$ for all $i \in N\} \Rightarrow \{z\ \mathbf{R}(E)\ z'\}$.*

Our next two axioms are properties of robustness of the social ordering with respect to changes in the set of agents. Separability is a familiar condition in welfare economics (see for instance Fleming (1952)) and in social choice (see d'Aspremont and Gevers (1977)), requiring that agents who are indifferent between two alternatives do not influence social preferences over these alternatives. Our version of the axiom, due to Fleurbaey and Maniquet (2001), focuses on agents who receive the same bundle in two allocations. It demands that removing or adding such agents do not modify the social ordering of such allocations.

Separability *Let $z, z' \in Z(N, R)$. Let $S \subset N$ and suppose that $z_{N \setminus S} = z'_{N \setminus S}$. Then, $\{z\ \mathbf{R}(E)\ z'\} \Leftrightarrow \{z_S\ \mathbf{R}(S, R_S, C)\ z'_S\}$.*

Our third axiom, Replication Invariance, requires that the social ranking of two allocations be preserved in any economy obtained by merely replicating agents and rescaling the cost function. Some notation is needed to make this statement precise. If r is a positive integer, an *r-replica* of a society (N, R) is any society (N', R') such that $|R'^{-1}(R^0)| = r\,|R^{-1}(R^0)|$ for every $R^0 \in \mathcal{R}$. This condition implies, in particular, that $|N'| = r\,|N| = rn$. A *replica* of (N, R) is any society which is an r-replica of (N, R) for some r. We use a similar terminology for economies and allocations. Thus, an economy $E' = (N', R', C')$ is an r-replica of $E = (N, R, C)$ if i) (N', R') is an r-replica of (N, R) and ii) $C' = rC$. If $z \in Z(N, R)$, an allocation

$z' \in Z(N', R')$ is an r-replica of z if, for all $R^0 \in \mathcal{R}$, all $i \in R^{-1}(R^0)$ and $i' \in R'^{-1}(R^0), z(i) = z'(i')$. We will often use the symbol $r.E$ to denote an arbitrary r-replica of an economy E, and $r.z$ to denote an arbitrary r-replica of an allocation z.

Replication Invariance *Let r be a positive integer, and $E' = r.E$ an r-replica of E. Let z, $z' \in Z(N, R)$ and let $r.z$, $r.z' \in Z(N', R')$ be r-replicas of z and z' respectively. Then,* $\{z \ \mathbf{R}(E) \ z'\} \Rightarrow \{r.z \ \mathbf{R}(r.E) \ r.z'\}$.

Finally, we conclude with a formal statement of the Excessive Welfare Transfer Principle discussed in the introduction. We defined the unanimity welfare level of an agent, say, i, to be the level she would enjoy at an efficient allocation if the others shared her preference and everyone was treated equally. Observe that this unanimity level depends only upon i's preference, the number of agents in the economy, and the cost function: it is obtained by maximizing R_i subject to C/n. Consider now an allocation where i's welfare is below her unanimity level while some other agent j's welfare is above his unanimity level. A transfer of private good from i to j should decrease social welfare. Formally, if $R_i \in \mathcal{R}$ and $C \in \mathcal{C}$, let $z_i^*(R_i, C)$ be the best element of R_i under C, that is, the unique bundle $z_i = (x_i, y_i)$ such that $x_i \geq C(y_i)$ and $z_i' R_i z_i$ for every $z_i' = (x_i', y_i')$ such that $x_i' \geq C(y_i')$. Our axiom reads as follows.

Excessive Welfare Transfer Principle *Let $z, z' \in Z(N, R)$ and let $i, j \in N$. Suppose that $z_{N \setminus ij} = z'_{N \setminus ij}, y_i = y_i', y_j = y_j'$* and *$x_i + x_j = x_i' + x_j'$. Then,* $\{z_i^*(R_i, C/n) P_i z_i P_i z_i'$ *and* $z_j' P_j z_j P_j z_j^*(R_j, C/n)\} \Rightarrow \{z \ \mathbf{P}(E) \ z'\}$.

4 Some implications

The Excessive Welfare Transfer Principle is the only equity axiom in our list. In conjunction with the other axioms, however, it implies other equity properties. Two such properties are established in the lemmata below, which constitute the building blocks for our main theorem in the next section.

In income inequality measurement theory, the Pigou-Dalton transfer principle requires that an income transfer from a rich agent to a poorer agent be considered a social improvement (provided the latter agent remains poorer than the former). This principle is compatible with any degree of income inequality aversion (see Chakravarty (1990)). In our richer model where income can no longer be equated with welfare, the Excessive Welfare Transfer

Principle is one of several plausible extensions of the Pigou-Dalton principle. As such, it is compatible with any degree of welfare inequality aversion. But quite interestingly, combining it with our other axioms precipitates an extreme form of welfare inequality aversion captured in the following condition. Here and throughout the section, an arbitrary economy $E = (N, R, C)$ is again fixed.

