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Credibility and Strategic Learning in Networks

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Abstract

This paper studies a model of diffusion in a fixed, finite connected network. There is an interested party that knows the quality of the product or idea being propagated and chooses an implant in the network to influence other agents to buy or adopt. Agents are either "innovators", who adopt immediately, or rational. Rational consumers buy if buying rather than waiting maximizes

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expected utility. We consider the conditions on the network under which optimal diffusion of the good product with probability one is a perfect Bayes equilibrium. The structure of the entire network turns out to be important. We also discuss various inoptimal equilibria.

1 Introduction

1.1 Main features

This paper studies a model of diffusion and social learning (of what we call broadly "technology") in a connected network with the following essential features:

1. An individual, known henceforth as the firm, has private information about the quality of the technology it seeks to diffuse to potential consumers. This firm is outside the network but might choose to pay some agent in the network to propagate its product or idea. The agent so chosen is referred to here as an "implant".

2. The network is populated by agents or players, one at each node. Each agent observes the actions of his or her neighbors over time and makes a decision on whether to adopt the technology or not. These players come in two varieties, innovators, who always adopt the technology, and standard players, each of whom is fully rational and makes a decision on whether to adopt or not based on utility maximization.

3. The network is given exogenously; agents with direct links can engage in restricted communication with each other, however agents who are not directly linked can have no communication with each other. What the firm knows about individual agent interactions is restricted to the direct neighbours of the implant chosen, if any. In particular, the firm is unaware who is an innovator and who is not before it chooses the implant. In the extensions, we consider relaxing this requirement.

4. The structure of the network is common knowledge to the firm and to all agents.

The main novelty of this paper is that the players rationally decide whether the communications they receive from their neighbors are credible or not. Learning therefore occurs through strategic choices by agents and this affects whether diffusion occurs to the whole population or dies out within some finite distance of the origin. As far as we know, ours is the first paper to study this issue in the context of networks in any detail.

1.2 Motivation

There are several different economic problems that motivated us in studying this issue, though the model we end up with does not fit every aspect of each of these motivating problems. One example comes from a New York Times magazine article about viral marketing. The article discusses a company called bzz.com. The article mentions that the company would "implant" agents, with good connections, to sell products like books, CDs or party food items to their friends and social "neighbors". The company would provide talking points to the agents, who would then "recommend" the product to their neighbors.

Another example is suggested by Munshi's [21] empirical study of social learning in the Green Revolution in India. The government or its representative wants to push new high-yielding varieties of seeds for wheat and rice.[1] It chooses an individual in the community whose adoption of the new technology will have the most widespread impact. Neighbors of this individual observe the percentage of the farm acreage he devotes to the new varieties and each neighbor then makes a similar decision for his or her own farm, which is then observed by neighbors of neighbors and so on.[2] In small villages, it is not unreasonable to assume common knowledge of the network and acreage planted is easily observable, so this example fits some key features of the model.

The use of implants or "farmer facilitators" in such settings has also been reported.

"To spread the word [about soil micronutrients], Karnataka hired, on a seasonal basis, "farmer facilitators" from within communities rather than outsiders, on the assumption that villagers were more likely to listen to their peers than strangers."[3]

Of course, our model is not going to capture all the characteristics associated with these examples. However, our attempt has been to construct a tractable model which captures the main features underlying these models.

[1]This example may seem inappropriate if one views the government as a benevolent agent because that would rule out its promoting a bad technology. However, governments in developing countries are sometimes said to be "bought out" by "big" business houses. Also, there is no reason why private entities such as companies producing new types of agricultural inputs cannot replace the government in Munshi's story.

[2]A farmer is convinced about the efficacy of the seeds in some way, maybe by testing them, before devoting substantial acreage to them instead of the existing varieties.

[3]Reported in "The Guardian", March 13, 2013.

1.3 The model, a brief verbal description

The players in this model are the *seller* or the *firm,* which knows the quality of the product it seeks to sell and potential buyers who are arranged in a *fixed exogenous network,* whose structure is common knowledge. We shall use this terminology to describe the motivating examples in the last subsection. For instance, the seller could be a medical technology company seeking adoption by doctors of a particular method of surgery, with the buyers being the doctors. The given network is the exogenous structure of social communication among the doctors in a particular specialty.

