Bernoulli Without Bayes: A Theory of Utility-Sophisticated Preferences under Ambiguity

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Abstract

A decision-maker is utility-sophisticated if he ranks acts according to their expected utility whenever such comparisons are meaningful. We characterize utility sophistication in cases in which probabilistic beliefs are not too imprecise, and show that in these cases utility-sophisticated preferences are completely determined by consequence utilities and event attitudes captured by preferences over bets. The Anscombe-Aumann framework as employed in the classical contributions of Schmeidler (1989) and Gilboa-Schmeidler (1989) can be viewed as an important special case. For the class of utility sophisticated preferences with sufficiently precise beliefs, we also propose a definition of revealed probabilistic beliefs that overcomes the limitations of existing definitions.

Keywords: expected utility, ambiguity, probabilistic sophistication, revealed probabilistic beliefs.

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1. INTRODUCTION

Expected tility theory rests on two pillars of consequentialist rationality: the existence of a unique subjective probability measure underlying all decisions (the "Bayes principle"), and the consistent valuation of outcomes by cardinal utilities (the "Bernoulli principle"). Both of these assumptions have been challenged. On the one hand, as illustrated by the Ellsberg paradox, it is frequently not possible to represent a decision-maker's betting preferences in terms of a well-defined subjective probability measure; in such cases, decision-makers are said to view certain events as "ambiguous". On the other hand, faced with given probabilities, utilities and probabilities may not combine linearly, as in the Allais paradox and related phenomena; such decision-makers are sometimes referred to as exhibiting "probabilistic risk-attitudes".

While a descriptively fully adequate model of decision-making will need to incorporate both phenomena, for modelling purposes it is often desirable to zoom in on one of these two departures from the expected utility paradigm. To this purpose, Machina-Schmeidler (1992) have introduced the notion of probabilistic sophistication which precludes all phenomena of ambiguity but does not constrain the nature of probabilistic risk-attitudes. In the present paper, we introduce a complementary notion of utility sophistication which precludes all phenomena deriving from probabilistic risk-attitudes but does not constrain the decision-maker's attitudes towards ambiguity.

Besides this analytical motivation, the notion of utility sophistication has also an important normative purpose. Since the underlying Bernoulli principle is conceptually clearly distinct from the Bayes principle, one can formulate a normative position on which departures from the Bayes principle are rationally justifiable while departures from the Bernoulli principle are not. Such a position seems in fact quite attractive. On the one hand, it can be doubted that the precision of beliefs required by the Bayes principle is normatively mandated; indeed, it can even be argued that in situations of partial or complete ignorance rational decision making *cannot* rationally be based well-defined subjective probabilities (see the classical literature on complete ignorance surveyed in Luce-Raiffa (1957) as well the subsequent contributions of Jaffray (1989) and Nehring (1991,2000)). On the other hand, while it is frequently argued that departures from the Bernoulli principle are rationally permissible, we are not aware of an argument that would rationally *mandate* departures from the Bernoulli principle, i.e. in particular, departures from expected utility in the presence of probabilities. Moreover, the typical examples of departures from the Bernoulli principle such as the Allais paradox can be interpreted as "real but not rational", by attributing them to cognitive distortions in the processing of probabilities as in Kahneman-Tversky's (1979) prospect theory, or as "rational but merely apparent", by appealing to the existence of implicit psychological payoffs (cf. for example Broome (1991) and Caplin-Leahy (2001)). The present paper articulates this normative "Bernoulli without Bayes" position axiomatically but will not defend it further.

Broadly speaking, we shall view an agent as "utility-sophisticated" if he compares acts in terms of their expected utility "whenever possible". Importantly, since the possibility of such comparisons depends on the agents' beliefs, utility sophistication must be defined *relative* to a specified set of probabilistic beliefs (at least initially). We shall thus model probabilistic beliefs as a separate entity, specifically as partial orderings over events (likelihood relations) represented by a (closed convex) set of admissible probability measures Π. The specified likelihood relation will be viewed as describing non-exhaustively some but not necessarily all of the decision maker's beliefs; the leading example of such beliefs derives from the existence of a continuous randomization device as implicit in the Anscombe-Aumann (1963) approach to decision making under uncertainty.

Given a cardinal utility function u (obtained from risk preferences), an agent is *utility-sophisticated* with respect to the set of admissible priors Π if the agent prefers any act f over another act g whenever the expected utility of f weakly exceeds that of g with respect to any admissible prior. Utility sophistication implies expected utility maximization over unambiguous acts (acts whose induced distribution does not depend on the prior), but is substantially stronger since it also restricts preferences over ambiguous acts. This added strength proves crucial for its analytical power.

The first and foremost task of the paper is to provide axiomatic foundations. The crucial axiom replacing Savage's Sure-Thing Principle is an axiom of "Trade-off Consistency".[1] The main result of the paper, Theorem 1, derives utility sophistication from this axiom in the presence of arguably weak regularity assumptions on preferences, assuming both a rich set of consequences and a sufficiently rich (specifically: "equidivisible") likelihood relation.

Theorem 1 also shows that under these assumptions, utility-sophisticated preferences over general acts are uniquely determined by preferences over bets and the decision-maker's cardinal valuation of outcomes. This powerful reduction property parallels that of probabilistically sophisticated preferences which are uniquely determined by preferences over lotteries and the decision-maker's subjective probability measure. The reduction property greatly simplifies the task of developing more specific models of decision-making under ambiguity, since it focusses attention on the

[1]Its basic idea can be described as follows. Consider two acts f and g whose outcomes differ on only two *equally likely* events A and B such that f yields a better outcome in event A and g yields a better outcome in event B. Suppose also that we already have obtained a ranking of utility differences from the decision-maker's preferences over unambiguous acts. Tradeoff Consistency requires that if the utility gain from the outcome of f over that of g in the event A exceeds the utility gain from g over f in the event B, then f is preferred to g. More precisely, Tradeoff Consistency requires that preferences over acts can be rationalized consistently in this manner by an appropriate ranking of utility differences .

relatively simple class of betting preferences.[2]

By not assuming Savage's axiom P4, our main representation theorem allows for betting preferences over events to depend on the "stakes" of the bets involved. This generality is important since in the presence of ambiguity, P4 cannot be taken to be a requirement of rationality; indeed, there is a live interest in stake-dependent preference models (see e.g. Epstein-Le Breton (1993), Klibanoff et al. (2005)). When betting preferences are stake-dependent, they reflect both event attitudes (beliefs and ambiguity attitudes) as well as, more indirectly, consequence attitudes (utilities). The Stake Independence axiom P4 is thus necessary to achieve a separation of consequence and event attitudes as determinants of overall preferences. In Theorem 2 we show that it is also sufficient, and characterize the restrictions that betting preferences must satisfy to be consistent with utility sophistication.

In this analysis, utility sophistication has been defined relative to a hypothetically or situationally given likelihood relation. Is it possible to eliminate reference to beliefs as an independent, non-behavioral construct, and define utility sophistication in purely behavioral terms? While we suggest that this cannot be done without arbitrariness for preferences that are merely utility-sophisticated relative to sparse likelihood relations, we propose a definition of "revealed utility sophistication" which implies that the set of probabilistic beliefs relative to which the given preference ordering is construed as utility-sophisticated (in the above relative sense) must be "rich" and show that the proposed definition has attractive properties.

As an important side-benefit, refining and modifying earlier and related work (Ghirardato et al. (2004), Nehring (1996, 2001)), this allows one to define "revealed prob-

[2]One particularly important type of further specialization is based on the modelling of ambiguity attitudes (ambiguity aversion versus ambiguity seeking). In a companion paper, we argue that these can indeed be modelled naturally in terms of assumptions relating betting preferences and partial proabilistic beliefs (cf. Nehring 2001).

abilistic beliefs" in a natural manner. We argue that the restriction of this definition to revealed utility-sophisticated preferences makes it immune to the interpretative ambiguities that characterized these earlier contributions.

Comparison to the existing literature.—

While the existing literature has not yet attempted to define a distinct notion of utility sophistication, we show in section 5.2 that, after translation into the present framework, many models of decision making under ambiguity in the Anscombe-Aumann framework give rise to revealed utility-sophisticated preferences, starting from the seminal contributions of Schmeidler (1989) and Gilboa-Schmeidler (1989).

Other contributions, especially Ghirardato-Marinacci (2002) and Ghirardato et al. (2004), assume a utility-sophisticated viewpoint by assuming in the *interpretation* of their definitions and axioms that all departures from expected utility can be attributed to ambiguity. However, as argued by Epstein-Zhang (2001) and discussed further in sections 6 and 7, such an interpretational assumption may be arbitrary or inappropriate.

Organization of the paper.—

The remainder of the paper is organized as follows. In section 2, we introduce likelihood relations and their multi-prior representation, as well as basic assumptions on preferences maintained throughout. We then define the notion of utility sophistication and characterize it axiomatically (section 3), paying particular attention to the case of "stake-independent" (P4) betting preferences (section 4). In section 5, we study utility sophistication in various preference models in the literature, and establish a close link to standard models in the Anscombe-Aumann (1963) framework; inter alia, we show that CEU preferences are never utility-sophisticated with respect to a rich likelihood relation unless they are SEU. Section 6 quantifies out the likelihood relation to arrive at a definition of "revealed utility-sophisticated beliefs" and shows it to satisfy important desiderata. The definition naturally suggests an accompanying definition of "revealed probabilistic beliefs" as discussed in section 7. All proofs are contained in the appendix.

2. BACKGROUND

2.1. Equidivisible Likelihood Relations

Since utility sophistication is to be defined relative to a specified set of probabilistic beliefs, we shall model a decision maker in terms of two entities, a preference relation $\succsim$ over Savage acts and a comparative likelihood relation $\trianglerighteq$ describing some or all of his probabilistic beliefs. Formally, a likelihood relation is a partial ordering $\trianglerighteq$ on an algebra of events Σ in a state space Ω, with the instance $A \trianglerighteq B$ denoting the DM's judgment that A is at least as likely as B. We shall denote the symmetric component of $\trianglerighteq$ ("is as likely as") by $\equiv$. For now, we shall treat the likelihood relation as an independent primitive. To emphasize its typically non-exhaustive interpretation, the likelihood relation is frequently referred to as the decision-maker's (belief) "context"; an important example is the existence an independent randomization device as described in Example 1 below. The viability of the non-exhaustive interpretation of $\trianglerighteq$ will be formally supported in section 6, Proposition 6; in that section it is also shown how a fully behavioral definition of utility sophistication can be obtained by "quantifying out" the context $\trianglerighteq$. For further discussion of the general approach, see Nehring (2006) where the framework of "decision-making in the context of probabilistic beliefs" has been introduced.

A *prior* π is a finitely additive, non-negative set-function on Σ such that $\pi(\Omega) = 1$.

Given a likelihood relation $\trianglerighteq$, let $\Pi_{\trianglerighteq}$ denote its set of *admissible priors* defined by

$$\pi \in \Pi_{\trianglerighteq} \text{ if and only if, for all } A, B \in \Sigma,\ A \trianglerighteq B \text{ implies } \pi(A) \geq \pi(B).$$

For any $\trianglerighteq$, $\Pi_{\trianglerighteq}$ is a closed convex set in the product (or weak*) topology. Conversely, any non-empty set of priors Π induces an associated likelihood relation $\trianglerighteq_{\Pi}$ given by the unanimity condition

$$A \trianglerighteq_{\Pi} B \text{ if and only if } \pi(A) \geq \pi(B) \text{ for all } \pi \in \Pi.$$

A likelihood relation $\trianglerighteq$ is *coherent* if there exists $\Pi \neq \varnothing$ such that $\trianglerighteq = \trianglerighteq_{\Pi}$; in that case, $\trianglerighteq$ is said to be derived from Π. Clearly, if $\trianglerighteq$ is derived from Π, it is also derived from the closed convex hull of Π; it is therefore without loss of generality to assume sets of priors to be closed and convex. Furthermore, it is easily verified that $\trianglerighteq$ is coherent if and and only if it is derived from the set $\Pi_{\trianglerighteq}$; the set $\Pi_{\trianglerighteq}$ will therefore be referred to as *the* multi-prior representation of $\trianglerighteq$.[3]

A central role in the following will be played by likelihood relations with a convex-ranged multi-prior representation. The set of priors Π is *convex-ranged* if, for any event $A \in \Sigma$ and any $\alpha \in (0,1)$, there exists an event $B \in \Sigma$, $B \subseteq A$ such that $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi$. It is easily verified that a coherent likelihood relation $\trianglerighteq$ defined on a $\sigma-$algebra is derived from a convex-ranged set of priors if and only if it is *equidivisible*, i.e. if and only if, for all events $A \in \Sigma$, there exists an event $B \in \Sigma$ such that $B \subseteq A$ and $B \equiv A \backslash B$. Nehring (2006) contains an axiomatization of coherent equidivisible likelihood relations, and shows that they are derived from a unique closed convex set of priors, namely $\Pi_{\trianglerighteq}$; the latter result implies that, for any convex-ranged Π, $\Pi = \Pi_{(\trianglerighteq_{\Pi})}$. Slightly abusing terminology, we will refer to likelihood relations with

[3]Note that in general there may exist multiple closed convex sets Π from which a given coherent likelihood relation is derived; in such cases (which are precluded by the convex-rangedness assumption to follow) there is a loss of information in representing beliefs by likelihood relations rather than by sets of priors.

convex-ranged multi-prior representation $\Pi_{\trianglerighteq}$ as equidivisible, whether or not they are defined on a σ-algebra.[4] Evidently, the superrelation of any equidivisible relation is equidivisible.

Equidivisible likelihood relations are characterized by a rich set of unambiguous and conditionally unambiguous events. Say that $B \in \Sigma$ is *unambiguous given* A if, for some $\alpha \in [0,1]$, $\pi(B) = \alpha\pi(A)$ for all $\pi \in \Pi_{\trianglerighteq}$. Let Λ_A denote the family of events that are unambiguous given A; clearly, Λ_A is closed under finite disjoint union and under complementation, but not necessarily under intersection. An event A is *null* if $A \equiv \varnothing$, or, equivalently, if $\pi(A) = 0$ for all $\pi \in \Pi_{\trianglerighteq}$. For any non-null A and any $\pi \in \Pi_{\trianglerighteq}$, let $\overline{\pi}(./A)$ denote the restriction of $\pi(./A)$ to Λ_A, with $\overline{\pi}(B/A)$ denoting the unambiguous conditional probability of B given A. We will say that B is *unambiguous* if it is "unambiguous given Ω", and write Λ for Λ_Ω, as well as $\overline{\pi}$ for $\overline{\pi}(./\Omega)$. As a matter of further notion, let $\pi^-(A) = \min_{\pi\in\Pi} \pi(A)$ and $\pi^+(A) = \max_{\pi\in\Pi} \pi(A)$ the lower and upper probabilities of event A. Also, the indicator function associated with event A will be denoted by 1_A.

Example 1 (Continuous Randomization Device). The following translates the widely used Anscombe-Aumann (1963) framework as a likelihood relation. Consider a product space $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 is a space of "generic states" , and Ω_2 a space of "random states" with associated algebra Σ_1 and $\sigma-$algebra Σ_2, respectively. Let η denote a convex-ranged[5], finitely additive prior over random events Σ_2 . The "continuity" and stochastic independence of the random device are captured by the following coherent likelihood relation $\trianglerighteq_{AA}$ defined on the product algebra $\Sigma = \Sigma_1 \times \Sigma_2$; note that any $A \in \Sigma_1 \times \Sigma_2$ can be written as $A = \sum_i S_i \times T_i$, where the $\{S_i\}$ form a

[4]On an algebra, convex-rangedness is characterized by a bit more than equidivisibility proper; convex-rangedness on algebras arises naturally in the Anscombe-Aumann context $\trianglerighteq_{AA}$ defined in Example 1 below.