Excessive Welfare Aversion *Let* $z, z' \in Z(N, R)$ *and let* $i, j \in N$*. Suppose that* $z_{N\setminus ij} = z'_{N\setminus ij}$*. Then,* $\{z_i^*(R_i, C/n) P_i z_i P_i z'_i$ *and* $z'_j P_j z_j P_j z_j^*(R_j, C/n)\} \Rightarrow \{z \; \mathbf{P}(E) \; z'\}$.

We stress that this condition is a drastic strengthening of our original Excessive Welfare Transfer Principle. It regards as a social improvement any arbitrarily small welfare gain obtained by an agent below her unanimity level at the expense of a potentially very large welfare loss by an agent above his unanimity level.

Lemma 1 *If a social ordering function satisfies Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle, then it satisfies Excessive Welfare Aversion.*

The second equity property that we derive from our axioms is a variant of the fundamental idea that equals should be treated equally. Agents are said to be equals if they have the same preferences; such agents are said to be treated equally if they receive bundles they deem equivalent. Consider now the following extension of the equal treatment condition: if two agents with identical preferences receive bundles on different indifference curves, increasing the welfare of the worse-off agent and reducing that of the other agent is a social improvement, as long as the latter remains at least as well-off as the former.

Inequality Aversion among Equals *Let* $z, z' \in Z(N, R)$ *and let* $i, j \in N$*. Suppose that* $R_i = R_j$ *and* $z_{N\setminus ij} = z'_{N\setminus ij}$*. Then,* $\{z'_j P_i z_j R_i z_i P_i z'_i\} \Rightarrow \{z \; \mathbf{P}(E) \; z'\}$.

Lemma 2 *If a social ordering function satisfies Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle, then it satisfies Inequality Aversion among Equals.*

5 Cost-calibrated welfare egalitarianism

It turns out that the axioms introduced in section 3 force us to use a very specific type of social ordering functions: allocations must be ranked according to the "cost-calibrated" welfare of the worst-off agent. To make this statement precise, we introduce a last piece of notation.

For each preference R_i and every cost function C, denote by $a_0(R_i, C)$ the largest number a for which R_i's best bundle subject to the cost function C/a involves no production: $a_0(R_i, C) = \sup\{a > 0 : (0,0)\ R_i\ (C(y_i)/a, y_i) \text{ for all } y_i \in \mathbb{R}_+\}$, with the convention that $\sup \emptyset = 0$. For each $a > 0$, define the set $\Gamma(a) = \{(x_i, y_i) \in \mathbb{R} \times \mathbb{R}_+ : x_i \geq C(y_i)/a\}$, and let $\Gamma(0) := \{(x_i, 0) : x_i \geq 0\}$ and $\Gamma(\infty) := \{(0, y_i) : y_i \in \mathbb{R}_+\}$. The restriction of R_i to the subset of admissible bundles admits a unique numerical representation $u(R_i, C, .) : Z(R_i) \to \mathbb{R}$ such that

$$\max_{z_i \in \Gamma(a)} u(R_i, C, z_i) = a \text{ for all } a \geq a_0(R_i, C).$$

We call the number $u(R_i, C, z_i)$ agent i's *cost-calibrated welfare level* at bundle z_i. By definition of this number, agent i is indifferent between receiving the bundle z_i or maximizing R_i subject to the cost function $C/u(R_i, C, z_i)$. This is illustrated on Figure 1. Note that $u(R_i, C, (0,0)) = a_0(R_i, C)$ and $u(R_i, C, (0, y)) \to \infty$ as $y \to \infty$.

A social ordering function $\mathbf{R}$ is a *cost-calibrated maximin function* if, for each economy, the ordering of allocations it prescribes is consistent with the application of the maximin criterion to the vectors of cost-calibrated welfare levels generated by these allocations. More precisely: for any $E = (N, R, C)$ and all z, $z' \in Z(N, R)$,

$$\min_{i \in N} u(R_i, C, z_i) > \min_{i \in N} u(R_i, C, z'_i) \Rightarrow z\ \mathbf{P}(E)\ z'.$$

A prominent example is the *cost-calibrated leximin function* $\mathbf{R}_L$. Let $\succsim^n_L$ denote the usual leximin ordering of $\mathbb{R}^n_+$: for any $w, w' \in \mathbb{R}^n_+$, $w \succsim^n_L w'$ if and only if the smallest coordinate of w is greater than the smallest coordinate of w', or they are equal but the second smallest coordinate of w is greater than the second smallest coordinate of w', and so on. The social ordering function $\mathbf{R}_L$ ranks the admissible allocations for any given economy by applying the leximin ordering to the corresponding vector of cost-calibrated welfare levels: for any $E = (N, R, C)$ and z, $z' \in Z(N, R)$,

$$z\ \mathbf{R}_L(E)\ z' \Leftrightarrow (u(R_1, C, z_1), ..., u(R_n, C, z_n)) \succsim^n_L (u(R_1, C, z'_1), ..., u(R_n, C, z'_n)).$$

Theorem *The cost-calibrated leximin function* $\mathbf{R}_L$ *satisfies Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle. Conversely, every social ordering function satisfying these four axioms is a cost-calibrated maximin function.*

Before we turn to the proof, a brief discussion of this result may be useful.