A buyer finds out the quality of the product and then makes a recommendation to her neighbors if she finds that the product is "good". Notice that we rule out the possibility of *negative* recommendations. In many contexts, this is not a bad assumption. For instance, a doctor may not want to publicize the fact that a particular method of treatment has not worked. Thus "no recommendation" is noisy bad news and a "recommendation" is possibly good news.[4]

The seller can choose to "seed" the network by paying an agent at any given node in the network to give a positive recommendation about the seller's product. Only one node in the network can be so "seeded" and, as mentioned earlier, is referred to in the paper as an *implant. Note that an implant's identity is known only to the firm and the implant and not to any of the potential buyers.* Thus a potential buyer cannot distinguish between another ordinary buyer and an implant of the firm.

Buyers have ex ante beliefs about whether the product is good and can be one of two types. There is some probability that a buyer at a given node is an "innovator", who will try the new product immediately; with the complementary probability a buyer is normal, in that she makes a rational decision on whether to buy or not. We assume (to avoid trivialities) that the ex ante belief is below the threshold required to induce the second type of buyers to purchase the product.

Buyers who are not innovators and who receive recommendations from their neighbors have to form posterior beliefs about the quality of the product, and then decide whether to buy the product. Each purchase gives the seller a unit profit (prices are assumed to be fixed), and future payoffs are discounted; so if buying is optimal, buying now is better than not buying, or waiting in the case of non-myopic buyers.

The game proceeds as follows: In the first period, the type of the seller is first drawn and is private information for the firm. Given the type, the seller chooses at most one implant. At the same time, each buyer observes whether he or she

[4]In the Extensions section, we briefly discuss what happens if we allow agents to make negative recommendations.

is an innovator or not. Each buyer observes only his or her own type, the type of the buyer being either "innovator" or "rational". The probability distributions of buyer types and seller types are common knowledge. Innovators then buy and make recommendations or not. The implant, if there is one, may decide to make a recommendation immediately to his or her neighbors, or to wait until a later period. Each buyer, implant or not, can speak only once.

In any period, buyers who did not buy in the earlier periods observe recommendations or the absence of recommendations from their neighbors, make Bayesian inferences from these events and decide whether to buy or not. The time at which an implant makes a recommendation is a strategic choice. However, since all payoffs in future periods are discounted with a fixed, common discount factor $\delta < 1$, the G-type implant has no incentive to defer his recommendation and will therefore speak in period one. Since the network is finite, all information will percolate through the network in finite time after either an implant or an innovator speaks. We can therefore treat the game as finite by choosing an appropriately long horizon, based on the maximum traversal time in the network.

This is therefore a finite game of asymmetric information and we focus on the Perfect Bayesian Equilibria of this game, which are known to exist for finite games. Our principal interest is in studying the conditions under which the product of good quality will diffuse with probability one throughout the network, and in the fastest possible time - we call this the *optimal diffusion equilibrium* (ODE). In particular, we investigate the types of network structures that are consistent with the existence of an ODE. We will also look at conditions under which the good product will diffuse throughout the network with probability one, even though it may not do so in the fastest possible time. We will call such an equilibrium a *complete diffusion equilibrium* (CDE). Obviously, an ODE is a CDE.

Our analysis shows that the two types of firm will follow very different targeting strategies in equilibrium. A firm that produces a good quality product will be interested in the extent and speed of diffusion over time. In an ODE, such a firm will want to place its implant at a node that maximizes a measure of centrality which we will denote as "diffusion centrality". On the other hand, a firm producing a bad quality product is more myopic, because it knows that its product will not sell beyond one period. So, this firm will want to place its implant at nodes that have the highest number of connections. However, the bad quality firm must also ensure that its agent's recommendation is credible. For instance, if consumers know that the bad quality firm implants a particular node with probability one, then that node's recommendation is less likely to be credible.