[5]That is, $\{\eta\}$ is convex-ranged in Σ_2.

finite partition of Ω_1 :[6]

$$\sum_i S_i \times T_i \trianglerighteq_{AA} \sum_i S_i \times T_i' \text{ if and only if } \eta(T_i) \geq \eta(T_i') \text{ for all } i.$$

Clearly, there exists a unique set of priors Π_{AA} representing $\trianglerighteq_{AA}$; indeed, Π_{AA} is simply the set of all product-measures $\pi_1 \times \eta$ where π_1 ranges over all finitely additive measures on Σ_1. Note that the convex-rangedness of Π_{AA} is a straightforward consequence of the convex-rangedness of η.

In general, a decision-maker will have further probabilistic beliefs captured by a likelihood relation $\trianglerighteq$ that strictly contains the context $\trianglerighteq_{AA}$; this relation evidently inherits the equidivisibility of $\trianglerighteq_{AA}$.

Example 2 (Limited Imprecision). A particular way to formalize the intuitive notion of a limited extent of overall ambiguity is to assume that Σ is a σ-algebra and that Π is the convex hull of a finite set Π' of non-atomic, countably additive priors. Due to Lyapunov's (1940) celebrated convexity theorem, Π is convex-ranged. The priors $\pi \in \Pi'$ can be interpreted as a finite set of hypotheses a decision-maker deems reasonable without being willing to assign precise probabilities to them. Finitely generated sets of priors occur naturally, for example, when an individual bases his beliefs on the views of a finite set of experts who have precise probabilistic beliefs $\trianglerighteq_i$ but disagree with each other. The decision maker may naturally want to respect all instances of expert agreement; these are represented by the unanimity relation $\trianglerighteq_I = \cap_{i \in I} \trianglerighteq_i$ which is evidently finitely generated.

In the following, when it is necessary to refer to asymmetric likelihood comparisons, rather than using simply the asymmetric component $\triangleright$ of $\trianglerighteq$, it is often more appropriate to use the "uniformly more likely" relation $\triangleright\triangleright$, where $A \triangleright\triangleright B$ if

[6]The relation $\trianglerighteq_{AA}$ is easily characterized axiomatically; see Nehring (2006).

$\min_{\pi\in\Pi_{\trianglerighteq}}(\pi(A)-\pi(B))>0$. In general, $\rhd\rhd$ is a proper subrelation of $\rhd$. For further discussion and a characterization of $\rhd\rhd$ in terms of $\trianglerighteq$ for equidivisible contexts, see Nehring (2006).

2.2 Maintained Assumptions on Preferences

Consider now a DM described by a preference ordering over acts $\succsim$ and a coherent likelihood relation $\trianglerighteq$; we will typically write Π for $\Pi_{\trianglerighteq}$. Let X be a set of *consequences*. An *act* is a finite-valued mapping from states to consequences, $f:\Omega\rightarrow X$, that is measurable with respect to the algebra of events Σ; the set of all acts is denoted by $\mathcal{F}$. A *preference ordering* $\succsim$ is a weak order (complete and transitive relation) on $\mathcal{F}$. An act is *unambiguous* if it is measurable with respect to the system of unambiguous events Λ; the set of all unambiguous acts is denoted by $\mathcal{F}^{ua}$. The restriction of $\succsim$ to $\mathcal{F}^{ua}$ represents the decision maker's risk preferences.

We shall write $[x_1 \text{ on } A_1; x_2 \text{ on } A_2; ...]$ for the act with consequence x_i in event A_i; constant acts $[x \text{ on } \Omega]$ are typically referred to by their constant consequence x. To prepare the ground for the subsequent analysis, we now introduce the basic substantive and regularity assumptions that will be maintained throughout.

The belief context constrains most directly preferences over bets. A *bet* is a pair of acts with the same two outcomes, i.e. a pair of the form $([x \text{ on } A; y \text{ on } A^c], [x \text{ on } B; y \text{ on } B^c])$. Fundamental is the following rationality requirement on the relation between preferences and probabilistic beliefs.

Axiom 1 (Compatibility) *For all $A,B\in\Sigma$ and $x,y\in X$:*

i) $[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c]$ *if* $A\trianglerighteq B$ *and* $x\succsim y$, *and*

ii) $[x \text{ on } A; y \text{ on } A^c] \succ [x \text{ on } B; y \text{ on } B^c]$ *if* $A\rhd\rhd B$ *and* $x\succ y$.

Throughout, preferences will be assumed to be eventwise monotone in the following weak version of Savage's axiom P3.

Axiom 2 (Eventwise Monotonicity) *For all acts* $f \in \mathcal{F}$, *consequences* $x, y \in X$ *and events* $A \in \Sigma$: $[x$ *on* $A; f(\omega)$ *elsewhere*$] \succsim [y$ *on* $A; f(\omega)$ *elsewhere*$]$ *whenever* $x \succsim y$.

The following condition ensures that the set of consequences is sufficiently rich.

Axiom 3 (Solvability) *For any* $x, y \in X$ *and* $T \in \Lambda$, *there exists* $z \in X$ *such that* $z \sim [x, T; y, T^c]$.

For expositional simplicity, especially in the stake-dependent case, we shall assume throughout that consequences are bounded in utility.

Axiom 4 (Boundedness) *There exist* x^-,$x^+ \in X$ *such that, for all* $x \in X$, $x^- \precsim x \precsim x^+$.

To obtain a real-valued representation, some Archimedean property is usually assumed. The following is sufficiently strong to help deliver the main result, Theorem 1, below. Note that it is defined relative to the belief context and presumes its equidivisibility. Substantively, as confirmed by the upcoming representation result, Proposition 1, it asserts that if acts are changed on events of sufficiently small upper probability, strict preference does not change.

Axiom 5 (Archimedean) *For any* $x, y \in X$ *such that* $x \succsim y$ *and any acts* $f = [x$ *on* A, y *on* B; f *otherwise*$]$ *and* g *such that* $f \succ g$ *(resp.* $f \prec g$*) and such that* A *is unambiguous given* $A + B$, *there exists an event* C *that is unambiguous given* $A + B$ *such that* $C \lhd\lhd A$ *and* $f' = [x$ *on* C, y *on* $(A+B)\backslash C$; f *otherwise*$] \succ g$ *(resp.* $f' \prec g$*).*

Since axioms 3 through 5 will usually show up together in the following results, it is convenient to refer to a preference ordering satisfying these three axioms as *regular.*

Let $\mathcal{Z}$ denote the set of finite-valued, Σ-measurable functions $Z : \Omega \rightarrow [0,1]$. Using the above axioms, we will now establish a basic representation theorem that ensures the existence of a utility function u mapping X onto the unit interval together with an evaluation functional I: $\mathcal{Z} \rightarrow [0,1]$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$, for all $f, g \in \mathcal{F}$.

I is *normalized* if $I(c1_\Omega) = c$ for all $c \in [0,1]$ and $I(1_T) = \overline{\pi}(T)$ for all $T \in \Lambda$. Note that for normalized I, u is calibrated in terms of probabilities, i.e. satisfies $u(z) = \overline{\pi}(T)$ whenever $z \sim [x^+, T; x^-, T^c]$.[7] I is *monotone* if $I(Y) \geq I(Z)$ whenever $Y \geq Z$ (pointwise); I is *compatible with* $\trianglerighteq$ if $I(1_A) \geq I(1_B)$ whenever $A \trianglerighteq B$ and $I(1_A) > I(1_B)$ whenever $A \rhd\rhd B$; I is *event-continuous* if, for any $x, y \in X$, $Z \in \mathcal{Z}$, $E \in \Sigma$, $A \in \Lambda_E$ with $A \subseteq E$ and any increasing sequence $\{A_n\}$ of events contained in A such that $\overline{\pi}(A_n/E)$ converges to $\overline{\pi}(A/E)$, $I(x1_{A_n} + y1_{E\setminus A_n} + Z1_{E^c})$ converges to $I(x1_A + y1_{E\setminus A} + Z1_{E^c})$.

Proposition 1 *Let $\trianglerighteq$ be a equidivisible belief context. The following two statements are equivalent:*

i) the preference ordering $\succsim$ is compatible with $\trianglerighteq$, eventwise monotone and regular (Archimedean, solvable, and bounded).

ii) there exist an onto utility-function $u : X \rightarrow [0,1]$ and a functional $I : \mathcal{Z} \rightarrow [0,1]$ that is monotone, event-continuous and compatible with $\trianglerighteq$ such that

$$f \succsim g \text{ if and only if } I(u \circ f) \geq I(u \circ g), \text{ for all } f, g \in \mathcal{F}.$$

There is a unique pair (u, I) satisfying ii) such that I is normalized.

In the sequel, preferences over bets will play a special role. We shall frequently but not always assume that preferences over bets depend only on the events involved, not on the stakes. This is captured by Savage's axiom P4.

[7] To see this, $z \sim [x^+, T; x^-, T^c]$ implies $I(u(z)1_\Omega) = I(1_T)$. Thus by the two normalization conditions $u(z) = I(u(z)1_\Omega) = I(1_T) = \overline{\pi}(T)$.

Axiom 6 (Stake Independence, P4) *For all $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$ and all $A, B \in \Sigma$:*

$$[x \text{ on } A; y \text{ on } A^c] \succsim [x \text{ on } B; y \text{ on } B^c] \text{ iff } [x' \text{ on } A; y' \text{ on } A^c] \succsim [x' \text{ on } B; y' \text{ on } B^c].$$

We will frequently use the notation $A \succsim_{bet} B$ for the preference $[x^+ \text{ on } A; x^- \text{ on } A^c] \succsim [x^+ \text{ on } B; x^- \text{ on } B^c]$. This notation is primarily motivated by the stake-independent case in which the relation $\succsim_{bet}$ completely summarizes the DM's beliefs and ambiguity attitudes.[8] If preferences are utility-sophisticated, this turns out to be the case even when betting preferences are stake-dependent.

Compatibility of betting preferences with a given belief context ensures a ranking of bets on unambiguous events T according to their unambiguous probability $\overline{\pi}(T)$. Under the assumptions of Proposition 1, there exists a unique set-function $\rho : \Sigma \to [0,1]$ representing $\succsim_{bet}$ that is additive on unambiguous events and has $\rho(\Omega) = 1$; ρ assigns to each event the probability $\overline{\pi}(T)$ of any unambiguous event to which it is indifferent. If I is normalized, clearly $\rho(A) = I(1_A)$. The properties on I introduced above translate naturally into properties of ρ. In particular, ρ is *compatible with* $\trianglerighteq$ if $\rho(A) \geq \rho(B)$ whenever $A \trianglerighteq B$ and $\rho(A) > \rho(B)$ whenever $A \rhd\rhd B$; finally, ρ is *event-continuous* if, for any disjoint $B, E \in \Sigma$, any $A \in \Lambda_E$ with $A \subseteq E$ and any increasing (respectively decreasing) sequence $\{A_n\}$ of events contained in (resp. containing) A such that $\overline{\pi}(A_n/E)$ converges to $\overline{\pi}(A_n/E)$, $\rho(A_n + B)$ converges to $\rho(A + B)$.

[8]In the stake-independent case, $\succsim_{bet}$ is formally equivalent to Savage's "revealed likelihood" relation; such an interpretation is not warranted in the presence of ambiguity, however, since the relation incorporates not merely beliefs in this case (however construed) but also ambiguity attitudes.

3. UTILITY SOPHISTICATED PREFERENCES

The fundamental goal of this paper is to provide axiomatic foundations for the intuitive notion of a decision-maker who departs from expected-utility *only* for reasons of ambiguity. This idea can be formulated transparently with reference to exogenously specified belief context $\trianglerighteq$ in terms of the following property of utility sophistication.

Definition 1 (Utility Sophistication) *The preference relation* $\succsim$ *is utility-sophisticated with respect to* Π *if there exists* $u : X \to \mathbf{R}$ *such that* $f \succsim g$ *(resp.* $f \succ g$*) whenever* $E_\pi u \circ f \geq E_\pi u \circ g$ *(resp.* $E_\pi u \circ f > E_\pi u \circ g$*) for all* $\pi \in \Pi$; $\succsim$ *is utility-sophisticated with respect to the context* $\trianglerighteq$ *if it is utility-sophisticated with respect to* $\Pi_{\trianglerighteq}$.[9]

To motivate the key axiom underlying utility sophistication, consider first the ranking of unambiguous (risky) acts for which utility sophistication entails EU maximization with respect to the probability measure $\overline{\pi}$. Specifically, consider choices among unambiguous acts f and g with two outcomes, each of which with subjective probability one half, and assume that $f = [x \text{ on } A; y \text{ on } A^c]$ and $g = [x' \text{ on } A; y' \text{ on } A^c]$ with $x \succ x'$, $y' \succ y$ and $A \equiv A^c$. According to a classical interpretation of expected utility theory, a DM should choose f over g exactly if he assesses the utility gain from x over x' to exceed the utility loss of obtaining y rather than y'. The preference of f over g by a DM committed to this principle reveals a greater utility gain from x over x' than from y' over y. Thus, if the DM chooses $f = [x \text{ on } A; y \text{ on } A^c]$ over $g = [x' \text{ on } A; y' \text{ on } A^c]$, consistency requires that he also choose the act $[x \text{ on } E; y \text{ on } E^c]$ over $[x' \text{ on } E; y' \text{ on } E^c]$, where E is any other event that is equally likely to its complement,

[9] By the uniqueness of the multi-prior representation of equidivisible contexts mentioned above, for convex-ranged Π, utility-sophistication with respect to Π is the same as utility-sophistication with respect to $\trianglerighteq_\Pi$. However, without equidivisibility, it may be that $\Pi \subsetneq \Pi_{(\trianglerighteq_\Pi)}$, so that utility-sophistication with respect to a set of priors cannot be equated to utility-sophistication with respect to the associated likelihood relation.

$E \equiv E^c$.[10]

The following "Trade-off Consistency" axiom generalizes this consistency requirement to choices of the form $f = [x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}]$ versus $g = [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$ whenever the events A and B are judged equally likely ($A \equiv B$), whether or not they are unambiguous themselves. Since the relative probabilities of the events A and B are judged to be equal, the comparison between the acts f and g boils down to a comparison of the respective utility gains as the decisive decision criterion also in this more general case. In order to compare the acts f and g, the DM simply does not need to consider his (possibly imprecise) assessment of the likelihood of the union $A + B$, nor the payoffs in states outside $A + B$. This motivates the following rationality axiom according to which the DM's preferences must be consistently rationalizable in terms of utility differences in the manner just described.

Axiom 7 (Tradeoff Consistency) *For all $x, y, x', y' \in X$ such that $x \succsim x'$, acts $f, g \in \mathcal{F}$ and events A disjoint from B and A' disjoint from B' such that $A \equiv B \rhd\!\rhd \emptyset$ and $A' \equiv B'$:*

if $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$,

then $[x \text{ on } A'; y \text{ on } B'; g(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'; y' \text{ on } B'; g(\omega) \text{ elsewhere}]$.

Note the restriction to events A and B of strictly positive lower probability; it ensures that the premise "$[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$" implies that the utility advantage of x over x' is not smaller than that of y' over y if the latter is positive. Note also that, for equidivisible contexts $\trianglerighteq$,

[10] This consistency requirement is in fact axiom 2 of Ramsey's (1931) seminal contribution. Conditions requiring consistency of trade-offs across choices have been used elsewhere in the axiomatizations of SEU and CEU theory; see in particular Wakker (1989).

Trade-off Consistency entails Eventwise Monotonicity.[11]

For Trade-off Consistency to allow for ambiguity, the restriction to equally likely rather than merely indifferent events A and B respectively A' and B' is crucial. Indeed, if one replaced this clause by a weaker one requiring these events to be indifferent as bets ($A \sim_{bet} B$ and $A' \sim_{bet} B'$), the resulting stronger axiom would force betting preferences to satisfy the additivity condition

$$A \sim_{bet} B \text{ if and only if } A + C \sim_{bet} B + C, \text{ for any } A, B, \text{ and } C,$$

and thereby impose SEU.