1) In the classical literature on the maximin and leximin criteria, individual utilities are given *a priori*: see, for instance, Hammond (1976). By contrast, no utility information is given in our model. The leximin criterion could therefore be applied to any utility representation of individual preferences. The significance of the second part of the theorem is therefore twofold: not only do our axioms lead us to rank allocations by applying the maximin criterion to vectors of corresponding welfare levels, they also force us to use a specific (endogenous) welfare measure based on the cost-calibrated representation of preferences.

2) The two statements in our theorem are not exact converse to each other. On the one hand, $\mathbf{R}_L$ is not the only social ordering function satisfying the four stated axioms. Other examples include functions that agree with $\mathbf{R}_L$ whenever the latter does not declare a tie between two allocations, but do break some such ties. Consider an economy $E = (N, R, C)$ and an admissible allocation z. For every agent i, the definition of the cost-calibrated numerical representation of R_i ensures that $z_i I_i z_i^*(R_i, C/u(R_i, C, z_i))$. Denote by $y_i^*(R_i, C/u(R_i, C, z_i))$ the quantity of the non-rival good in the bundle $z_i^*(R_i, C/u(R_i, C, z_i))$. Define the social ordering function $\mathbf{R}'_L$ as follows: for every economy E and any two admissible allocations z, z', let $z\ \mathbf{R}'_L(E)\ z'$ if and only if either i) $z\ \mathbf{P}_L(E)\ z'$ or ii) $z\ \mathbf{I}_L(E)\ z'$ and $(y_1^*(R_1, C/u(R_1, C, z_1)),$ $..., y_n^*(R_n, C/u(R_n, C, z_n))) \succsim_L^n (y_1^*(R_1, C/u(R_1, C, z'_1)), ..., y_n^*(R_n, C/u(R_n, C,$ $z'_n)))$. It is not difficult to verify that $\mathbf{R}'_L$ satisfies our four axioms. This example may be modified to obtain a social ordering function that satisfies the axioms but disagrees with some of the strict preferences recommended by $\mathbf{R}_L$.

On the other hand, not all cost-calibrated maximin functions satisfy our four axioms. For instance, a cost-calibrated maximin function which breaks ties between allocations by preferring the one where the highest cost-calibrated welfare is lower, violates Unanimity.

3) The four axioms stated in the theorem are independent.

A social ordering function satisfying all of them but Unanimity is the following dual of the cost-calibrated welfare leximin function: first minimize the largest cost-calibrated welfare level; in case of a tie, minimize the second largest, and so on.

If Separability is not required, we may use a numerical representation of preferences that depends not only on the cost function, C, but also on the number of agents, n. For any number a, let $\Lambda(C, n, a) = \{(x_i, y_i) \in \mathbb{R} \times \mathbb{R}_+ : x_i \geq (C(y_i)/n) - a\}$. Note that Λ is replication-invariant: $\Lambda(rC, rn, a) = \Lambda(C, n, a)$ for any positive integer r. Define the numerical representation $v(R_i, C, n, .) : z(R_i) \to \mathbb{R}$ of preference R_i by the condition that $\max_{z_i \in \Lambda(C,n,a)} v(R_i, C, n, z_i) = a$. Rank allocations in the economy (N, R, C) by applying the leximin criterion to the vectors of welfare levels generated through this new representation.

If Replication Invariance is not imposed, apply the leximin criterion to numerical representations that depend on C in such a way that they coincide with the cost-calibrated representations $u(R_i, C, .)$ only on those bundles z_i for which $u(R_i, C, z_i) = 1/n$ for some integer n.

Finally, a social ordering function satisfying all axioms but the Excessive Welfare Transfer Principle is the non-rival good welfare leximin function defined in Maniquet and Sprumont (2002). It compares allocations by applying the leximin criterion to the vectors of "non-rival good welfare" levels. If $R_i \in \mathcal{R}$ and $z_i \in Z_i(R_i)$, agent i's non-rival good welfare from bundle z_i is defined as the unique number $y_i^0 \in \mathbb{R}_+$ such that $z_i I_i(0, y_i^0)$.

4) In addition to the properties stated in the theorem, the cost-calibrated welfare leximin function $\mathbf{R}_L$ possesses many others. For instance, it satisfies the inter-profile condition of "Responsiveness", which requires that a social preference for an allocation z over an allocation z' be preserved if all agents' upper contour sets at (the bundle they receive at) z shrink while their upper contour sets at z' expand. For a discussion of inter-profile conditions in economic models, see Fleurbaey and Maniquet (1996) or Le Breton and Weymark (2001).

On the other hand, $\mathbf{R}_L$ violates a property of welfare inequality aversion introduced in Maniquet and Sprumont (2002) under the name of "Free Lunch Aversion". Consider an allocation at which two agents, i and j, consume the same quantity of the non-rival good. Suppose that i's private good contribution is positive but j's contribution is negative. Since j already enjoys a "free lunch", a transfer of private good from i to j should be deemed to decrease social welfare. Formally:

Free Lunch Aversion *Let* $E = (N, R, C)$ *be an arbitrary economy,* $z = (x, y)$, $z' = (x', y') \in Z(N, R)$, *and* $i, j \in N$. *Suppose that* $y_i = y_j, y = y'$, $x_{N \setminus ij} = x'_{N \setminus ij}$, *and* $x_i + x_j = x'_i + x'_j$. *Then,* $\{0 \leq x_i < x'_i$ *and* $x'_j < x_j \leq 0\} \Rightarrow \{z \ \mathbf{P}(E) \ z'\}$.