The optimal behavior of the two types of firm determine whether an ODE or

CDE exists or not. Our analysis reveals some counter-intuitive results. For instance, a popular theme in the existing literature on diffusion of products or viruses (in epidemiology, transferred over to the analysis of "viral" marketing) suggests that it is optimal for the seller to choose an "influential" member of the population to be its representative.[5] A "naive" view is that highly connected individuals are more likely to be influential. However, our analysis shows that if any individual is "too influential" in the sense of being connected to *everyone*, then an ODE cannot exist. This is because both types of firms would target such an individual, thereby destroying the credibility of her recommendation. Of course, optimal diffusion would typically be guaranteed in contexts where the *credibility of recommendations* is not at stake. This is one point in which the strategic element in the problem has bite. It also turns out for somewhat subtler reasons that *larger* (in a sense to be described later) networks are more likely to support optimal diffusion.

In section 4, we discuss the role of the assumption that consumers may be innovators with positive probability. In particular, we show that diffusion becomes harder if this probability is zero. Section 5 focuses on the line network. Although this is a very specific network, we focus on it because this network illustrates very well several of the issues that the paper is trying to emphasize. In particular, we show that if the line is longer than a threshold value, then the only PBE are ODE under a mild restriction on out of equilibrium beliefs. this result is a partial justification of why we focus on ODE. Section 7 provides a partial characterization of networks which support ODEs. Section 8 discusses the sense in which "too well connected" networks cannot sustain optimal diffusion. In an Extensions section, we briefly discuss the robustness of our results do some modifications of the model.

We conclude this subsection with a point about the sequential rationality requirement. We can go very far with just Bayesian updating, especially if the probability that a buyer is an innovator is positive. However, there are cases in which probability zero events have to be considered (off the proposed equilibrium path) and the way these are handled will determine what the equilibrium is. We will discuss such beliefs wherever needed in the paper.

[5] This has given rise to the development of algorithms to locate the most influential members in a network. See Richardson and Domingos [23] and Kempe [19].

1.4 Related literature

There has been a voluminous literature on diffusion of innovations arising from different causes.[6] The paper that is closest to ours is Galeotti and Goyal [13]. However, unlike our work, Galeotti and Goyal do not consider a seller with private information or Bayesian buyers who take into account how their current state of information reveals what is happening in unobserved parts of the network.

Our work is also related to social learning in networks, as in Bala and Goyal [1], Chatterjee and Xu [2].[7] The main difference between our work and these papers is that we have a strategic seller who has private information about quality and is trying to manipulate the diffusion, whilst none of the other papers do. Also our buyers are rational Bayesians. It is not surprising therefore that our equilibria are quite different.

Since the main novelty about this paper is the set-up with a motivated, privately-informed seller and rational buyers, we briefly explain here the features that arise from this modeling choice. First, there is the credibility of recommendations, which depends on the equilibrium decisions of good and bad types of seller. Second, there is the implant having to take into account not only the number of possible recipients of the recommendation, but also the neighbors of these recipients. (If none of these neighbors of neighbors makes a recommendation, this is noisy bad news and affects credibility differently for different individuals.) Third, a potential buyer analyzes, from the equilibrium strategies, what is happening in the parts of the network she cannot directly observe and this affects what she infers from the absence or presence of a recommendation from a neighbor.

One effect of these three factors is that the *entire* network is important. That is, we cannot describe the results in terms of simple parameters such as the degree distribution or the diameter or the connectivity. These do not provide detailed enough structure to account fully for rational behavior.

2 The formal model

In this section, we describe the basic model. The set $N = \{1, 2, \ldots, n\}$ represents the set of consumers. The structure of interactions between the set of consumers is represented by means of a graph Γ in which the nodes are elements of N and $ij \in \Gamma$

[6] See for instance, Peyton Young [26], Draief and Massoulie [7], Durrett [8], Conley and Udry [6].

[7] See Goyal[14] for an illuminating survey of papers on learning in networks. Other related papers are Ellison[9], Ellison and Fudenberg[10], Ellison and Fudenberg[11].

if consumers i and j can communicate with each other. There is a firm F, which is interested in selling its product. The product is either of type $G(ood)$ or $B(ad)$. Firm F knows the type of its product and other agents only know the common knowledge probability that the product is good, p. We assume the graph is connected, so there is a path through which diffusion can occur between any two individuals.