Trade-off Consistency becomes particularly powerful if the underlying belief context is equidivisible. For in this case not only does it entail utility sophistication, utility sophistication itself becomes particularly powerful, as it implies that a DM's multi-act preferences are determined by his preferences over unambiguous acts together with his preferences over bets. Mathematically, this is the consequence of the existence of a non-linear expectation operator that reflects the DM's ambiguity attitudes.

The key to deriving this built-in expectation operator is the mixture-space structure induced by equidivisible belief contexts as introduced in Nehring (2006). With each $Z \in \mathcal{Z}$, one can associate an equivalence class $[Z]$ of events $A \in \Sigma$ as follows. Let $A \in [Z]$ if there exists a partition $\{E_i\}$ of Ω such that $Z = \sum z_i 1_{E_i}$, and such that, for all $i \in I$ and $\pi \in \Pi : \pi(A \cap E_i) = z_i \pi(E_i)$. Note that $[Z]$ is non-empty by the convex-rangedness of Π. Moreover, it is easily seen that for any two $A, B \in [Z]$: $\pi(A) = \pi(B)$ for all $\pi \in \Pi$, and thus $A \equiv B$. Hence by Compatibility also $A \sim_{bet} B$. One therefore arrives at a well-defined ordering of random variables $\widehat{\succsim_{bet}}$ on $\mathcal{Z}$ by

[11]Indeed, for equivisible contexts, Eventwise Monotonicity is simply Tradeoff Consistency restricted to cases in which $x = y$, $x' = y'$, $A + B = \Omega$ and $A \equiv B$, with $A' + B'$ ranging over all events $E \in \Sigma$. It is for the purpose of enabling this implication that we have not required the condition $y' \succsim y$ in the definition of Tradeoff Consistency.

setting

$$Y \widehat{\succsim_{bet}} Z \text{ if } A \succsim_{bet} B, \text{ for any } A \in [Y] \text{ and } B \in [Z].$$

Let $\widehat{\rho}$ denote the associated unique extension of ρ to $\mathcal{Z}$ given by

$$\widehat{\rho}(Z) = \rho(A) \text{ for any } A \in [Z]. \tag{1}$$

Again, by the construction of the mixture-space, this is well-defined, and one has

$$Y \widehat{\succsim_{bet}} Z \text{ if and only if } \widehat{\rho}(Y) \geq \widehat{\rho}(Z).$$

Clearly, by Compatibility, $\widehat{\rho}$ is a monotone, normalized evaluation functional on $\mathcal{Z}$. We shall call $\widehat{\rho}(Z)$ the "intrinsic integral" of Z.

We are now in a position to state the main result of the paper.

Theorem 1 *Let $\trianglerighteq$ be an equidivisible belief context. The following three statements are equivalent:*

1. The preference ordering $\succsim$ is regular, trade-off consistent and compatible with $\trianglerighteq$.

2. The preference ordering $\succsim$ is Archimedean and utility-sophisticated with respect to $\trianglerighteq$, for some onto function $u : X \to [0,1]$.

3. There exists an onto function $u : X \to [0,1]$ and an event-continuous set-function ρ compatible with $\trianglerighteq$ with associated intrinsic integral $\widehat{\rho}$ defined by (1) such that, for all $f, g \in \mathcal{F}$:

$$f \succsim g \text{ iff } \widehat{\rho}(u \circ f) \geq \widehat{\rho}(u \circ g).$$

Theorem 1 achieves two things. First of all, it delivers an axiomatic foundation for utility-sophisticated preferences when the underlying belief context is equidivisible and when the set of consequences is rich; both of these assumptions are used essentially in the derivation. As a significant surplus value, it shows that utility sophistication in the equidivisible case entails the existence of an intrinsic-integral representation. This implies that multi-act preferences are completely determined by event attitudes (captured by betting preferences and represented by ρ) and consequence attitudes (capture by preferences over unambiguous acts and represented by u).[12] By consequence, all departures from SEU can be fully explained by non-additivity of betting preferences.

4. SEPARATING EVENT ATTITUDES FROM CONSEQUENCE ATTITUDES

As an important dimension of its generality, Theorem 1 does not assume Stake Independence (Savage's axiom P4). While in the context of probabilistically sophisticated preferences P4 is typically viewed as a rationality axiom expressing consistency of revealed likelihood judgements, this interpretation is no longer viable under ambiguity, since in this more general context betting preferences may reflect not just likelihood judgments but also ambiguity attitudes.

We submit that, having lost its original rationale, under ambiguity P4 can no longer be viewed as a rationality condition as there does not seem to be anything genuinely "inconsistent" or even strange in stake dependence. For example, in the context of an Ellsberg urn experiment, a decision maker may well prefer a bet of \$1 on a draw from an urn with unknown composition (getting \$0 otherwise) over a bet of \$1 on an event with an objective probability of 40%, and exhibit at the same time the opposite preference once the stakes are raised to \$10,000 (versus \$0). Such preferences can naturally interpreted as reflecting greater ambiguity aversion at greater possible

[12] Since $\widehat{\rho}$ is constructed from ρ with reference to the context $\trianglerighteq$, it may appear that the context also plays a role in determining $\succsim$. We shall show however below that this is not the case, at least in the standard case of preferences satisfying P4; this follows immediately from the final assertion in Proposition 2 below.

gains. The bottom line is that P4 should be viewed as a well-behavedness rather than rationality condition.[13]

In the absence of P4, betting preferences over extreme stakes represented by $\succsim_{bet}$ fail to describe preferences over bets with intermediate stakes. However, if preferences are utility-sophisticated, $\succsim_{bet}$ determines the intrinsic integral $\widehat{\rho}$, and thus all preferences (in particular: all betting preferences) are determined once consequence / risk attitudes captured by u are given. As a result, preferences over bets with intermediate stakes will partly depend on these attitudes. By modus tollens, Stake Independence P4 is therefore necessary for a clean separation of consequence and pure event attitudes (beliefs and ambiguity attitudes). In this section, we will show that Stake Independence is also sufficient for such a separation and characterize the restrictions on stake-independent betting preferences imposed by utility sophistication.

First, P4 turns out to be equivalent to the following invariance properties of betting preferences.

Axiom 8 (Union Invariance) *For any $T \in \Lambda$ and any $A, B \in \Sigma$ disjoint from T :* $\quad A \succsim_{bet} B$ *if and only if* $A + T \succsim_{bet} B + T.$

Axiom 9 (Splitting Invariance) *For any $A, B \in \Sigma$ and any partitions of A and B into equally likely subevents $\{A_1, ..., A_n\}$ and $\{B_1, ..., B_n\}$,with $A_i \equiv A_j$ and $B_i \equiv B_j$ for all $i, j \leq n$,* $\quad A \succsim_{bet} B$ *if and only if* $A_1 \succsim_{bet} B_1$.

The two invariance axioms are intuitive and have intrinsic appeal even in the absence of utility sophistication. In view of their appeal, it is not surprising that both conditions have some *incognito* precedents in the literature. On the one hand, Epstein-Zhang (2001) effectively build Union Invariance into their very definition of an event

[13] Such a view also resolves the conflict between dynamic consistency and P4 outside SEU observed by Epstein-Le Breton (1993). Moreover, note that restricted to bets on unambiguous events, P4 still obtains as an implication of compatibility with the underlying belief context, and does not need to be assumed independently.

T as "revealed unambiguous".[14] Splitting Invariance as well is not entirely new, as it can be reformulated as a restriction on betting preferences over independent events. Say that events A and B are independent if $\pi(B/A) = \pi(B/A^c)$ for all $\pi \in \Pi$. If preferences are compatible with the equidivisible context $\trianglerighteq$ as maintained, then it can be shown easily that they satisfy Splitting Invariance if and only if

$$\rho(A \times B) = \rho(A)\rho(B) \tag{2}$$

for all $A \in \Sigma$ and $B \in \Lambda$ such that A and B are independent. In defining "product capacities" for independent events, authors such as Ghirardato (1997) and Hendon et al. (1996) have appealed to generalizations of (2) that allow both events A and B to be ambiguous.

Alternatively, P4 can be characterized in terms of "constant-linearity" of the evaluation functional I. An evaluation functional I (in particular $\widehat{\rho}$) is *constant-additive* if $I(Y + c1_\Omega) = I(Y) + c$; I is *positively homogeneous* if $I(\alpha Y) = \alpha I(Y)$ for any $\alpha \in [0,1]$; I is *constant-linear* if it is constant-additive and positively homogeneous. Again, this condition is of independent interest and has been studied in the literature, especially by Ghirardato et al. (2004) . Note that, for two-outcome acts $[x$ on $A; y$ on $A^c]$ with $x \succsim y$, a constant-linear intrinsic integral has the following simple "biseparable" representation (Ghirardato-Marinacci (2001))

$$\widehat{\rho}(u \circ f) = u(x)\rho(A) + u(y)(1 - \rho(A)).$$

Constant Linearity can be viewed as a cardinal stake-invariance property of multi-act preferences. The following result derives this property from the weaker and arguably more primitive ordinal P4 property, assuming utility sophistication.

[14] That is to say, Epstein-Zhang's definition of revealed unambiguous events is such that Union Invariance (applied to revealed unambiguous events instead of Λ) holds by definition.

Theorem 2 *Suppose $\succsim$ is regular, trade-off consistent and compatible with the equidivisible context $\trianglerighteq$. Then the following three statements are equivalent.*

1. *$\succsim$ satisfies P4.*
2. *I is constant-linear.*
3. *$\succsim$ satisfies Union and Splitting Invariance.*

In the Appendix, we demonstrate the implications 2) $\Longrightarrow$ 1), 1) $\Longrightarrow$ 3), and 3) $\Longrightarrow$ 2). The first implication 2) $\Longrightarrow$ 1) is valid for any constant-linear evaluation functional I, without reference to an equidivisible belief context. The second implication 1) $\Longrightarrow$ 3) relates two different properties of betting preferences, making essential use of utility sophistication. Finally, the implication 3) $\Longrightarrow$ 2) mirrors the invariance properties of betting preferences in corresponding properties of the intrinsic integral $\widehat{\rho}$; utility sophistication closes the circle via the identity $I = \widehat{\rho}$.[15]

Theorem 2 entails the desired separation of event attitudes from consequence valuations, as formalized by the following result. Note that while Theorem 2 shows that utility sophistication imposes Union- and Splitting Invariance on stake-independent betting preferences, the following Proposition 2 adds that these are in fact the *only* restrictions on betting preferences imposed by utility sophistication. In this result, $\succsim_{ua}$ represent given (EU maximizing) risk-preferences while $\succsim_{\mathcal{B}}$ represents given betting preferences; the two must agree on the set of bets on unambiguous events. The result asserts that these are jointly consistent with utility sophistication if and only if $\succsim_{\mathcal{B}}$ satisfies Union- and Splitting-Invariance, and that in this case they determine

[15] It may seem a bit surprising that utility sophistication entails non-trivial restrictions on betting preferences given stake-independence. To see how this is possible, note that while utility sophistication by iteslf does not restrict betting preferences *for given stakes x* and y, it does constrain betting preferences across stakes, even in the absence of P4. The existence of such restrictions explains how the imposition of *further* restrictions on betting preferences across stakes such as P4 can entail restrictions on betting preferences for given stakes.

the overall preference ordering uniquely.

Proposition 2 *Let $\trianglerighteq$ be an equidivisible context. Let $\succsim_{ua}$ be a preference ordering on unambiguous acts $\mathcal{F}^{ua}$ that is trade-off consistent, regular, and compatible with $\trianglerighteq$ restricted to Λ. Furthermore, let $\succsim_{\mathcal{B}}$ be a complete and transitive relation on Σ that is Archimedean and compatible with $\trianglerighteq$ such that $(\succsim_{ua})_{bet}$ agrees with the restriction of $\succsim_{\mathcal{B}}$ to $\Lambda \times \Lambda$. Then the following two statements are equivalent:*

1. $\succsim_{\mathcal{B}}$ satisfies Union and Splitting Invariance with respect to $\trianglerighteq$.

2. There exists a preference ordering $\succsim$ on all of $\mathcal{F}$ that is stake-independent, Archimedean and tradeoff-consistent with respect to $\trianglerighteq$ and whose restrictions to $\mathcal{F}^{ua}$ and $\succsim_{bet}$ agree with $\succsim_{ua}$ and $\succsim_{\mathcal{B}}$, respectively.

The preference ordering specified in (2) is unique. Furthermore, this ordering does not depend on the context $\trianglerighteq$.[16]

5. UTILITY SOPHISTICATION IN PARTICULAR MODELS

5.1 Models with EU aggregators

Note that Utility Sophistication can be reformulated essentially equivalently using the notion of an aggregator Ψ of the expected-utility values under the admissible priors in Π. To this purpose, let $\mathbb{E}_{\Pi} : \mathcal{Z} \to [0,1]^{\Pi}$ denote the evaluation operator given by $\mathbb{E}_{\Pi}(Z) = (E_{\pi}Z)_{\pi\in\Pi}$ for $Z \in \mathcal{Z}$; for any act f, $\mathbb{E}_{\Pi} u \circ f$ is the vector of expected utilities $(E_{\pi} u \circ f)_{\pi\in\Pi}$ induced by f.[17] Clearly, $\mathbb{E}_{\Pi}(\mathcal{Z})$ is a convex subset of $[0,1]^{\Pi}$. An EU aggregator is simply a monotone mapping $\Psi : \mathbb{E}_{\Pi}(\mathcal{Z}) \to [0,1]$. Then, up to minor technicalities, Utility Sophistication with respect to Π is equivalent to

[16] The latter statement is made formally precise in the proof of Proposition 2.

[17] Note also that in view of the Krein-Milman theorem one could restrict attention to the subvector $(E_{\pi} u \circ f)_{\pi\in Ext(\Pi)}$.

the existence of an EU aggregator Ψ, an operator $\mathbb{E}_\Pi$ and a utility-function u such that

$$f \succsim g \text{ if and only if } \Psi\left(\mathbb{E}_\Pi\left(u \circ f\right)\right) \geq \Psi\left(\mathbb{E}_\Pi\left(u \circ g\right)\right), \text{ for any } f, g \in \mathcal{F}.$$

A variety of models in the literature can be put in this form. For example, the Minimum Expected Utility ("MEU") model due to Gilboa-Schmeidler (1989) corresponds to the aggregator

$$\Psi\left(U\right) = \min_{\pi \in \Pi} U_\pi.$$

A natural generalization admitting ambiguity-seeking is derived from evaluating acts according to the entire range of expectations under Π, with

$$\Psi\left(U\right) = \Upsilon\left(\min_{\pi \in \Pi} U_\pi, \max_{\pi \in \Pi} U_\pi\right),$$

for some monotone function Υ; see e.g. Jaffray (1989). Such "Interval EU" preferences satisfy P4 if and only if Υ is linear, i.e. if and only only if Ψ can be written as

$$\Psi\left(U\right) = \alpha \max_{\pi \in \Pi} U_\pi + \left(1 - \alpha\right) \min_{\pi \in \Pi} U_\pi,$$

which is Hurwicz's classical optimism-pessimism criterion.[18] Dubbed α-MEU, it has been axiomatized by Ghirardato et al. (2004) and Kopylov (2002).

Klibanoff et al. (2005) study preferences associated with aggregators

$$\Psi\left(U\right) = E_\mu \phi\left(U_\pi\right),$$

where μ is a probability-measure on Π, and $\phi : \mathbf{R} \to \mathbf{R}$ is strictly increasing and continuous (typically smooth). If ϕ is smooth, then bets $[x \text{ on } A; y \text{ on } A^c]$ with small stakes (i.e. with $u(x) - u(y)$ close to zero), are evaluated approximately according to $E_\mu \pi$. Thus, such preferences will satisfy P4 if and only if ϕ is linear, i.e. SEU.

[18]This follows from Theorem 2 together with Ghirardato et al.'s (2004) demonstration of the equivalence of the constant-linearity of I and the constant-α representation.

Another fairly diverse class of preferences with an EU aggregator representation has been studied in Siniscalchi's (2003) "plausible priors" model; all of these satisfy P4 by construction.