That $\mathbf{R}_L$ violates Free Lunch Aversion, however, is only a consequence of a more fundamental incompatibility. No social ordering function satisfies Free Lunch Aversion and the Excessive Welfare Transfer Principle. Figure 2 shows an economy $E = (\{1, 2\}, R, C)$ and two allocations $z, z' \in Z(\{1, 2\}, R)$ such that i) $y_1 = y'_1 = y'_2 = y_2$, ii) $x_1 + x_2 = x'_1 + x'_2$, iii) $x_1 < x'_1 \leq 0 \leq x'_2 < x_2$, and iv) $z^*_1(R_1, C/2) P_1 z_1 P_1 z'_1$ and $z'_2 P_2 z_2 P_2 z^*_2(R_2, C/2)$. By Free Lunch Aversion, $z' \ \mathbf{P}(E) \ z$, whereas the opposite strict social preference is implied by the Excessive Welfare Transfer Principle.

6 Proofs

We begin by noting an obvious but useful consequence of Unanimity: for all $E = (N, R, C)$ and for all $z, z' \in Z(N, R)$, $\{z_i \, I_i \, z'_i$ for all $i \in N\} \Rightarrow \{z \ \mathbf{I}(E) \ z'\}$. We call this property Unanimous Indifference.

Proof of Lemma 1. Let $\mathbf{R}$ satisfy Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle. Suppose, by way of contradiction, that $\mathbf{R}$ violates Excessive Welfare Aversion. Let $E = (N, R, C)$, $i, j \in N$, and $z, z' \in Z(N, R)$ be such that $z_{N \setminus ij} = z'_{N \setminus ij}$,

$$z^*_i(R_i, C/n) P_i z_i P_i z'_i, \ \ z'_j P_j z_j P_j z^*_j(R_j, C/n), \tag{1}$$

but

$$z' \ \mathbf{R}(E) \ z. \tag{2}$$

By Unanimous Indifference, we may assume that $y_i = y'_i$ and $y_j = y'_j$. By strict monotonicity of preferences, $x'_i > x_i$ and $x'_j < x_j$.

Let r be a positive integer large enough to guarantee that

$$z'_j \ P_j \ z_j \ P_j \ z^*_j(R_j, rC/(rn + 1)). \tag{3}$$

As illustrated on Figure 3, choose now four bundles $z^1_0, z^2_0, z^3_0, z^4_0$ such that $y^1_0 = y^2_0$, $y^3_0 = y^4_0$ and construct a preference R_0 such that

$$0 < x^1_0 - x^2_0 < r(x'_i - x_i), \tag{4}$$

$$x_0^3 - x_0^4 = r(x_j - x_j'), \tag{5}$$

and

$$z^*(R_0, rC/(rn+1)) \ P_0 \ z_0^2 I_0 z_0^4 \ P_0 \ z_0^1 I_0 z_0^3 \ P_0 \ z^*(R_0, C/n). \tag{6}$$

Let $E^1 = (N^1, R^1, rC)$ be an r-replica of E with $N \subseteq N^1$. By Replication Invariance and (2),

$$r.z' \ \mathbf{R}(E^1) \ r.z.$$

Construct $E^2 = (N^2, R^2, rC)$ by adding to society (N^1, R^1) one agent, say, 0, endowed with the preference R_0 constructed above. Formally, $N^2 = N^1 \cup \{0\}$ (so that the original agent set N is a subset of N^2), $R^2_{N^1} = R^1$, and $R^2_0 = R_0$. By Separability,

$$(z_0^3, r.z') \ \mathbf{R}(E^2) \ (z_0^3, r.z).$$

There are $rn + 1$ agents in the economy E^2 and the cost function is rC. Combining (3), (6) and (5), we may therefore use the Excessive Welfare Transfer Principle r times to conclude that $(z_0^4, r.z_j, r.z'_{N\backslash j}) \ \mathbf{P}(E^2) \ (z_0^3, r.z')$. Therefore,

$$(z_0^4, r.z_j, r.z'_{N\backslash j}) \ \mathbf{R}(E^2) \ (z_0^3, r.z).$$

Remove agent j (a member of N) from N^2 to obtain $E^3 = (N^2\backslash j, R^2_{N^2\backslash j}, rC)$. By Separability,

$$(z_0^4, (r-1).z_j, r.z'_{N\backslash j}) \ \mathbf{R}(E^3) \ (z_0^3, (r-1).z_j, r.z_{N\backslash j}),$$

and thus, by Unanimous Indifference,

$$(z_0^2, (r-1).z_j, r.z'_{N\backslash j}) \ \mathbf{R}(E^3) \ (z_0^1, (r-1).z_j, r.z_{N\backslash j}). \tag{7}$$