All buyers have an initial probability $p^0 \equiv p$ that the product is of the good type. There are *two* types of consumers. Buyers of the first type - we refer to them as the *innovators*- buy immediately. This could, for instance, be because they derive a relatively high utility from the good product or a not very low utility from the bad product. The second type of buyers (the *normal* types) get utilities g and $-b$ from the G and B type products. Throughout the paper, we retain the following assumption.

Assumption 1

$$p < \bar{p} \equiv \frac{b}{b+g}$$

Each consumer buys the product at most once. The firm gets 1 unit for each item purchased and 0 if an item is not purchased. There are no capacity constraints on the number of items sold.

Future payoffs are discounted by δ for F and for the consumers.

Payoffs to firm F and implant.

Conditional on there being an implant, we assume that the implant is an employee of the firm. The fixed cost c of hiring the implant is borne solely by the firm. The implant chooses his or her strategy in order to maximize the firm's payoff. If k_τ items are purchased at time τ, the firm's payoff is $\sum_\tau \delta^{\tau-1}k_\tau - c$, if there is an implant and $\sum_\tau \delta^{\tau-1}k_\tau$ if there is none. Note that this expression does not depend on the type of the firm. Whether a firm is G or B will affect the expected payoff through the conditional probability that a neighbour of a current period buyer purchases in the following period.

The time line: Nature draws the type of F and this is revealed only to F. F chooses a site i to place one "implant" at a cost c or decides not to use an implant. The implant, if any, is paid to pass on a recommendation to his neighbors in Γ. If i is not an implant, she can be an "innovator" in which case she tries the new product immediately. The probability that any site i is an innovator is $\rho < 1$ and the event that "i is an innovator" is independent of other events "$j \neq i$ is an innovator". All this takes place, in sequence, at $t = 0$. At $t = 1$, any i who is an innovator makes a recommendation to his neighbors. An implant of B type can choose any time $t \geq 0$ (also $t \leq T$) to make a recommendation ("speak").[8] The implant of the G type is assumed to speak immediately. The neighbors receiving the recommendation might choose to buy the product or not to buy it. At $t = \tau$, any site who is either an implant or has tried the product and found it good, after receiving a recommendation in $\tau - 1$, can make a recommendation to neighbors. A site i does not observe if neighbors have *received* recommendations or have chosen to buy the product- she only observes whether recommendations are made by the neighbors themselves.

There is no exogenous time limit on the game; however, as pointed out earlier, since there are a finite number of neighbors, each speaks at most once and the G type implant, if there is one, speaks in period 1, the game must end in finite time. That is, all credible information transmission will end in finite time.

In view of Assumption 1, the second type of consumer will not buy the good unless she revises her probability belief about the good. If her updated belief (following a recommendation received by her) in some period t is $p^t = \bar{p}$, she will be indifferent between buying and not buying the good. We assume that buyers are myopic - they make their purchase decision as soon as they receive one recommendation. A non-myopic buyer could wait to receive more information (both recommendation and absence of a recommendation convey information in equilibrium).

Strategies and equilibrium

The strategic players in this paper are (i) the firm, (ii) the implant and (iii) the standard (non-innovative) buyers. To describe the strategies, let the set of nodes in the network be denoted by V. Let ϕ denote 'No implant'. Let $\mathbf{d}_i \in \{0, 1\}$ denote buyer $i's$ action to not buy or buy respectively and $s_i \in \{0, 1\}$ denote $i's$ action to speak (1) or not (0).

It is more convenient to describe the behavioral strategies for the players.

Firm: The firm chooses, given its type, whether to have an implant and where in the network to place him. That is, the firm chooses a mapping $\{G, B\} \rightarrow \Delta(V \cup \phi)$. It could possibly randomise between having an implant and not having one.