Finally, somewhat outside the present framework by considering choice-functions rather than weak orders, in Nehring's (1991,2000) "Simultaneous Expected Utility" model $\Psi(U)$ is the lexicographic minimum of appropriately renormalized expected utilities; the renormalization allows an interpretation of the solution as a bargaining solution among alternative selves associated with the extremal priors $ext(\Pi)$. All of the above contributions are situated in variants of the Anscombe-Aumann (1963) framework. This is no accident: indeed, by translating these contributions into the present setting, we will see that the typical assumptions made there imply utility sophistication with respect to the AA context $\trianglerighteq_{AA}$.

5.2 Utility Sophistication in the Anscombe-Aumann Framework

The Anscombe-Aumann (1963) framework is distinguished by taking acts to be mappings from states to probability distributions of consequences, rather than simply as mappings from states to consequences as in the Savage (1954) framework. These probability distributions are interpreted as objective probabilities of the realizations of an external random device ("roulette lotteries") that is not part of the explicitly modeled state space. In section 2, we have restated this description as an equidivisible context $\trianglerighteq_{AA}$. We will begin by showing how a preference relation over Savage acts can be redescribed as a preference relation over Anscombe-Aumann (AA-) acts and vice versa.

Formally, an AA-act F is a finite-valued Σ_1-measurable mapping from the subjective state space Ω_1 to the set of probability distributions on X with finite support $\mathcal{L}$. Let $\mathcal{F}^{AA}$ denote their set. Denoting elements of $\mathcal{L}$ by $q = (q^x)_{x \in X}$, with q^x as the probability of obtaining x under q, one can write $F = [q_1 \text{ on } S_1; q_2 \text{ on } S_2; ...]$ in analogy to the notation for Savage acts. Note that since Σ is the product algebra of Σ_1 and Σ_2, any Savage act f can be written in the form $[x_{i,j} \text{ on } S_i \times T_{i,j}]_{i \leq n, j \leq n_i}$ for appropriate n and $\{n_i\}_{i \leq n}$. One can thus associate with any Savage act $f = [x_{ij} \text{ on } S_i \times T_{ij}]$ the AA-act $F(f) = [p_i \text{ on } S_i]$, with $p_i^x = \sum_{j \leq n_i, x_{ij} = x} \eta(T_{i,j})$; the AA-act $F(f)$ associates with any subjective state $\omega \in \Omega_1$ the lottery that yields the consequence x with unambiguous (subjective) probability entailed by the likelihood judgments $\trianglerighteq_{AA}$. By the equidivisibility of $\trianglerighteq_{AA}$, this mapping is onto, i.e. any AA-act is the image of some Savage act.

In order to associate with the given preference relation $\succsim$ over Savage acts a well-defined preference relation over AA acts, one needs to extend the assumption that preferences are compatible with respect to the context $\trianglerighteq_{AA}$ in the following natural way.[19]

Axiom 10 (Strong Compatibility) *For all $f \in \mathcal{F}$, $x, y \in X$ and $A, B \subseteq C \in \Sigma$:*

i) $[x \text{ on } A; y \text{ on } C \backslash A;\ f \text{ elsewhere}] \succsim [x \text{ on } B; y \text{ on } C \backslash B;\ f \text{ elsewhere}]$ *if* $A \trianglerighteq B$ *and* $x \succsim y$, *and*

ii) $[x \text{ on } A; y \text{ on } C \backslash A;\ f \text{ elsewhere}] \succ [x \text{ on } B; y \text{ on } C \backslash B;\ f \text{ elsewhere}]$ *if* $A \rhd\!\rhd B$ *and* $x \succ y$.

Note that Compatibility is simply Strong Compatibility restricted to the case of $C = \Omega$; in turn, Strong Compatibility is entailed by Utility Sophistication.[20]

The lottery p *stochastically dominates* the lottery q if, for all $y \in X$, $\sum_{x: x \succsim_{AA} y} p^x \geq \sum_{x: x \succsim_{AA} y} q^x$; p *stochastically dominates* q *strictly* if at least one of these inequalities is strict. The AA-act $F = [p_i \text{ on } S_i]$ *(strictly) stochastically dominates* the AA-act $F = [q_i \text{ on } S_i]$ if p_i (strictly) stochastically dominates q_i for every i.

[19] Strong Compatibility is called "Likelihood Consequentialism" in Nehring (2006) where it has been introduced and discussed in greater detail.

[20] The second, strict part of Strong Compatibility plays no role in the following; we have included it only to ensure that Stong Compatibility entails Compatibility.

Fact 1 *The following two conditions are equivalent for a weak order* $\succsim$ *on* $\mathcal{F}$*:*

i) $\succsim$ *is strongly compatible with* $\trianglerighteq_{AA}$

ii) For all f, g *such that* $F(f)$ *stochastically dominates* $F(g)$ *(resp. strictly stochastically dominates)* $f \succsim g$ *(resp.* $f \succ g$*).*

It is immediate from part ii) that if $\succsim$ is strongly compatible with $\trianglerighteq_{AA}$, any f, f' such that $F(f) = F(f')$ must be indifferent. One thus obtains a well-defined weak order on $\mathcal{F}^{AA}$ by setting

$$F \succsim_{AA} G :\Leftrightarrow f \succsim g \text{ for any } f \text{ and } g \text{ such that } F = F(f) \text{ and } G = F(g).$$

Furthermore, $\succsim_{AA}$ respects stochastic dominance. The following result is therefore a straightforward corollary of Fact 1; it implies that preferences over Savage acts that are strongly compatible with the context $\trianglerighteq_{AA}$ and preferences over AA-acts that respect AA Stochastic Dominance are essentially the same object.

Proposition 3 *If the preference ordering over Savage acts are strongly compatible with the context* $\trianglerighteq_{AA}$*, the associated preference ordering* $\succsim_{AA}$ *respects AA stochastic dominance. Conversely, if the preference ordering* $\overline{\succsim}$ *over AA-acts respects stochastic dominance, there exists a unique preference ordering* $\succsim$ *that is strongly compatible with* $\trianglerighteq_{AA}$ *such that* $\overline{\succsim} = \succsim_{AA}$ *.*

We will now show that the standard assumptions on AA preferences in contributions such as Schmeidler (1989) and Gilboa-Schmeidler (1989) amount to utility sophistication of the corresponding preferences over Savage acts. These assumptions are summarized by the following three axioms.

Axiom 11 (Monotonicity) *For all acts* $F \in \mathcal{F}^{AA}$*, lotteries* $p, q \in \mathcal{L}$ *and events* $S \in \Sigma_1$: $[p \text{ on } S; F(\omega) \text{ elsewhere}] \succsim_{AA} [q \text{ on } S; F(\omega) \text{ elsewhere}]$ *whenever* $p \succsim q$.

Axiom 12 (Lottery Independence) *For all lotteries* $p, q, r \in \mathcal{L}$ *and all* $\alpha \in (0,1]$: $p \succsim_{AA} q$ *if and only if* $\alpha p + (1-\alpha)r \succsim_{AA} \alpha q + (1-\alpha)r$.

Axiom 13 (Certainty Independence) *For all acts* $F, G \in \mathcal{F}^{AA}$, *constant acts (lotteries)* $H \in \mathcal{F}^{AA}_{const} = \mathcal{L}$ *and all* $\alpha \in (0,1]$: $F \succsim_{AA} G$ *if and only if* $\alpha F + (1-\alpha)H \succsim_{AA} \alpha G + (1-\alpha)H$.[21]

The two main results of the paper, Theorem 1 and 2 yield the following result.

Proposition 4 *Suppose that the preference ordering* $\succsim$ *is regular and strongly compatible with the context* $\trianglerighteq_{AA}$. *Let* $\succsim_{AA}$ *denote the associate preference ordering over AA-acts. Then*

i) $\succsim$ *satisfies Trade-off Consistency if and only if* $\succsim_{AA}$ *satisfies Monotonicity and Lottery Independence. Furthermore,*

ii) $\succsim$ *satisfies Trade-off Consistency and Stake Independence (P4) if and only if* $\succsim_{AA}$ *satisfies Monotonicity and Certainty Independence.*

Proposition 4 yields a subjective, epistemic foundation of the standard modelling of ambiguity in the AA framework. All axioms are conditions on preferences over Savage acts, some of them formulated in relation to a given belief context. Since all uncertainty is treated on par as part of a single state space, all purely behavioral assumptions carry their usual, transparent meaning. By contrast, the original AA framework treats objective and subjective uncertainty differently; while the trick of including the objective uncertainty in the consequences is mathematically neat, it rather drastically changes the meaning of standard assumptions such as Monotonicity which turns out to be much stronger than usual.[22] On the other hand, Certainty

[21] As usual, this mixture-operation is defined pointwise.

[22] This intransparency potentially affects assumptions made within this framework to characterize specific preference models. Epstein (1999), for example, criticizes Schmeidler's (1989) and Gilboa-

Independence, which looks rather ad hoc and has no direct counterpart in the Savage framework, can be replaced by the transparent and standard assumption of stake independence.

The above epistemic subjective rendering of the AA setup is different from the recent preference-based translation by Ghirardato et al. (2003). The key to their work is a preference-based definition of utility-mixtures. It allows them to appeal in a Savage setting to axioms and results that are mathematically analogous to those formulated originally in an AA setting. However, since these axioms and results now refer to different objects, namely Savage rather than Anscombe-Aumann acts, they have rather different content.

5.3 Choquet Expected Utility

A main contribution of Theorem 1 was to show that utility sophistication with respect to an equidivisible context implies the determination of preferences over general multi-valued acts from preferences over unambiguous (risky) acts and preferences over bets expressed by the intrinsic integral $\widehat{\rho}$. The Choquet Expected Utility (CEU) model which ranks acts according to the Choquet integral of utilities $\int u \circ f d\nu$ is the main alternative model in the literature with this property.[23] In contrast to utility

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Schmeidler's (1989) definition of ambiguity aversion as too restrictive and/or inapplicable in a Savage setting. In Nehring (2001), we have formulated a definition of ambiguity aversion in terms of betting preferences, and show that it yields Schmeidler's *in the utility-sophisticated case.* In the absence of utility-sophistication, for example in the context of the CEU model, the new definition has however none of the restrictive and undesirable implications critized by Epstein.

Likewise, one obtains Schmeidler's (1989) "mixed CEU" model by imposing "Comonotonic Independence" restricted to non-random (Σ_1-measurable) acts; this simple observation throws light on the well-known fact that Schmeidler's model is quite distinct from proper CEU models as formulated in a Savage framework (Gilboa 1987, Sarin-Wakker 1992).

[23]This property comes out especially clearly in Sarin-Wakker's (1992) axiomatization based on a Cumulative Dominance axiom which explicitly constructs multi-act preferences from preferences

sophistication, the CEU model is designed to also allow for departures from expected utility in the absence of ambiguity, accommodating for example the Allais (1953) paradox. If one writes the non-normalized capacity ν as $\phi \circ \rho$, such departures are reflected in the non-linearity of ϕ.

When are Choquet preferences utility-sophisticated? While this can happen when the underlying context is not equidivisible, it never happens under equidivisibility in the presence of any ambiguity.

Proposition 5 *Suppose that a CEU preference ordering $\succsim$ is utility-sophisticated relative to the equidivisible context $\trianglerighteq$; then $\succsim$ is in fact SEU.*

To illustrate the incompatibility of CEU preferences with utility sophistication, consider a decision-maker with CEU preferences ordering who is an EU maximizer over unambiguous acts (i.e. with $\phi = id$). Let A be an "ambiguous" event with respect to which the decision-maker is ambiguity averse a la Ellsberg, i.e. for which $\rho(A) + \rho(A^c) < 1$. Denote consequences in (non-normalized) utiles, and take $B \subseteq A^c$ such that $B \equiv A^c \backslash B$. Let T be any unambiguous event such that $T \equiv T^c$. At issue is the comparison of the constant act 1_Ω, and the act f given as

$$[1 \text{ on } A, 2 \text{ on } B, 0 \text{ on } A^c \backslash B].$$

Conditional on A^c, this act entails an unambiguous 50-50 lottery with utility-payoffs 2 or 0. Since $1_\Omega \sim [2 \text{ on } T, 0 \text{ on } T^c]$ by assumption, Trade-off Consistency implies

$$f \sim 1_\Omega.$$

Since the act f has therefore expected utility of 1 for every prior $\pi \in \Pi$, on the Bernoulli Principle, the act f has a certainty equivalent of 1 irrespective of the

over bets.

DM's ambiguity attitudes, even though the probabilities of its outcomes are ambiguous. By contrast, a CEU maximizer evaluates the act f as if its valuation (certainty equivalent) was ambiguous. Indeed, one easily computes that $\int u \circ f d\nu = 2\rho(B) + 1[\rho(A+B) - \rho(B)] = \rho(A+B) + \rho(B)$. If one assumes for simplicity that betting preferences are based on lower probabilities (i.e. $\rho(E) = \min_{\pi \in \Xi} \pi(E)$ for all E, for some $\Xi \subseteq \Pi$), this implies[24] that in fact

$$\int u \circ f d\nu = 1 + \frac{1}{2}(\rho(A) + \rho(A^c) - 1) = 1 - \frac{1}{2}\left(\max_{\pi \in \Xi} \pi(A) - \min_{\pi \in \Xi} \pi(A)\right), \tag{3}$$

and therefore

$$f \prec 1_\Omega. \tag{4}$$

Thus the ambiguity of the individual outcomes leads the ambiguity-averse CEU decision maker to discount the act f relative to its unambiguous expected utility. Moreover, this discount is proportional to the ambiguity of A measured by $\max_{\pi \in \Xi} \pi(A) - \min_{\pi \in \Xi} \pi(A)$, and can be large (up to $\frac{1}{2}$).

Proposition 5 reveals a fundamental incompatibility between rank-dependence and the Bernoulli principle.[25] This incompatibility extends to non-P4 generalizations of CEU such as Cumulative Prospect Theory, and does not hinge on equidivisibility of the context. If the context is not equidivisible, there often exist some non-degenerate utility-sophisticated CEU preferences, but their set will in many cases still be fairly degenerate.[26]

[24]Indeed, one easily computes that $\rho(A+B) = \frac{1}{2}(1 + \min_{\pi \in \Xi} \pi(A))$ and $\rho(B) = \frac{1}{2}\min_{\pi \in \Xi} \pi(A^c)$.

[25]At first glance, Proposition 5 might seem to conflict with a well-known result of Schmeidler (1989) who showed that the CEU and MEU models coincide for convex capacities. Proposition 5 thus implies that capacities that are compatible with an equidivisible context cannot be convex, which can also be easily verified directly.

[26]For example, let μ_1 and μ_2 denote probability measures on two distinct subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \Sigma$, and let $\mathcal{A}$ denote the smallest algebra containing both $\mathcal{A}_1$ and $\mathcal{A}_2$. Let Π denote the set of priors π

5.4. A Two-Way Classification of Preference Models

The above discussion allows one to classify a variety of preference models proposed in the literature according to their utility sophistication and stake-independence. To tie in directly with the literature, we assume that all models are situated in the epistemized version of the traditional AA model analyzed in section 5.2 using Proposition 4. In particular, utility sophistication is taken to properly mean "utility sophistication relative to the context $\trianglerighteq_{AA}$".