Finally, let $z_i'' = (x_i' - (x_0^1 - x_0^2)/r, \ y_i)$. By (4) and strict monotonicity of R_i, $z_i P_i z_i''$. Therefore, recalling that $z_{N\backslash ij} = z'_{N\backslash ij}$, (7) and Unanimity yield

$$(z_0^2, (r-1).z_j, r.z'_{N\backslash j}) \ \mathbf{R}(E^3) \ (z_0^1, (r-1).z_j, r.z_i'', r.z_{N\backslash ij}).$$

In view of (1) and (6), and since there are rn agents in the economy E^3, this contradicts the Excessive Welfare Transfer Principle. ■

The following simple observation will be used in the two remaining proofs. Let $E = (N, R, C)$ be an arbitrary economy and let I, J be two nonempty disjoint subsets of N. Let $z, z' \in Z(N, R)$ be such that $z_{N\backslash(I\cup J)} = z'_{N\backslash(I\cup J)}$. If a choice function $\mathbf{R}$ satisfies Excessive Welfare Aversion, then $\{z_i^*(R_i, C/n)$

$P_i\ z_i\ P_i\ z'_i$ for all $i \in I$ and $z'_j\ P_j\ z_j\ P_j\ z^*_j(R_j, C/n)$ for all $j \in J\} \Rightarrow \{z\ \mathbf{P}(E)\ z'\}$. We omit the formal proof of this fact, which merely consists in applying the axiom repeatedly.

Proof of Lemma 2. Let $\mathbf{R}$ satisfy Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle. By Lemma 1, $\mathbf{R}$ also satisfies Excessive Welfare Aversion. Suppose, by way of contradiction, that $\mathbf{R}$ violates Inequality Aversion among Equals. Let $E = (N, R, C)$, $z, z' \in Z(N, R)$ and $i, j \in N$ be such that $R_i = R_j$, $z_{N\backslash ij} = z'_{N\backslash ij}$,

$$z'_j P_i z_j R_i z_i P_i z'_i, \tag{8}$$

but

$$z'\ \mathbf{R}(E)\ z. \tag{9}$$

In (8), we may assume without loss of generality that $z_j P_i z_i$. Indeed, if $z_j I_i z_i$, our assumptions on preferences guarantee that there exists a bundle t_i such that $z_j P_i t_i P_i z'_i$. Unanimity and (9) then ensure that $z'\ \mathbf{R}(E)\ (z_{N\backslash i}, t_i)$, and we need only replace z with $(z_{N\backslash i}, t_i)$ in the argument below. Now, since $z_j P_i z_i$, there exist two positive integers s and r such that

$$z'_j P_i z_j P_i z^*_i(R_i, rC/s) P_i z_i P_i z'_i. \tag{10}$$

Clearly, we may assume that $s \geq 2$. It is convenient to distinguish two cases.

Case 1: $r = 1$.

If $s < n$, choose $S \subset N$ such that $i, j \in S$ and $|S| = s$. By Separability and (9), $z'_S\ \mathbf{R}(S, R_S, C)\ z_S$. In view of (10), this contradicts Excessive Welfare Aversion since $r = 1$.

If $s \geq n$, choose $S \supseteq N$ such that $|S| = s$. Let $R''_{S\backslash N} \in \mathcal{R}^{S\backslash N}$ and $z''_{S\backslash N} \in Z(S\backslash N, R''_{S\backslash N})$. By Separability and (9), $(z', z''_{S\backslash N})\ \mathbf{R}(S, (R, R''_{S\backslash N}), C)\ (z, z''_{S\backslash N})$, which again contradicts Excessive Welfare Aversion.

Case 2: $r > 1$.

Let $E' = (N', R', C')$ be an r-replica of E such that $i, j \in N'$. By Replication Invariance and (9), $r.z'\ \mathbf{R}(E')\ r.z$. On the other hand, since $C' = rC$, (10) reads $z'_j P_i z_j P_i z^*_i(R_i, C'/s) P_i z_i P_i z'_i$. Using the observation preceding the proof of Lemma 2, we may now repeat the argument in Case 1 with $N', R', C', r.z$, and $r.z'$ playing the roles of N, R, C, z, and z', respectively. ■

Proof of the Theorem. The proof that $\mathbf{R}_L$ satisfies our four axioms is straightforward; we establish here the second part of our theorem. Let thus

$\mathbf{R}$ satisfy Unanimity, Separability, Replication Invariance, and the Excessive Welfare Transfer Principle. By lemmata 1 and 2, $\mathbf{R}$ also satisfies Excessive Welfare Aversion and Inequality Aversion among Equals.

Now, let $E = (N, R, C)$ and $z, z' \in Z(N, R)$ be such that

$$\min_{i \in N} u(R_i, C, z_i) > \min_{i \in N} u(R_i, C, z_i'). \tag{11}$$

Suppose, contrary to the claim, that

$$z' \; \mathbf{R}(E) \; z. \tag{12}$$

We will derive a contradiction.

Assume, without loss of generality, that $\min_{i \in N} u(R_i, C, z_i') = u(R_1, C, z_1')$. To alleviate notations, let us write u_i instead of $u(R_i, C, .)$ for every $i \in N$. Thus, $u_i(z_i) > u_1(z_1')$ for all $i \in N$. Because of Unanimity, we need only consider the case where

$$u_j(z_j') > \max_{i \in N} u_i(z_i) \text{ for all } j \in N \backslash 1. \tag{13}$$

This is without loss of generality. Indeed, if (13) does not hold, we simply choose $t, t' \in Z(N, R)$ such that i) $u_i(t_i') \geq u_i(z_i')$ and $u_j(t_j) \leq u_j(z_j)$ and ii) the conditions obtained from (11) and (13) by replacing z with t and z' with t' are satisfied. By Unanimity, $t' \; \mathbf{R}(E) \; z'$ and $z \; \mathbf{R}(E) \; t$, so that by (12) and transitivity of $\mathbf{R}$, $t' \; \mathbf{R}(E) \; t$. We may then merely replace z' with t' in the argument below.

Let p, q be two positive integers such that

$$\min_{i \in N} u_i(z_i) > q/p > u_1(z_1'). \tag{14}$$

We may and do choose $p, q > n$. Let $E^0 = (N^0, R^0, C^0)$ be a p-replica of E such that $N \subset N^0$. There are pn agents in this economy, and $C^0 = pC$. By Replication Invariance, (12) implies that

$$p.z' \; \mathbf{R}(E^0) \; p.z,$$

where $p.z$, $p.z' \in Z(N^0, R^0)$ are p-replicas of z and z' respectively. Choose now $z_1''' \in Z(1, R_1)$ and $z_{N\backslash 1}'' \in Z(N \backslash 1, R_{N \backslash 1})$ such that

$$u_i(z_i) > u_i(z_i'') > q/p > u_1(z_1''') > u_1(z_1') \text{ for all } i \in N \backslash 1. \tag{15}$$

Such allocations exist by continuity of the preferences. The construction of z_1''' and $z''_{N\backslash 1}$ is illustrated in Figure 4, where $i \in N\backslash 1$.

Construct an economy $E^1 = (N^1, R^1, C^1)$ by "adding" to society (N^0, R^0) a disjoint 1-replica of society $(N\backslash 1, R_{N\backslash 1})$. Formally, N^1 is the union of N^0 and some disjoint agent set M, $R^1_{N^0} = R^0$, (M, R^1_M) is a 1-replica of $(N\backslash 1, R_{N\backslash 1})$, and $C^1 = C^0$. There are $pn + (n-1)$ agents in this economy. By Separability,

$$(p.z' \ , \ 1.z''_{N\backslash 1}) \ \mathbf{R}(E^1) \ (p.z \ , \ 1.z''_{N\backslash 1}), \tag{16}$$

where we recall that $p.z$, $p.z' \in Z(N^0, R^0)$ and where $1.z''_{N\backslash 1} \in Z(M, R^1_M)$ is a 1-replica of the allocation $z''_{N\backslash 1} \in Z(N\backslash 1, R_{N\backslash 1})$. By Inequality Aversion among Equals, and recalling (13) and (15), the better allocation in (16) can be improved upon by replacing the bundle of every agent i with z_i, except for p "replicas" of agent 1. Formally, choosing a set $N^1_1 \subseteq (R^1)^{-1}(R_1) \subset N^1$ of cardinality $|N^1_1| = p$, we have

$$(p.z'_1, (p+1).z_{N\backslash 1}) \ \mathbf{R}(E^1) \ (p.z, \ 1.z''_{N\backslash 1}),$$

where $p.z'_1 \in Z(N^1_1, R^1_{N^1_1})$ is a p-replica of z'_1 and $(p+1).z_{N\backslash 1} \in Z(N^1\backslash N^1_1, R^1_{N\backslash N^1_1})$ is a $(p{+}1)$-replica of $z_{N\backslash 1}$. Construct now an economy $E^2 = (N^2, R^2, C^2)$ from E^1 by "removing p replicas" of each $i \in N\backslash 1$. Formally, $N^2 = N^1_1 \cup (N\backslash 1)$, $R^2 = R^1_{N^2}$, and $C^2 = C^1$. There are $p{+}(n{-}1)$ agents in this economy. By Separability,

$$(p.z'_1, \ z_{N\backslash 1}) \ \mathbf{R}(E^2) \ (p.z_1, \ z''_{N\backslash 1}),$$

where $p.z_1, p.z'_1 \in Z(N^1_1, R^2_{N^1_1})$ and $z_{N\backslash 1}, z''_{N\backslash 1} \in Z(N\backslash 1, R^2_{N\backslash 1})$. Next, by Unanimity,

$$(p.z'_1, \ z_{N\backslash 1}) \ \mathbf{R}(E^2) \ (p.z'''_1, \ z''_{N\backslash 1}), \tag{17}$$

because (14) and the definition of 1 imply that $u_1(z_1) > q/p > u_1(z'_1)$, so that (15) yields $u_1(z_1) > u_1(z'''_1)$.