Note that on this understanding, the MEU model due to Gilboa-Schmeidler (1989) does not coincide with the class of multi-prior preferences over Savage acts (as axiomatized by Casadesus et al. (2000) and Ghirardato et al. (2003)) that are strongly compatible with $\trianglerighteq_{AA}$.[27] As noted in section 5.3, CEU preferences (and rank-dependent more generally) can be compatible with the $\trianglerighteq_{AA}$ as an equidivisible context only when they are degenerate (SEU). We have discussed various stake-dependent (non-P4) preference models and their utility sophistication above. In particular, we noted that utility-smooth preferences will in general violate P4. By contrast, event-smooth preferences (i.e. preferences that are locally linear in events) as in the work of Machina (2004) and Epstein (1999) do not conflict with P4 per se; for example, event-smoothness imposes only mild conditions on CEU preferences. However, in view of the identity of utility-evaluation functional I and the implicit integral $\widehat{\rho}$ under utility sophistication, event-smoothness and utility-smoothness are essentially the same thing for utility-sophisticated preferences; this implies the absence event-smooth pref-

that agree with μ_i on $\mathcal{A}_i$. It follows from the analysis in Nehring (1999) that if a CEU preference relation is utility-sophisticated with respect to Π, the representing capacity must be additive on all of $\mathcal{A}$. Of course, utility-sophistication by itself carries no such implication, as evidenced by the MEU model; nor does the assumption of CEU preferences: in general, many CEU preferences are strongly compatible with $\trianglerighteq_{\Pi}$, without eliminating ambiguity about events in $\mathcal{A}\backslash(\mathcal{A}_1 \cup \mathcal{A}_2)$.

[27]Example 3 below illustrates the difference.

erences that are utility-sophisticated as well as stake-independent but do not coincide with SEU.[28] Finally, while stake-independence is generally lost under conditioning, utility sophistication is preserved.

	–	**Utility Sophistication**
Non-P4	Cumulative Prospect Theory (KT 92) Utility-Smooth Preferences Event-Smooth Preferences (Machina 04) Conditional Pref. (Epstein-Le Breton 93)	Interval Expected Utility (Jaffray 89) Utility-Smooth Preferences (KMM 05) Event-Smooth Preferences (Machina 04) Conditional Util. Soph. Preferences
P4	CEU (Gilboa 89, Sarin-Wakker 92) Event-Smooth Preferences (Machina 04)	CEU (Schmeidler 89) MEU (Gilboa-Schmeidler 89) α-MEU (GMM 02, Kopylov 02) Plausible Priors (Siniscalchi 03) Simultaneous EU (Nehring 91, 00)
Prob. Soph.	Machina-Schmeidler (92)	Subjective EU (Savage 54)

Table 1: Two-Way Classification of Preference Models

6. UTILITY SOPHISTICATION DE-RELATIVIZED

Utility Sophistication has been defined relative to an exogenously specified set of likelihood comparisons $\trianglerighteq$. A priori, it is quite possible, however, that a DM may be utility-sophisticated relative to a non-exhaustive likelihood relation $\trianglerighteq^0$ while at the same time failing to be utility-sophisticated relative to a richer, apparently equally

[28]This claim would appear to be robust to the particular formalization of event-smoothness adopted. Epstein (1999) and Machina (2004) define event-smoothness relative to an additive reference measure; in the present setting, it would be natural to use the sub-additive upper-probability π^+_{AA} for this purpose.

appropriate belief relation $\trianglerighteq$. Indeed, this possibility arises trivially in the case of the "vacuous relation" $\trianglerighteq^0 = \trianglerighteq_\varnothing$ given by $A \trianglerighteq_\varnothing B$ iff $A \supseteq B$. For a richer example, consider CEU preferences $\succsim$ that are SEU on unambiguous acts but not globally, and that are strongly compatible with respect to some equidivisible context $\trianglerighteq$. Then $\succsim$ is utility-sophisticated relative to the restriction $\trianglerighteq^0$ of $\trianglerighteq$ to unambiguous events, but, in view of Proposition 5, $\succsim$ cannot be utility-sophisticated with respect to $\trianglerighteq$ itself.

Such examples may create the impression that one needs to know his "true," exhaustively described beliefs $\trianglerighteq^\top$ in order to meaningfully ascertain whether a decision maker is genuinely utility-sophisticated. This would raise obvious conceptual and/or pragmatic issues, especially in view of the fact that there is no generally accepted notion of how to infer these beliefs from his preferences, and would thus seriously threaten the applicability of the entire notion of utility sophistication.

Importantly, however, the problem does not arise when the set of non-exhaustively ascribed beliefs $\trianglerighteq^0$ is sufficiently rich, specifically: when it is equidivisible, for example when $\trianglerighteq^0$ is the AA context. Suppose that preferences are utility-sophisticated with respect to $\trianglerighteq_{AA}$, and consider any superrelation $\trianglerighteq \supseteq \trianglerighteq_{AA}$ that is consistent with the decision-maker's preferences in that $\succsim$ is strongly compatible with $\trianglerighteq$. We will show that preferences must then be utility-sophisticated not just with respect to $\trianglerighteq^0 = \trianglerighteq_{AA}$, but with respect to $\trianglerighteq$, whatever $\trianglerighteq$ may be! In the language of "true beliefs", this means that if it is known that the decision maker's true beliefs $\trianglerighteq^\top$ contain an equidivisible likelihood relation $\trianglerighteq^0$, it suffices to check for utility sophistication with respect to $\trianglerighteq^0$ to determine whether the decision maker is utility-sophisticated with respect to the true beliefs $\trianglerighteq^\top$. This is established by the following result.

Proposition 6 *i) Suppose $\succsim$ is trade-off consistent with respect to the equidivisible context $\trianglerighteq^0$ and strongly compatible with $\trianglerighteq \supseteq \trianglerighteq^0$. Then $\succsim$ is trade-off consistent with respect to $\trianglerighteq$ as well.*

ii) Suppose $\succsim$ is regular and utility-sophisticated with respect to the equidivisible *context $\trianglerighteq^0$ and strongly compatible with $\trianglerighteq \supseteq \trianglerighteq^0$. Then $\succsim$ is regular and utility-sophisticated with respect to $\trianglerighteq$ as well.*

A Fully Behavioral Definition of Utility Sophistication

While Proposition 6 successfully dispenses with the need to identify the decision maker's true beliefs, it still leaves utility sophistication relative to the belief context $\trianglerighteq^0$. Is it possible to overcome this belief-relativity completely and provide a useful notion of "revealed utility sophistication" that is well-defined in terms of preferences alone? More specifically, is it possible to ascertain in terms of preferences alone whether a decision maker behaves in accordance with the Bernoulli principle?

What is sought is a behavioral *definition* roughly on par with the definition of probabilistic sophistication as a behavioral criterion of satisfaction of the Bayes principle. The following example – discussed before in Epstein-Zhang (2001) and Ghirardato-Marinacci (2002) – shows that satisfaction of the Bernoulli principle cannot always be inferred unambiguously from observed behavior, hence that one cannot hope to arrive at a satisfactory definition of revealed utility sophistication that is decisive in all cases.

Example 3 (Probabilistic Sophistication or Utility Sophistication?) Let μ be a convex-ranged probability measure on some event space (Ω, Σ), and $\phi : [0,1] \rightarrow [0,1]$ an increasing, strictly convex function mapping the unit interval onto itself. Suppose that preferences have a CEU representation with capacity $\phi \circ \mu$, and are thus probabilistically sophisticated in the sense of Machina-Schmeidler (1992). One natural explanation of these preferences is that the DM has probabilistic beliefs given by μ (in line with the Bayes principle) and is "probabilistically risk-averse". However, there is a competing explanation, namely that the decision-maker behaves according to the Bernouilli priniciple but is ambiguity-averse, evaluating acts according to the minimum expected utility of the core of the capacity ν, $\Xi = \{\pi : \pi \geq \nu\}$[29]; such preferences are utility-sophisticated relative to any $\Pi \supseteq \Xi$. There seems to be no basis for privileging one explanation over the other on the basis of preferences alone. At most, one could argue in favor of a "convention" by postulating the primacy of one criterion over the other a priori, for example by declaring probabilistic sophistication to reveal absence of ambiguity by definition.[30]

A key feature of Example 3 is sparseness of the beliefs relative to which utility sophistication can be attributed. But as Proposition 6 has just shown, the illustrated conflict could not arise if the decision maker was utility-sophisticated relative to a sufficiently rich (equidivisible) likelihood relation. In addition, richness in the form of equidivisibility of beliefs is crucial for the full analytical power of utility sophistication due to the entailed reduction property asserted by the main result of the paper, Theorem 1. Indeed, this reduction property entails that multi-act preferences can be *fully explained* as utility-sophisticated evaluation determined by risk- and betting-preferences. Preferences are, one might say, as utility-sophisticated as they can possibly be; any remaining gap to SEU is fully accounted for by the ambiguity revealed in betting preferences. The crucial role of the richness of the underlying belief context motivates the following definition of "revealed utility sophistication".

Definition 2 (Revealed Utility Sophistication) *The preference relation $\succsim$ is re-*

[29] The capacity ν is easily seen to be convex; the existence of an equivalent MEU representation follows therefore from a result by Schmeidler (1989).

[30] This seems to be the line taken by Epstein-Zhang (2001). Ghirardato-Marinacci (2002), on the other hand, argue for the convention aligned with the second interpretation, explaining very clearly the unavoidability of a conventional element in the absence of non-behavioral information. They also point out how the existence of exogeneously identified unambiguous events allows one to distinguish between the two possible interpretations of a probabilistically sophisticated DM.

vealed utility-sophisticated if it is utility-sophisticated relative to some equidivisible context $\trianglerighteq$.

If preferences satisfy Revealed Utility Sophistication, this can be taken to indicate satisfaction of the Bernoulli principle. By contrast, if preferences violate this condition, this may be due either to a genuine violation of the Bernoulli principle, or to insufficiently rich beliefs, or both. As there is evidently a provisional element in the proposed definition in that equidivisibility is merely sufficient, but not strictly necessary to obtain the conclusions of Theorem 1 and Proposition 6; improvements of the definition by weakenings of the equidivisibility assumption are thus imaginable.

To make the proposed definition applicable, an operational criterion of its satisfaction is highly desirable. We will now provide such a criterion for the special case of stake-independent preferences. The assumption of stake-independence is helpful since it can be shown to imply the existence of a *unique* maximal likelihood relation $\succsim^*_{bet}$ relative to which a given preference ordering is utility-sophisticated;[31] from this one immediately the equivalence of revealed utility sophistication to equidivisibility of the relation $\succsim^*_{bet}$.

For preparation, a bit of further background is needed. If preferences are P4, then in order to be utility-sophisticated relative to an equidivisible context, they must be constant-linear in view of Theorem 2 . If they are indeed constant-linear, the existence of such a context can be determined via the following notion of a maximal independent subrelation $\succsim^*$ of the given preference relation $\succsim$. Define the following mixture-operation $\alpha f \oplus (1-\alpha) g$ on the space of acts: for $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha) g$ denotes any act h such that, for all $\omega \in \Omega$, $u(h_\omega) = \alpha u(f_\omega) + (1-\alpha) u(g_\omega)$; note that by Eventwise Monotonicity the choice of the act h is immaterial. Since utility functions in a constant-linear representation are unique up to positive affine transformation, this

[31]While we believe the provided criterion to be applicable also in the stake-dependent case, this needs to be verified in future research.

mixture-operation is well-defined in terms of preferences; Ghirardato et al. (2003) provide a directly behavioral definition. A (possibly incomplete) relation $\succsim'$ is *independent* if, for all f, g, h and $\alpha \in (0,1]$, $f \succsim' g$ if and only if $\alpha f \oplus (1-\alpha) h \succsim' \alpha g \oplus (1-\alpha) h$. In Nehring (2001), we have obtained (a version of) the following result, a version of which can also be found in Ghirardato et al. (2004, Propositions 4 and 5).[32] The step from i) to ii) follows from Bewley's (1986) Theorem.

Proposition 7 *Suppose that the preference ordering $\succsim$ has a constant-linear representation $I \circ u$ such that $u(X)$ is convex.*

i) Then there exists a unique maximal independent subrelation $\succsim^*$, with

$f \succsim^* g$ if and only for all h and all $\alpha \in (0,1]$, $\alpha f \oplus (1-\alpha) h \succsim \alpha g \oplus (1-\alpha) h$.

ii) There exists a unique closed, convex set of priors Π^* such that

$$f \succsim^* g \text{ if and only } E_\pi u \circ f \geq E_\pi u \circ g \text{ for all } \pi \in \Pi^*. \tag{5}$$

In particular, Π^* is the unique minimal set of closed, convex of priors Π such that $\succsim$ is utility-sophisticated with respect to Π and u.

Furthermore, $\succsim^*_{bet}$ is the unique maximal coherent likelihood relation $\trianglerighteq$ such that $\succsim$ is utility-sophisticated with respect to $\trianglerighteq$ and u.[33]

Proposition 7 entails the following operational characterization of revealed utility sophistication.

[32] A first version of this result was presented in the talk Nehring (1996) which made use of a different version of condition i); the exact version of the characterization of $\succsim^*$ in i) was arrived at independently by Ghirardato et al. (2004).

[33] The last claim has no counterpart in Ghirardato et al. (2004). To verify it, note that $\succsim^*_{bet} = \trianglerighteq_{(\Pi^*)}$ by the representation (5). Since by the definition of coherence for any coherent context $\trianglerighteq$ such that $\trianglerighteq \not\subseteq \succsim^*_{bet}$, one has $\Pi_{\trianglerighteq} \not\supseteq \Pi^*$ (for otherwise $\trianglerighteq = \trianglerighteq_{(\Pi_{\trianglerighteq})} \subseteq \trianglerighteq_{(\Pi^*)} = \succsim^*_{bet}$), and the claim follows.

Proposition 8 *Suppose that the preference ordering $\succsim$ has a constant-linear representation $I \circ u$ such that $u(X)$ is convex. Then the following three conditions are equivalent:*

1. *$\succsim$ is revealed utility-sophisticated;*
2. *Π^* is convex-ranged;*
3. *$\succsim^*_{bet}$ is equidivisible.*

In the case of multi-prior (MEU) preferences with set of priors Ξ, Π^* is easily verified to be equal to $\Pi^* = \Xi$; see Ghirardato et al. (2004, Proposition 16) for a published proof. Thus the preferences in example 3 are not revealed utility-sophisticated, since Ξ fails to be convex-ranged there.

7. REVEALED UNAMBIGUOUS BELIEFS

Proposition 8 suggests a natural definition of "revealed probabilistic beliefs", namely $\succsim^*_{bet}$. Not only are these beliefs the largest (most precise) likelihood relation that is consistent with assuming utility sophistication with respect to them, by Proposition 6 it is not possible to impute a strictly larger, more precise likelihood relation relative with which preferences are at least strongly compatible.[34]

Definition 3 (Revealed Probabilistic Beliefs) *Suppose that the preference ordering $\succsim$ is revealed utility-sophisticated and satisfies P4, with constant-linear representation $I \circ u$ such that $u(X)$ is convex. Then $\succsim^*_{bet}$ defines the decision maker's* revealed probabilistic beliefs.

Consider, for example, the counterpart to certainty-independent preferences in the AA-framework. In view of Proposition 4, such preferences are utility-sophisticated

[34] These two claims require $\succsim^*_{bet}$ to be regular.

with respect to $\trianglerighteq_{AA}$ as well as stake-independent. They are thus revealed utility-sophisticated, with revealed probabilistic beliefs $\succsim^*_{bet} \supseteq \trianglerighteq_{AA}$. The existence of an independent randomization device is thus revealed through preferences rather than postulated on a non-behavioral basis.

In earlier work (Nehring (1996), see also Nehring (1999) and Nehring (2001)) as well as in the rich contribution by Ghirardato et al. (2004), analogous definitions (with $\succsim^*$ instead of $\succsim^*_{bet}$) have been put forward without restriction to revealed utility-sophisticated preferences. These earlier definitions are subject to the criticism that they somewhat arbitrarily attribute to ambiguity what could be attributed with equal legitimacy to failures of utility sophistication; in Example 3, for instance, in view the agent's "probabilistic sophistication", a strong case can be made for attributing complete probabilistic beliefs to the agent. This position is consistent with Definition 3, since $\succsim^*_{bet}$ fails to be equidivisible and is therefore not viewed as identifying the DM's probabilistic beliefs.