Now, either $p + n - 1 > q$ or $p + n - 1 \leq q$. In the former case, construct $E^3 = (N^3, R^3, C^3)$ from E^2 by "removing $p - q + n - 1$ replicas of agent 1". Formally, choose $N^3_1 \subset N^1_1$ with $|N^3_1| = q - n + 1$, and let $N^3 = N^3_1 \cup (N\backslash 1)$, $R^3 = R^2_{N^3}$, and $C^3 = C^2$. There are exactly q agents in the economy E^3. (Note that $0 < p - q + n - 1 < p$ because we chose $p, q > n$.) By Separability,

$$((q-n+1).z'_1 \ , \ z_{N\backslash 1}) \ \mathbf{R}(E^3) \ ((q-n+1).z'''_1 \ , \ z''_{N\backslash 1}), \tag{18}$$

where $(q-n+1).z'_1, (q-n+1).z'''_1 \in Z(N_1^3, R^3_{N_1^3})$. But recalling the definition of the cost-calibrated numerical representations u_j, and since $C^3 = pC$, (15) means that $z_1^*(R_1, C^3/q)P_1 z'''_1 P_1 z'_1$ and $z_i P_i z''_i P_i z_i^*(R_i, C^3/q)$ for all $i \in N/1$. According to the observation preceding the proof of Lemma 2, and since $q = |N_3|$, (18) contradicts Excessive Welfare Aversion.

In the case where $p+n-1 \leq q$, construct $E^4 = (N^4, R^4, C^4)$ from E^2 by "adding $q-p-n+1$ replicas" of an arbitrary agent, say, agent 1. Formally, N^4 is the union of N^2 with some disjoint set L, $R^4_{N^2} = R^2$, and $R^4_L(i) = R_1$ for all $i \in L$, and $C^4 = C^2$. Again, there are q agents in this economy. Giving to all agents in L the same bundle z_0 and applying Separability, we get from (17) that

$$((q-p-n+1).z_0\ ,\ p.z'_1,\ z_{N\backslash 1})\ \mathbf{R}(E^4)\ ((q-p-n+1).z_0\ ,\ p.z'''_1\ ,\ z''_{N\backslash 1}),$$

which again contradicts Excessive Welfare Aversion. ■

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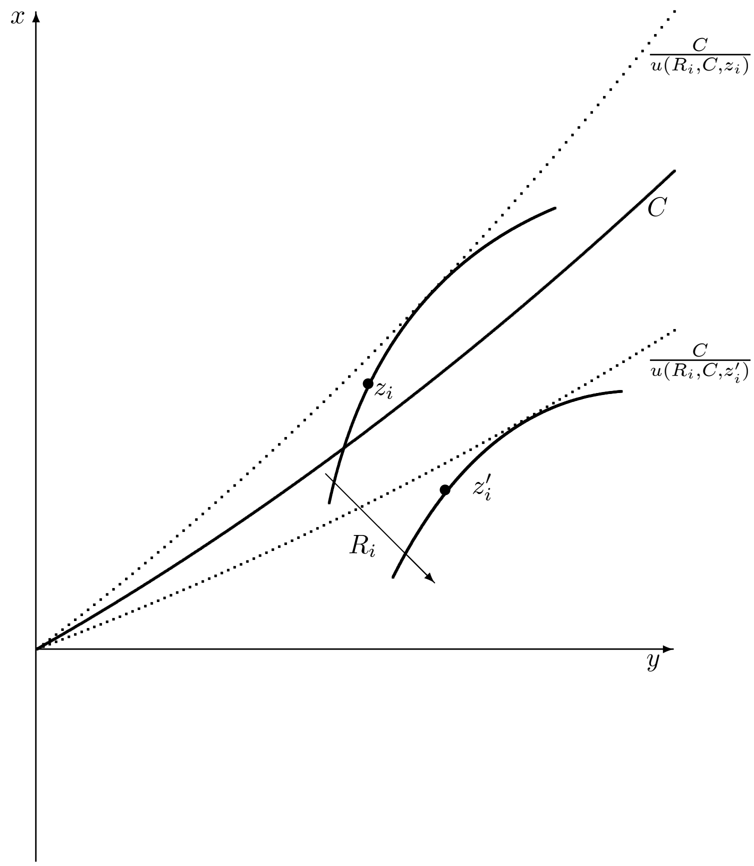