Could one conceive of cases in which the proposed definition gives a questionable, or even the intuitively wrong answer? To seriously compete with $\succsim^*_{bet}$ as a candidate for belief attribution, such counterexamples would presumably have to exhibit beliefs $\trianglerighteq^{\#}$ with which preferences are strongly compatible and which are as rich or "richer" than $\succsim^*_{bet}$. While there is evidently some latitude in determining what should count as "rich" in this context, an obvious natural candidate is equidivisibility. We have not yet been able to come up with an example of this kind in which there exists a competing, equidivisible $\trianglerighteq^{\#}$ with which $\succsim$ is strongly compatible but not utility-sophisticated; note in particular that, by Proposition 6, $\trianglerighteq^{\#}$ can never strictly include $\succsim^*_{bet}$. It seems likely that such examples exist only in exceptional circumstances, if they exist at all.

Furthermore, even if such examples can be produced, with $\trianglerighteq^{\#}$ broadly on par with $\succsim^*_{bet}$ in terms of its richness as a likelihood relation, there remain strong grounds for privileging the $\succsim_{bet}^{*}$ as the more compelling representation of the decision maker's probabilistic beliefs. First of all, since $\succsim_{bet}^{*}$ is the unique maximal relation relative to which preferences are utility-sophisticated, preferences cannot be utility-sophisticated relative to $\trianglerighteq^{\#}$ at the same time. Imputing the beliefs $\succsim_{bet}^{*}$ rather than $\trianglerighteq^{\#}$ thus renders the decision maker's behavior overall more rational. And secondly, due to the reduction property associated with utility sophistication relative to an equidivisible context, the likelihood relation $\succsim_{bet}^{*}$, together with the revealed utility sophistication, explains the decision maker's preferences globally (without gap, as it were), while $\trianglerighteq^{\#}$, tied to preferences merely via Strong Compatibility, constrains and thus explains preferences only partially.

Restricting the domain of the definition of revealed probabilistic beliefs as proposed has significant implications for the understanding of some of the major models of decision making under ambiguity. For example, as noted already, for MEU preferences with set of priors Ξ, the set of revealed priors given in Proposition 7 is $\Pi^{*} = \Xi$. However, examples such as Example 3 interfere with a straightforward interpretation of this set as the decision maker's beliefs. Yet such an interpretation constitutes a large part of the intuitive appeal of the MEU model in the first place. Here, the proposed domain restriction comes to rescue, by salvaging this interpretation for the case of multi-prior preferences associated with convex-ranged sets of priors Ξ. In particular, it salvages this interpretation for the original MEU model axiomatized by Gilboa-Schmeidler (1989) as reformulated here along the lines of section 5.

The proposed domain restriction also affects the analysis of CEU preferences, for the simple reason that, as evident from Proposition 5, it never applies to them. Thus, the unqualified claim in earlier work (Nehring (1999)) that "the" CEU model suffers from serious epistemic handicaps loses its basis. It can only be maintained for Schmeidler's (1989) version of the CEU model.

APPENDIX: PROOFS

Proof of Proposition 1. That ii) implies i) is straightforward; as to the Archimedean property, merely note that $I-$continuity implies an analogous property for decreasing sequences $\{A_n\}$ by switching the roles of x and y.

For the converse, take any $g \in \mathcal{F}$. By Eventwise Monotonicity and boundedness, $x^- \precsim g \precsim x^+$. By the convex-rangedness of $\trianglerighteq$, there exists a totally ordered chain of unambiguous events $\mathcal{T} \subseteq \Lambda$ such that, for any $T \in \Lambda$, there exists $T' \in \mathcal{T}$ such that $T' \equiv T$. Hence one can infer from the Archimedeanicity of $\succsim$ (applied to the case $A+B=\Omega$, i.e. $A \in \Lambda$) the existence of an event $T_g \in \Lambda$ such that $g \sim [x^+, T_g; x^-, T_g^c]$. By compatibility, all such events T_g have the same unambiguous probability $\overline{\pi}(T_g)$. Hence the mapping $V : g \to \overline{\pi}(T_g)$ is well-defined and represents $\succsim$ by construction. For any consequence/constant act z, set $u(z) := \overline{\pi}(T_z)$. By Eventwise Monotonicity, V can be written as $I \circ u$, with I monotone and compatible with $\trianglerighteq$; note that I is normalized by construction; moreover, the uniqueness claim is straightforward from Solvability which implies that u is onto.

It remains to verify that I is event-continuous. To do so, consider $\{A_n \in \Lambda_E\}$ and $A \in \Lambda_E$ such that $\overline{\pi}(A_n/E)$ converges to $\overline{\pi}(A/E)$ and such that the family is $\{A_n\} \cup A$ is ordered by set-inclusion. Take any $x, y \in X$ and $Z \in \mathcal{Z}$. W.l.o.g. $x \geq y$. It clearly suffices to show convergence of $I(x1_{A_n} + y1_{E \backslash A_n} + Z1_{E^c})$ for the case of $\{A_n\}$ being an increasing or decreasing sequence. The proof for both cases is analogous; assume the former, and suppose that the claim is false. I.e., in view of the monotonicity of I, suppose that $\sup_{n \in N} I(x1_{A_n} + y1_{E \backslash A_n} + Z1_{E^c}) < I(x1_A + y1_{E \backslash A} + Z1_{E^c})$. By normalization, there exist an event $T \in \Lambda$ such that $\sup_{n \in N} I(x1_{A_n} + y1_{E \backslash A_n} + Z1_{E^c}) < I(1_T) < I(x1_A + y1_{E \backslash A} + Z1_{E^c})$. Hence, by Archimedeanicity, there exist $A' \in \Lambda_E$ and $A' \lhd A$ such that $I(1_T) < I(x1_{A'} + y1_{E \backslash A'} + Z1_{E^c})$. But by the convergence assumption, $A' \trianglelefteq A_n$ for some n, hence $I(x1_{A'} + y1_{E \backslash A'} + Z1_{E^c}) \leq \sup_{n \in N} I(x1_{A_n} +$

$y1_{E\setminus A_n} + Z1_{E^c}) < I(1_T)$, a contradiction. $\square$

In the following Lemma, we state a key mathematical property of the intrinsic integral $\widehat{\rho}$ that will be used repeatedly in the sequel. Let $\mathcal{S}$ denote any finite partition of Ω into events $S_i \in \Sigma$. Say that $Z \in \mathcal{Z}$ is $\trianglerighteq$-unambiguous conditional on the finite partition $\mathcal{S}$ if, for all $S_i \in \mathcal{S}$, $Z1_{S_i}$ is Λ_{S_i}-measurable; let $\mathcal{Z}_{\mathcal{S}}$ denote their class. For $Z \in \mathcal{Z}_{\mathcal{S}}$, a expectation conditional on $\mathcal{S}$ is any random variable ζ such that

$$\begin{aligned}
\zeta(\omega) &= \sum_{z\in[\mathbf{0},\mathbf{1}]} z\pi(\{\omega' \in S_i \mid Z(\omega') = z\}/S_i) \text{ if } \omega \in S_i \text{ and } S_i \text{ is non-null, and} \\
\zeta(\omega) &= \text{arbitrary if } \omega \in S_i \text{ and } S_i \text{ is null;}
\end{aligned}$$

let the set of such ζ be denoted by $E(Z/\mathcal{S})$.

Lemma 1 (Characterization of Intrinsic Integral) *$\widehat{\rho}$ is the unique mapping $r : \mathcal{Z} \to [0,1]$ such that*

i) For any event $A \in \Sigma$, $r(1_A) = \rho(A)$, and

ii) **(Conditional Linearity)** *For any partition $\mathcal{S}$ and any $Z \in \mathcal{Z}_{\mathcal{S}}$, $r(Z) = r(\zeta)$ for any $\zeta \in E(Z/\mathcal{S})$.*

Note that Conditional Linearity implies in particular that $\widehat{\rho}$ restricted to unambiguous random variables is the ordinary expectation with respect to $\overline{\pi}$ or equivalently ρ.

<u>Proof of Lemma 1.</u>

It is immediate from its definition that $\widehat{\rho}$ satisfies i). To verify the Conditional Linearity of $\widehat{\rho}$, write Z as $\sum_{i,j} z_{ij}1_{A_{ij}}$ with $S_i = \sum_{j\leq n_j} A_{ij}$ for all i. Consider any C such that $\pi(C \cap A_{ij}) = z_{ij}\pi(A_{ij})$ for all i, j and all $\pi \in \Pi$; such C exist by the convex-rangedness of Π. Then $C \in [Z]$ by construction and fact, for all non-null S_i and all $\pi \in \Pi$,

$$\pi(C \cap S_i) = \sum_j \pi(C \cap A_{ij}) = \sum_j z_{ij}(\overline{\pi}(A_{ij}/S_i)\pi(S_i)) = \left(\sum_j z_{ij}\overline{\pi}(A_{ij}/S_i)\right)\pi(S_i).$$

From this evidently $C \in [\zeta]$ for any $\zeta \in E(Z/\mathcal{S})$. Thus indeed $C \in [Z] \cap [\zeta]$, and therefore

$$\widehat{\rho}\,(Z) = \rho(C) = \widehat{\rho}\,(\zeta)\,.$$

Conversely, assume that r satisfies i) and ii). Consider any $Z{=}\sum_i z_i 1_{S_i}$ and any $C \in [Z]$ such that $\pi\,(C \cap S_i) = z_i \pi\,(S_i)$ for all i, j and all $\pi \in \Pi$; such C exist by the convex-rangedness of Π. By construction of C, $1_C \in \mathcal{Z}_{\mathcal{S}}$ with $Z \in E(1_C/\mathcal{S})$. Hence

$$r\,(Z) = r\,(1_C) \;\; \text{(by ii)} = \rho(C) \;\; \text{(by i)} \;\; = \widehat{\rho}(Z),$$

which establishes that $r = \widehat{\rho}$. $\square$

Proof of Theorem 1.

iii) implies ii) To show that $\succsim$ is utility-sophisticated with respect to $\trianglerighteq$, take any f, g such that $E_\pi u \circ f \geq E_\pi u \circ g$ for all $\pi \in \Pi$, and take $A \in [u \circ f]$ and $B \in [u \circ g]$. By construction, $\pi(A) \geq \pi(B)$ for all $\pi \in \Pi$, and therefore by the compatibility of ρ

$$\widehat{\rho}\,(u \circ f) = \rho\,(A) \geq \rho\,(B) = \widehat{\rho}\,(u \circ g)\,,$$

i.e. $f \succsim g$. By the same token, if $E_\pi u \circ f > E_\pi u \circ g$ for all $\pi \in \Pi$, then $f \succ g$.

To verify that $\succsim$ is Archimedean, in view of Proposition 1 we need to verify that $\widehat{\rho}$ is event-continuous exploiting the event-continuity of ρ. Thus, take some $x, y \in X$, $Z \in \mathcal{Z}$, $E \in \Sigma$, $A \in \Lambda_E$ and some increasing sequence $\{A_n\}$ of events contained in A such that $\overline{\pi}\,(A_n/E)$ converges to $\overline{\pi}\,(A/E)$; we need to show that $\widehat{\rho}(x1_{A_n} + y1_{E\setminus A_n} + Z1_{E^c})$ converges to $\widehat{\rho}(x1_A + y1_{E\setminus A} + Z1_{E^c})$. By conditional linearity (Lemma 1),

$$\widehat{\rho}(x1_A + y1_{E\setminus A} + Z1_{E^c}) = \widehat{\rho}((\overline{\pi}\,(A/E)\,x + (1 - \overline{\pi}\,(A/E))\,y)\,1_E + Z1_{E^c})$$

and likewise

$$\widehat{\rho}(x1_{A_n} + y1_{E\setminus A_n} + Z1_{E^c}) = \widehat{\rho}((\overline{\pi}\,(A_n/E)\,x + (1 - \overline{\pi}\,(A_n/E))\,y)\,1_E + Z1_{E^c}).$$

Suppose $x \geq y$; the converse case is dealt with symmetrically.

Take $B \in [Z1_{E^c}]$, $A' \in [(\overline{\pi}(A/E)\,x + (1-\overline{\pi}(A/E))\,y)\,1_E]$ and an increasing sequence of events $A'_n \in [(\overline{\pi}(A_n/E)\,x + (1-\overline{\pi}(A_n/E))\,y)\,1_E]$ contained in A'. By construction,

$$\widehat{\rho}((\overline{\pi}(A/E)\,x + (1-\overline{\pi}(A/E))\,y)\,1_E + Z1_{E^c}) = \rho(A'+B),$$

as well as

$$\widehat{\rho}((\overline{\pi}(A_n/E)\,x + (1-\overline{\pi}(A_n/E))\,y)\,1_E + Z1_{E^c}) = \rho(A'_n+B)$$

Note that by definition, A' and the A'_n are unambiguous given E. Hence by the event-continuity of ρ, $\rho(A'+B) = \lim_{n\to\infty}\rho(A'_n+B)$, and therefore

$$\widehat{\rho}(x1_A + y1_{E\setminus A} + Z1_{E^c}) = \lim_{n\to\infty}\widehat{\rho}(x1_{A_n} + y1_{E\setminus A_n} + Z1_{E^c}),$$

as needed to be shown.

ii) implies i) It is clear that Utility Sophistication implies Compatibility. To verify Trade-off Consistency, take any $x, y, x', y' \in X$, $f, g \in \mathcal{F}$ and events A disjoint from B and A' disjoint from B' such that $A \equiv B \rhd\!\rhd \emptyset$ and $A' \equiv B'$ and such that $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$. By the assumption on A and B, for all $\pi \in \Pi$, $\pi(A) = \pi(B) > 0$; therefore, if it was the case that $u(x) + u(y) < u(x') + u(y')$, then the strict part of Utility Sophistication would imply that $[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \prec [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]$, which is false by assumption. Thus $u(x) + u(y) \geq u(x') + u(y')$, which implies by the non-strict part of Utility Sophistication that $[x \text{ on } A'; y \text{ on } B'; g(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'; y' \text{ on } B'; g(\omega) \text{ elsewhere}]$, as needed to be shown.

i) implies iii)

Since Trade-off Consistency implies Eventwise Monotonicity for equidivisible contexts as remarked in the text, by Proposition 1 there exist an onto function $u : X \to$

$[0,1]$ and a normalized functional $I : \mathcal{Z} \to [0,1]$ that is monotone, event-continuous and compatible with $\trianglerighteq$ such that $f \succsim g$ if and only if $I(u \circ f) \geq I(u \circ g)$, for all $f, g \in \mathcal{F}$. In particular, ρ is event-continuous as the restriction of I to indicator functions. It remains to show that $I = \widehat{\rho}$.

Step 1. We shall first consider the case of dyadic-valued utilities; a number is *dyadic* if $\alpha = \frac{\ell}{2^m}$, where m is natural or zero, and ℓ is an odd integer or zero; m will be referred to as the (dyadic) order of α denoted by $|\alpha|$. Let $\mathbf{D}$ denote the set of dyadic numbers in $(0,1]$.

Lemma 2 *For any $\alpha \in \mathbf{D}, w, x, y \in X, B \in \Sigma$, $A \in \Lambda_B$ with $A \subseteq B$ and $T \in \Lambda$ such that $\overline{\pi}(T) = \overline{\pi}(A/B) = \alpha$: if $w \sim [x, T; y, T^c]$, then $[w, B; f(\omega)$ elsewhere$] \sim [x, A; y, B \backslash A; f(\omega)$ elsewhere$]$.*

The Lemma is proved by induction on the order of α. If the order of α is 1, i.e. if $\alpha = \frac{1}{2}$, the assertion follows directly from Trade-off Consistency. Suppose thus that the Lemma has been shown for all dyadic coefficient α' with $|\alpha'| < |\alpha|$. Assume that $\alpha \geq \frac{1}{2}$; the case of $\alpha < \frac{1}{2}$ can be proved essentially identically. Then $\alpha = \frac{1}{2} + \frac{1}{2}\beta$, where β is dyadic with $|\beta| = |\alpha| - 1$.

Now define unambiguous events T_1, T_2, T_3 such that $T_1 + T_2 + T_3 = \Omega, T_2 + T_3 = T$, and $\overline{\pi}(T_2) = \frac{1}{2}\beta$. Since $\overline{\pi}(T) = \alpha$, one has also $\overline{\pi}(T_3) = \frac{1}{2}$ and $\overline{\pi}(T_2/T_1 + T_2) = \beta$. In parallel, define events $A_1, A_2, A_3 \in \Lambda_B$ such that $A_1 + A_2 + A_3 = B, A_2 + A_3 = A$, and $\overline{\pi}(A_2/B) = \frac{1}{2}\beta$. Since $\overline{\pi}(A/B) = \alpha$, one has also $\overline{\pi}(A_3/B) = \frac{1}{2}$ and $\overline{\pi}(A_2/A_1 + A_2) = \beta$. Such events exist by the convex-rangedness of Π.

Take any $D \in \Lambda$ such that $\overline{\pi}(D) = \beta$, and $z \in X$ such that $z \sim [x, D; y, D^c]$; such z exists by Solvability. Since $\overline{\pi}(T_2/T_1 + T_2) = \beta$, by the induction assumption this implies that

$$[z, T_1 + T_2; x, T_3] \sim [y, T_1; x, T_2; x, T_3],$$

hence by the assumption that $w \sim [x, T; y, T^c]$ and transitivity also that

$$[z, T_1 + T_2; x, T_3] \sim [w, T_1 + T_2; w, T_3]. \tag{6}$$

Writing $[x, A; y, B \backslash A; f(\omega) \text{ elsewhere}] = [y, A_1; x, A_2; x, A_3; f(\omega) \text{ elsewhere}]$, by the induction assumption one also has

$$[x, A; y, B \backslash A; f(\omega) \text{ elsewhere}] \sim [z, A_1 + A_2; x, A_3; f(\omega) \text{ elsewhere}].$$

By Trade-off Consistency and (6), in turn

$$[z, A_1 + A_2; x, A_3; f(\omega) \text{ elsewhere}] \sim [w, A_1 + A_2; w, A_3; f(\omega) \text{ elsewhere}].$$

Since $B = A_1 + A_2 + A_3$, we get by transitivity

$$[x, A; y, B \backslash A; f(\omega) \text{ elsewhere}] \sim [w, B; f(\omega) \text{ elsewhere}]$$

as desired.

Step 2. We shall next obtain the desired conclusion for the subset dyadic-valued functions $Y \in \mathcal{Z}$, which we shall abbreviate to $\mathcal{Z}_{\mathbf{D}}$. Thus, take any $Y = \sum_{i \leq n} y_i 1_{E_i} \in \mathcal{Z}_{\mathbf{D}}$; by solvability, there exists $f = [w_i, E_i]_{i \leq n} \in \mathcal{F}$ such that $u(w_i) = y_i$ for all i, so that $Y = u \circ f$. For each $i \leq n$, pick $A_i \subseteq E_i$ such that $\overline{\pi}(A_i/E_i) = u(w_i)$. By $n-$fold application of Lemma 2, $f \sim \left[x^+, \sum_{i \leq n} A_i; x^-, \left(\sum_{i \leq n} A_i\right)^c\right]_{i \leq n}$. Since $\sum_{i \leq n} A_i \in [Y]$ by construction, one obtains

$$I(Y) = I(u \circ f) = \rho(\sum_{i \leq n} A_i) = \widehat{\rho}(Y),$$

demonstrating that $I = \widehat{\rho}$ on $\mathcal{Z}_{\mathbf{D}}$.

Step 3.

This conclusion is extended to all of $\mathcal{Z}$ by an inductive continuity argument. Let $\mathcal{Z}_k$ the set of random variables $Y \in \mathcal{Z}$ such that in their canonical representation

$Y = \sum_{i \le n} y_i 1_{E_i}$ no more than k y_i's are not dyadic. Step 2 has established that $I = \widehat{\rho}$ on $\mathcal{Z}_{\mathbf{D}} = \mathcal{Z}_0$. Suppose therefore that $I = \widehat{\rho}$ on $\mathcal{Z}_k$; we need to show that $I = \widehat{\rho}$ on $\mathcal{Z}_{k+1}$. Take $Y = \sum_{i \le n} y_i 1_{E_i} \in \mathcal{Z}_{k+1}$, and assume w.l.o.g. that $y_1 \in (0, 1] \backslash \mathbf{D}$.

Take an increasing sequence $\{v_j\}$ in $\mathbf{D}$ converging to y_1, and take $B \in \left[\sum_{2 \le i \le n} y_i 1_{E_i}\right]$, $A \in [y_1 1_{E_1}]$ and an increasing sequence $\{A_j\}$ contained in A such that $A_j \in [v_j 1_{E_1}]$; such events exist by repeated applications of equidivisibility. Denote $Y_j := v_j 1_{E_1} + \sum_{2 \le i \le n} y_i 1_{E_i}$. Note that by construction, $A_j + B \in [Y_j]$ and $A + B \in [Y]$. By the event-continuity of ρ, $\lim_{j \to \infty} \rho(A_j + B) = \rho(A + B)$, and therefore

$$\lim_{j \to \infty} \widehat{\rho}(Y_j) = \lim_{j \to \infty} \rho(A_j + B) = \rho(A + B) = \widehat{\rho}(Y).$$

Likewise, take a decreasing sequence $\{v_j'\}$ in $\mathbf{D}$ converging to y_1, and denote $Y_j' := v_j' 1_{E_1} + \sum_{2 \le i \le n} y_i 1_{E_i}$. The same argument establishes that

$$\lim_{j \to \infty} \widehat{\rho}\left(Y_j'\right) = \widehat{\rho}(Y).$$

By the induction assumption, for all j,

$$\widehat{\rho}(Y_j) = I(Y_j) \text{ and } \widehat{\rho}\left(Y_j'\right) = I\left(Y_j'\right).$$

Hence, by the monotonicity of I,

$$\widehat{\rho}(Y) = \lim_{j \to \infty} \widehat{\rho}(Y_j) = \lim_{j \to \infty} I(Y_j) \le I(Y) \le \lim_{j \to \infty} I\left(Y_j'\right) = \lim_{j \to \infty} \widehat{\rho}\left(Y_j'\right) = \widehat{\rho}(Y),$$

which yields

$$\widehat{\rho}(Y) = I(Y)$$

as desired. $\square$

Proof of Theorem 2.

Step 1. constant-linearity of $\widehat{\rho}$ implies P4.

Take any $A, B \in \Sigma$ such that $\rho(A) \geq \rho(B)$, and any $x, y \in X$ with $u(y) < u(x)$. In view of Theorem 1, it suffices to show that $\widehat{\rho}(u \circ [x, A; y, A^c]) \geq \widehat{\rho}(u \circ [x, B; y, B^c])$. Indeed, this follows easily from the equalities

$$u \circ [x, A; y, A^c] = u(x)1_A + u(y)1_{A^c} = (u(x) - u(y))\, 1_A + u(y)1_\Omega,$$

whence by constant-linearity

$$\widehat{\rho}(u \circ [x, A; y, A^c]) = (u(x) - u(y))\, \rho(A) + u(y),$$

and similarly

$$\widehat{\rho}(u \circ [x, B; y, B^c]) = (u(x) - u(y))\, \rho(B) + u(y),$$

from which the desired conclusion follows immediately.

Step 2. P4 implies Union and Splitting Invariance.

Consider any $A \in \Sigma$, $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$, and $A' \in \Lambda_A$ as well as $B_1 \in \Lambda_A$ and $B_2 \in \Lambda_{A^c}$ (both disjoint from A') such that $\overline{\pi}(A'/A) = \alpha$ and $\overline{\pi}(B_1/A) = \overline{\pi}(B_2/A^c) = \beta$, and let $B = B_1 + B_2$.

Claim: $\rho(A' + B) = \alpha\rho(A) + \beta$.

Pick consequences y, x such that $u(y) = \beta$ and $u(x) = \alpha + \beta$. By utility sophistication and the conditional linearity property of $\widehat{\rho}$ (Lemma 1),

$$[x, A; y, A^c] \sim [x^+, A'+B_1; x^-, A \backslash (A' + B_1); x^+, B_2; x^-, A^c \backslash B_2] = [x^+, A'+B; x^-, (A' + B)^c].$$

Moreover, taking any $T \in \Lambda$ with $\overline{\pi}(T) = \rho(A)$, by P4,

$$[x, A; y, A^c] \sim [x, T; y, T^c],$$

and thus by transitivity

$$[x^+, A' + B; x^-, (A' + B)^c] \sim [x, T; y, T^c].$$

One computes $\widehat{\rho}\,(u \circ [x, T; y, T^c]) = E_{\overline{\pi}}\,(u \circ [x, T; y, T^c]) = (\alpha + \beta)\,\overline{\pi}\,(T) + \beta\pi^0\,(T^c) = \alpha\rho\,(A) + \beta$, whence

$$\rho(A' + B) = \widehat{\rho}(1_{A'+B}) = \widehat{\rho}\,(u \circ [x, T; y, T^c]) = \alpha\rho\,(A) + \beta,$$

verifying the claim.

Specialized to the case $\beta = 0$, the Claim clearly entails Splitting Invariance.

To obtain Union Invariance, choose any $A \in \Sigma$ and $C \in \Lambda$ disjoint from A. It clearly suffices to show that $\rho(A + C) = \rho(A) + \rho(C)$.

Take any $A' \in \Lambda_A$ such that $\overline{\pi}(A'/A) = \frac{1}{2}$ and any $C' \in \Lambda_C$ such that $\overline{\pi}(C'/C) = \frac{1}{2}$. Clearly, $C' \in \Lambda$ and $A' + C' \in \Lambda_{A+C}$ with $\overline{\pi}(A' + C'/A + C) = \frac{1}{2}$. Hence by Splitting Invariance,

$$\rho(A' + C') = \frac{1}{2}\left(\rho(A + C)\right). \tag{7}$$

Now choose $B_1 \in \Lambda_A$ and $B_2 \in \Lambda_{A^c}$ (both disjoint from A') such that $\overline{\pi}(B_1/A) = \overline{\pi}(B_2/A^c) = \frac{1}{2}\rho(C)$. Evidently, $B = B_1 + B_2 \in \Lambda$ with $\overline{\pi}\,(B) = \overline{\pi}\,(C') = \frac{1}{2}\rho(C)$. It is easily verified that therefore $A' + C' \equiv A' + B$, whence by Compatibility,

$$\rho\,(A' + C') = \rho\,(A' + B)\,. \tag{8}$$

Since $\frac{1}{2} + \frac{1}{2}\rho(C) \leq 1$, the Claim can be applied, yielding

$$\rho\,(A' + B) = \frac{1}{2}\rho(A) + \rho(B) = \frac{1}{2}\left(\rho(A) + \rho(C)\right). \tag{9}$$

Combining equations (7), (8), and (9) yields the desired result.

Step 3a) Union Invariance implies constant-additivity.

Take any $Y = \sum_{i \in I} y_i 1_{E_i}$ and $c \in [0,1]$ such that $Y + c1_\Omega \in \mathcal{Z}$. Since $Y \leq (1-c)1_\Omega$, there exist $A \in [Y]$ and $S, T \in \Lambda$ such that $\rho(S) = \rho(A) \leq 1 - c$, $\rho(T) = c$, and T is disjoint from both A and S. To see this, take $A = \sum_{i \in I} A_i$ with $A_i \in \Lambda_{E_i}$ and $\overline{\pi}(A_i/E_i) = y_i$, $S = \sum_{i \in I} S_i$ with $S_i \in \Lambda_{E_i}$ and $\overline{\pi}(S_i/E_i) = \rho(A)$, and $T = \sum_{i \in I} T_i$ with $T_i \in \Lambda_{E_i}$ and $\overline{\pi}(T_i/E_i) = c$ such that T_i is disjoint from both A_i and S_i, for all $i \in I$; such A_i, S_i, and T_i exist by the convex-rangedness of Π. Clearly, $A + T \in [Y + c1_\Omega]$. Since $A \sim_{bet} S$ by assumption, $A + T \sim_{bet} S + T$ by Union Invariance which in turn is tantamount to

$$\rho(A + T) = \rho(S + T) = \rho(S) + \rho(T) = \rho(A) + c.$$

Hence

$$\widehat{\rho}(Y + c1_\Omega) = \rho(A + T) = \rho(A) + c = \widehat{\rho}(Y) + c.$$

Step 3b) Splitting Invariance implies positive homogeneity

Take $Y \in \mathcal{Z}$ and rational $c = \frac{m}{n} \leq 1$, where m and n are natural numbers. Take $A \in [Y]$ and $T \in \Lambda$ such that $\overline{\pi}(T) = \widehat{\rho}(Y)$. By equidivisibility of $\trianglerighteq$ /convex-rangedness of Π, there exist partitions of A and T can be split into n equally likely subevents $\{A_1, ..., A_n\}$ and $\{T_1, ..., T_n\}$; by an argument paralleling that in i), the A_i can be chosen to belong to $[\frac{1}{n}Y]$, whence $\sum_{i \leq m} A_i \in [\frac{m}{n}Y]$. Since by construction $A \sim_{bet} T$, by Splitting Invariance $A_1 \sim_{bet} T_1$, and therefore by Splitting Invariance again $\sum_{i \leq m} A_i \sim_{bet} \sum_{i \leq m} T_i$. It follows that

$$\widehat{\rho}\left(\frac{m}{n}Y\right) = \rho\left(\sum_{i \leq m} A_i\right) = \overline{\pi}\left(\sum_{i \leq m} T_i\right) = \frac{m}{n}\overline{\pi}(T) = \frac{m}{n}\widehat{\rho}(Y),$$

which establishes positive homogeneity for rational α. This implies positive homogeneity for arbitrary α, since by monotonicity of $\widehat{\rho}$,

$$\begin{aligned} \alpha\widehat{\rho}(Y) &= \sup\{\beta\widehat{\rho}(Y) \mid \beta \leq \alpha, \beta \in \mathbf{Q}\} = \sup\{\widehat{\rho}(\beta Y) \mid \beta \leq \alpha, \beta \in \mathbf{Q}\} \\ &\leq \widehat{\rho}(\alpha Y) \leq \inf\{\widehat{\rho}(\beta Y) \mid \beta \geq \alpha, \beta \in \mathbf{Q}\} = \alpha\widehat{\rho}(Y), \end{aligned}$$

and thus $\widehat{\rho}(\alpha Y) = \alpha\widehat{\rho}(Y)$. $\square$

Proof of Proposition 2. The necessity of Union and Splitting Invariance follows from Theorem 2. The validity of the converse can be seen as follows. First, applying the proof of Theorem 1 to preferences over unambiguous acts $\succsim_{ua}$, one infers that these preferences have a SEU representation with utility function u, unique up to positive affine transformations. Likewise, applying the proof of Proposition 1, there exists a unique event-continuous ρ representing $\succsim_{\mathcal{B}}$ such that $\rho(T) = \overline{\pi}(T)$ for $T \in \Lambda$. Let $\widehat{\rho}$ denote the associated expectation operator given by (1). By the proof of the implication 3)$\Longrightarrow$2) of Theorem 2, $\widehat{\rho}$ is constant-linear. Define $\succsim$ by setting for all $f, g \in \mathcal{F}$:

$$f \succsim g \text{ iff } \widehat{\rho}(u \circ f) \geq \widehat{\rho}(u \circ g). \tag{10}$$

Clearly, by the implication 1)$\Longrightarrow$3) of Theorem 1, if an extension with the desired properties exists, it must be given by (10). Conversely, this preference ordering $\succsim$ is Archimedean and tradeoff-consistent by the implication 3)$\Longrightarrow$1) of Theorem 1. Since ρ agrees with $\overline{\pi}$ on Λ, the restriction of $\succsim$ to $\mathcal{F}^{ua}$ agrees with $\succsim_{ua}$. Furthermore, by construction $\succsim_{bet} = \succsim_{\mathcal{B}}$. Since $\widehat{\rho}$ is constant-linear, $\succsim$ satisfies P4 by the implication 2)$\Longrightarrow$1) of Theorem 2.