Figure 1

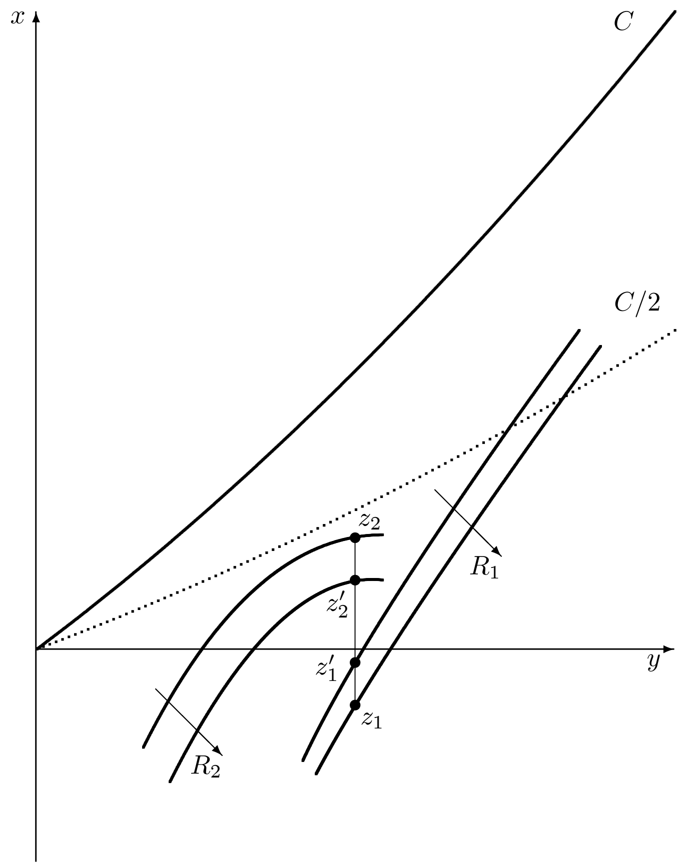

Figure 2

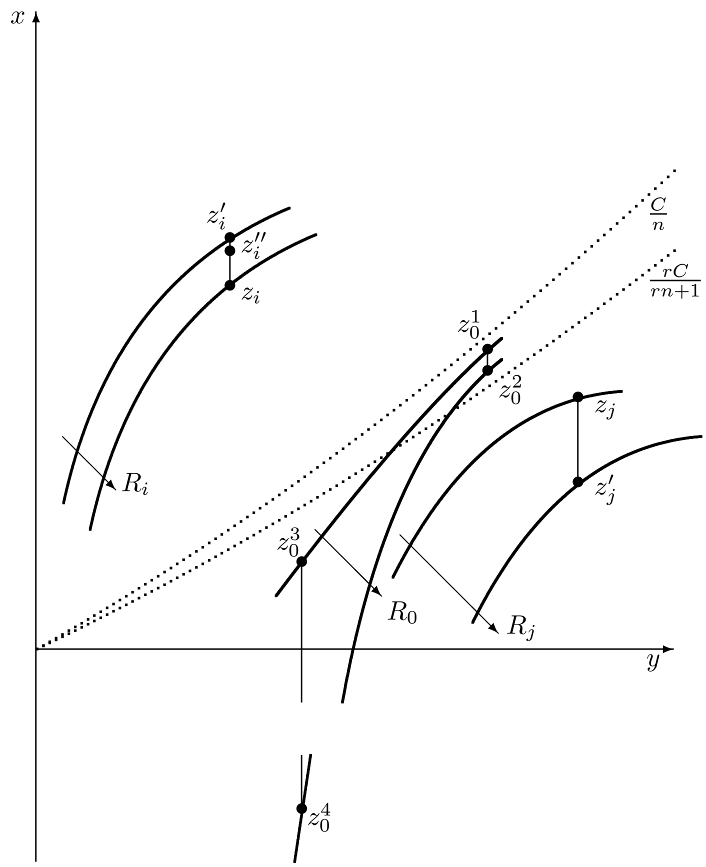

Figure 3

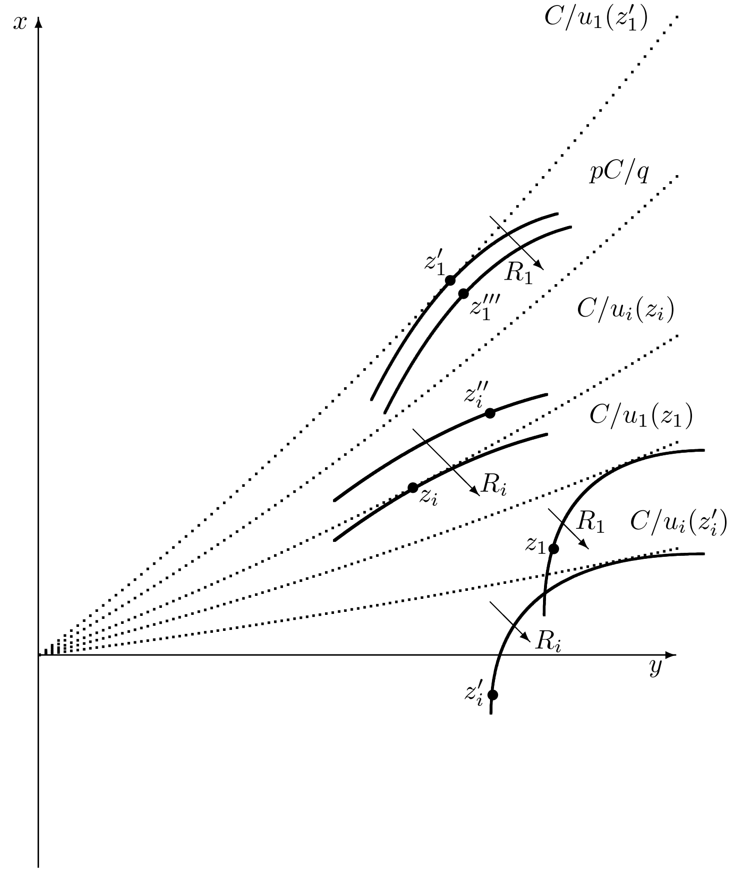

Figure 4