Finally, we need to show that the ordering $\succsim$ given in (10) does not depend on the context $\trianglerighteq$. That is, take two equidivisible contexts $\trianglerighteq^1$ and $\trianglerighteq^2$ with associated Λ^1 and Λ^2 relative to which $\succsim_{\mathcal{B}}$ is Archimedean, compatible and satisfies Union- and Splitting-Invariance, and take preference relations over unambiguous acts $\succsim_{ua}^1$ and $\succsim_{ua}^1$ with the same associated preferences over lotteries, hence with the same representing vNM utility function u. Let $\succsim^1$ and $\succsim^2$ denote the extensions to all Savage acts given by (10). Then we claim that in fact $\succsim^1 = \succsim^2$.

To see this, by Theorem 2, each $\succsim^i$ has a constant-linear representation $I \circ u$ with $u(X) = [0, 1]$, ensuring applicability of Proposition 7 in section 6 below. For

$i = 1, 2$, let $\succsim_{bet}^{*i}$ denote associated revealed likelihood relations. Since $\succsim^i$ is utility-sophisticated with respect to the equidivisible context $\trianglerighteq^i$ by construction and since $\succsim_{bet}^1 = \succsim_{bet}^2 = \succsim_{\mathcal{B}}$, by Proposition 7 evidently $\succsim_{bet}^{*1} = \succsim_{bet}^{*2}$. For $i = 1, 2, 3$ let $\Lambda^i, [.]^i, \rho^i, \widehat{\rho}^i$ denote the families of unambiguous events, equivalence class operators, normalized capacities and intrinsic integrals associated with $(\succsim_{\mathcal{B}}, \trianglerighteq^1)$, $(\succsim_{\mathcal{B}}, \trianglerighteq^2)$, and $(\succsim_{\mathcal{B}}, \succsim_{bet}^{*1} = \succsim_{bet}^{*2}$), respectively. By maximality of $\succsim_{bet}^{*1}$ and $\succsim_{bet}^{*2}$, evidently $\Lambda^3 \supseteq \Lambda^1 \cup \Lambda^2$ and $[Z]^3 \supseteq [Z]^1 \cup [Z]^2$ for all $Z \in \mathcal{Z}$. Thus, clearly $\rho^1 = \rho^3 = \rho^2$ and $\widehat{\rho}^1 = \widehat{\rho}^3 = \widehat{\rho}^2$. By Theorem 1 therefore $\succsim^1 = \succsim^2$. $\square$

Proof of Fact 1.

That ii) implies i) is straightforward.

The converse follows easily from showing that if $\succsim$ is strongly compatible with a complete $\trianglerighteq$ represented by a convex-ranged prior π, then $f \succsim g$ whenever $u \circ f$ stochastically dominates $u \circ g$. But viewing $\trianglerighteq$ as the revealed likelihood relation associated with $\succsim$, Strong Compatibility is just Machina-Schmeidler's (1992) Strong Comparative Probability axiom, and the present claim is the key step in their proof. $\square$

Proof of Proposition 4.

For $x \in X$, let δ_x denote the lottery putting probability 1 on x; also, for $q \in \mathcal{L}$, let $E(q) = \sum_{x \in X} x q_x$. In view of Proposition 1, it is w.l.o.g. to assume that $X = [0, 1]$ and $u(x) = x$ for all $x \in X$; this simply means that consequences are referred by the utilities they generate. As a result, acts $f \in \mathcal{F}$ can be identified with random variables $Z \in \mathcal{Z}$. If $\succsim_{AA}$ satisfies Lottery Independence, then the linearity assumption on u implies risk-neutrality, i.e., for all $q \in \mathcal{L}$, $q \sim_{AA} \delta_{E(q)}$, a simplification that will be used repeatedly in the proofs.

1. Trade-off Consistency of $\succsim$ implies Monotonicity and Lottery Independence of $\succsim_{AA}$

Lottery Independence follows immediately from EU maximization over unambiguous acts implied by Theorem 1. To verify Monotonicity, take any act $F \in \mathcal{F}^{AA}$, lotteries $p, q \in \mathcal{L}$ and event $S \in \Sigma_1$ such that $p \succsim_{AA} q$. By EU/expected value maximization over unambiguous acts, $p \sim_{AA} \delta_{E(p)}$ and $q \sim_{AA} \delta_{E(q)}$, hence $E(p) \geq E(q)$. Let Y and Z be Savage acts such that $F(Y) = [p$ on $S; F(\omega)$ elsewhere$]$, $F(Z) = [q$ on $S; F(\omega)$ elsewhere$]$, and $Y_{-S\times\Omega_2} = Z_{-S\times\Omega_2}$. By the Conditional Linearity property of the intrinsic integral (Lemma 1) and Theorem 1, $Y \sim [E(p)$ on $S \times \Omega_2, Y_{-S\times\Omega_2}]$ and $Z \sim [E(q)$ on $S \times \Omega_2, Z_{-S\times\Omega_2}]$. Since $E(p) \geq E(q)$, by Eventwise Monotonicity and transitivity therefore $Y \succsim Z$, which implies that $[p$ on $S; F(\omega)$ elsewhere$] \succsim_{AA} [q$ on $S; F(\omega)$ elsewhere$]$, as needs to be shown.

2. Monotonicity and Lottery Independence of $\succsim_{AA}$ imply Trade-off Consistency of $\succsim$.

In view of Theorem 1, we need to show that $I = \widehat{\rho}$. To do so, we need to show that for any $Z \in \mathcal{Z}$ there exists $A \in [Z]$ such that $Z \sim 1_A$ and thus $I(Z) = \rho(A)$.

Take any $Z \in \mathcal{Z}$ and write $Z = \sum_{i\leq n, j\leq n_i} z_{i,j} 1_{S_i\times T_{i,j}}$. By convex-rangedness of η, there exist for each $i \leq n, j \leq n_i$ events $T'_{i,j} \in \Sigma_2$ such that $T'_{i,j} \subseteq T_{i,j}$ and $\eta\left(T'_{i,j}\right) = z_{i,j}\eta(T_{i,j})$. Let $A := \sum_{i\leq n, j\leq n_i} S_i \times T'_{i,j}$. By construction, for all i, j, $A \cap (S_i \times T_{i,j}) = S_i \times T'_{i,j}$ and $\overline{\pi}(A/S_i \times T_{i,j}) = z_{i,j}$, whence $A \in [Z]$. Write $F(Z) = [p_i$ on $S_i]$ and $F(1_A) = [q_i$ on $S_i]$.

Since for all $i \leq n$,

$$E(p_i) = \sum_{j\leq n_i} z_{i,j}\eta(T_{i,j}) = \sum_{j\leq n_i} \eta\left(T'_{i,j}\right) = E(q_i),$$

by risk-neutrality

$$p_i \sim_{AA} q_i \text{ for all } i.$$

By Monotonicity, this implies

$$F(Z) \sim_{AA} F(1_A),$$

which, by the construction of $\succsim_{AA}$, yields $Z \sim 1_A$ as desired .

3. Trade-off Consistency and P4 of $\succsim$ imply Certainty Independence of $\succsim_{AA}$.

Take any AA-acts $F = [p_i \text{ on } S_i], G = [q_i \text{ on } S_i]$, any constant act $H = [q \text{ on } \Omega_1]$ and $\alpha \in (0,1]$.

Let $Y := \sum_i E(p_i) 1_{S_i \times \Omega_2}$ and $Z := \sum_i E(q_i) 1_{S_i \times \Omega_2}$. By Theorem 2, $\widehat{\rho}$ is constant-linear. Hence by Theorem 1,

$$Y \succsim Z \text{ if and only if } \alpha Y + (1-\alpha) E(q) \succsim \alpha Z + (1-\alpha) E(q). \tag{11}$$

By the Monotonicity and Lottery Independence (hence risk-neutrality) shown in part 1),

$$\begin{aligned}
F &\sim_{AA} [\delta_{E(p_i)} \text{ on } S_i] = F(Y), \\
G &\sim_{AA} [\delta_{E(q_i)} \text{ on } S_i] = F(Z), \\
\alpha F + (1-\alpha)H &\sim_{AA} [\delta_{\alpha E(p_i)+(1-\alpha)E(q)} \text{ on } S_i] = F(\alpha Y + (1-\alpha) E(q)), \text{ and} \\
\alpha G + (1-\alpha)H &\sim_{AA} [\delta_{\alpha E(q_i)+(1-\alpha)E(q)} \text{ on } S_i] = F(\alpha Z + (1-\alpha) E(q)).
\end{aligned}$$

In view of the equivalences established above, this translates back into the desired conclusion

$$F \succsim_{AA} G \text{ if and only if } \alpha F + (1-\alpha)H \succsim_{AA} \alpha G + (1-\alpha)H.$$

4. Monotonicity and Certainty Independence of $\succsim_{AA}$ imply P4 of $\succsim$.

Since $\succsim$ is trade-off consistent by part 2), in view Theorems 1 and 2 it suffices to establish the constant-linearity of the intrinsic integral $\widehat{\rho}$. Take any $Z \in \mathcal{Z}$, and $\alpha, c \in [0,1]$ such that $\alpha + c \leq 1$; we need to show that $\widehat{\rho}(\alpha Z + c1_\Omega) = \alpha \widehat{\rho}(Z) + c$ in any such case.

By definition of $\widehat{\rho}$,

$$Z \sim \widehat{\rho}(Z)\, 1_\Omega,$$

hence

$$F(Z) \sim_{AA} F(\widehat{\rho}(Z)\, 1_\Omega) = \boldsymbol{\delta}_{\widehat{\rho}(Z)}.$$

By Certainty Independence, it follows that

$$\begin{aligned} F\left(\alpha Z + c1_\Omega\right) &= \alpha F\left(Z\right) + (1-\alpha)\boldsymbol{\delta}_{\frac{c}{1-\alpha}} \\ &\sim_{AA} \alpha\boldsymbol{\delta}_{\widehat{\rho}(Z)} + (1-\alpha)\boldsymbol{\delta}_{\frac{c}{1-\alpha}} \\ &\sim_{AA} \boldsymbol{\delta}_{\alpha\widehat{\rho}(Z)+c} \\ &= F\left(\left(\alpha\widehat{\rho}(Z)+c\right)1_\Omega\right), \end{aligned}$$

whence from the definition of $\succsim_{AA}$,

$$\alpha Z + c1_\Omega \sim \left(\alpha\widehat{\rho}(Z)+c\right)1_\Omega,$$

and therefore

$$\widehat{\rho}(\alpha Z + c1_\Omega) = \alpha\widehat{\rho}(Z) + c$$

by the normalization of $\widehat{\rho}$. $\square$

Proof of Proposition 5.

Suppose that a CEU preference ordering $\succsim$ is utility-sophisticated relative to the equidivisible context $\trianglerighteq$. Since by EU maximization on unambiguous acts, $\nu = \rho$, we need to show that ρ is additive. Thus, take any disjoint events $A, B \in \Sigma$ as well as events $A' \subseteq A$ such that $A' \equiv A \backslash A'$ and $B' \subseteq B$ such that $B' \equiv B \backslash B'$. Specify consequences in utiles, and, for $z \in [0,1]$ let

$$f_z := [1 \text{ on } B', z \text{ on } A, 0 \text{ elsewhere}],$$

and

$$g_z := [\frac{1}{2} \text{ on } B, z \text{ on } A, 0 \text{ elsewhere}].$$

By construction, for all $\pi \in \Pi$, $E_\pi f_z = E_\pi g_z$, hence for all $z \in [0,1]$,

$$f_z \sim g_z$$

by utility sophistication.

Now

$$\int u \circ f_z d\nu = \rho(B') + z\left[\rho(A+B') - \rho(B')\right],$$

while, for $z \geq \frac{1}{2}$

$$\int u \circ g_z d\nu = z\rho(A) + \frac{1}{2}\left[\rho(A+B) - \rho(A)\right],$$

and for $z \leq \frac{1}{2}$

$$\int u \circ g_z d\nu = \frac{1}{2}\rho(B) + z\left[\rho(A+B) - \rho(B)\right].$$

Thus, $\int u \circ f_z d\nu = \int u \circ g_z d\nu$ for $z \in \{0, \frac{1}{2}, 1\}$ only if

$$\rho(A) = \rho(A+B') - \rho(B') = \rho(A+B) - \rho(B),$$

i.e. only if $\rho(A) + \rho(B) = \rho(A+B)$, as needed to be shown. $\square$.

Proof of Proposition 6.

Part 1). Take any $x, y, x', y' \in X$ such that $x \succsim x'$, acts $f, g \in \mathcal{F}$ and events A disjoint from B and A' disjoint from B' such that $A \equiv B \rhd\rhd \emptyset$, $A' \equiv B'$ and

$$[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}]. \tag{12}$$

By the equidivisibility of $\trianglerighteq^0$, there exist events A'' disjoint from B'' and A''' disjoint from B''' such that $A'' + B'' = A + B$, $A''' + B''' = A' + B'$, $A'' \equiv_0 B'' \rhd\rhd \emptyset$ and $A''' \equiv_0 B'''$.

Clearly by coherence and the fact that $\trianglerighteq \supseteq \trianglerighteq^0$, $A \equiv A''$ and $B \equiv B''$. Therefore by Strong Compatibility,

$$[x \text{ on } A; y \text{ on } B; f(\omega) \text{ elsewhere}] \sim [x \text{ on } A''; y \text{ on } B''; f(\omega) \text{ elsewhere}], \tag{13}$$

and

$$[x' \text{ on } A; y' \text{ on } B; f(\omega) \text{ elsewhere}] \sim [x' \text{ on } A''; y' \text{ on } B''; f(\omega) \text{ elsewhere}]. \tag{14}$$

Combining (12), (13) and (14) by transitivity therefore

$$[x \text{ on } A''; y \text{ on } B''; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A''; y' \text{ on } B''; f(\omega) \text{ elsewhere}]. \tag{15}$$

By Trade-off Consistency with respect to $\trianglerighteq^0$ thus

$$[x \text{ on } A'''; y \text{ on } B'''; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'''; y' \text{ on } B'''; f(\omega) \text{ elsewhere}]. \tag{16}$$

By the same token as above, $A' \equiv A'''$ and $B' \equiv B'''$, and therefore

$$[x \text{ on } A'''; y \text{ on } B'''; f(\omega) \text{ elsewhere}] \sim [x \text{ on } A'; y \text{ on } B'; f(\omega) \text{ elsewhere}], \tag{17}$$

and

$$[x' \text{ on } A'''; y' \text{ on } B'''; f(\omega) \text{ elsewhere}] \sim [x' \text{ on } A'; y' \text{ on } B'; f(\omega) \text{ elsewhere}]. \tag{18}$$

Combining (16), (17) and (18) by transitivity therefore

$$[x \text{ on } A'; y \text{ on } B'; f(\omega) \text{ elsewhere}] \succsim [x' \text{ on } A'; y' \text{ on } B'; f(\omega) \text{ elsewhere}] \tag{19}$$

as desired.

Part 2). Suppose $\succsim$ is utility-sophisticated, Archimedean, bounded, solvable with respect to the equidivisible $\trianglerighteq$ context that contains $\trianglerighteq^0$. Clearly, $\succsim$ is also Archimedean with respect to $\trianglerighteq$, since $\triangleright\triangleright$ contains $\triangleright\triangleright^0$; likewise, $\succsim$ is solvable with respect to $\trianglerighteq$. By Theorem 1 (2$\Longrightarrow$1), preferences are trade-off consistent with respect to $\trianglerighteq^0$. Since they are strongly compatible with $\trianglerighteq$ by assumption, they are also trade-off consistent with respect to $\trianglerighteq$ by part 1) . By Theorem 1 again (1$\Longrightarrow$2), preferences are utility-sophisticated with respect to $\trianglerighteq$. $\square$

Proof of Proposition 8.

The equivalence of (1) and (2) is immediate from Proposition 7. The equivalence of (2) and (3) follows from the fact that $\Pi_{\left(\succsim^*_{bet}\right)} = \Pi^*$, which in turn follows from the uniqueness of the multi-representation of equidivisible likelihood relations shown in Nehring (2006, Theorem 2).

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