Implant: We assume the implant has the same preferences as the type of firm that has hired him; he has to choose, given his type, when to speak or whether to remain silent. Let T be the set of possible time periods at which an implant could speak, given the structure of the network. At time t, $t \geq 1$, let h_{it} denote the private history of the agent at node i, the set of whose neighbors is denoted by $N_i(\Gamma)$. The history consists of a record of whether $j \in N_i(\Gamma) \cup i$ has made a positive recommendation about the new product at time $\tau < t$ or has not made any

[8] We discuss later what changes if the B-type implant is also required to speak in period 1.

recommendation; also whether i has bought the product or not before time t. Let the set of such histories for node i at time $t \in T$ be denoted by H_i^t. Let $s_i \in \{0, 1\}$ denote player $i's$ choice on whether to speak or not. Then the implant at node i has a behavioral strategy at time t, which is a mapping

$\{G, B\} \cup H_i^t \rightarrow \Delta(\{0, 1\})$.

Note, if agent i is an implant, he does not have to choose whether to buy or not. If h_i^t contains $s_i^\tau = 1$, the implant at i is constrained to choose $s_i^t = 0$.

Buyer: Given the history h_i^t, agent $i's$ behavioral strategy is to choose $\mathbf{d}_i \in \{0, 1\}$, possibly through randomization. Here $\mathbf{d}_i^t$ denotes the buying decision; $\mathbf{d}$ is not to be confused with d, the degree of a node. If $\mathbf{d}_i(h_i^\tau) = \mathbf{d}_i^\tau = 1$ for some $\tau < t$, then $\mathbf{d}_i(h_i^t) = 0$. Also, if h_i^t includes $j \in N_i(\Gamma)$ having spoken, then $\mathbf{d}_i^{t+\tau} = 0$, $\tau \geq 1$. That is, the myopic buyer decides immediately after receiving a recommendation to buy or not buy. If she does not buy immediately, she never buys. Her behavioral strategy is given by: $H_i^t \rightarrow \Delta(\{0, 1\})$.[9]

Note that the decision whether to recommend is *not* a strategic decision for the buyer; she recommends if she finds the product good and not if the product is bad. The implant, otherwise indistinguishable externally from a standard buyer, does have a strategic choice on when to recommend. However, the implant does not have a buying decision since he knows the type.

Equilibrium in this game is to be interpreted as Perfect Bayes Equilibrium (or sequential equilibrium, though we do not check for consistency of assessments). The requirements are: (i) Each agent, including the firm and implants but not including the innovators, updates beliefs according to Bayes' Theorem whenever possible and (ii) each agent maximizes her expected payoff at each (private) history given these beliefs. Out-of-equilibrium beliefs will be explicitly described when necessary.

Updating Beliefs

Let α and β be the mixed equilibrium strategies of types G and B respectively. Suppose consumer i receives a recommendation from her neighbor $i-1$ in period 1. If she receives no other recommendation, what is the probability that the product is G ? Let us denote this by $\eta_{i,i-1}^1$, where the superscript refers to the time the recommendation is received and the subscripts to the recipient and the sender of the recommendation.

Let $d_i(\Gamma) = |N_i(\Gamma)|$ be the *degree* of i in Γ. (Henceforth, whenever there is no ambiguity about Γ, we will simply write d_i, d_j, etc.)

[9] Note that this implies that if some node rejects a recommendation, that player's role in the game is over. The recommendation stops at that point along the path involving that player. There is no need to specify his strategy in the ensuing game, since he has no further role in it.

Since the derivation of the probability $\eta^1_{i,i-1}$ is somewhat tedious to check, we reproduce the actual calculation. The probability required is: Prob. [product is G| i is not an implant or innovator and none of the other neighbors other than $i-1$ has made a recommendation and $i-1$ has made a recommendation]

Let's call the conditioned event A and the conditioning event B.

Then, by Bayes' Theorem, P$(A \mid B) = P(B \mid A)P(A)/[P(B \mid A)P(A) + P(B \mid A^C)P(A^C)]$.

$=\frac{P(A \text{ and } B)}{P(A \text{ and } B)+P(A^C \text{ and } B)}$

For the event A and B, that is, the product is G *and* "i is not an implant or innovator and none of the other neighbors other than $i-1$ has made a recommendation and $i-1$ has made a recommendation", we multiply the prior probability that the product is G, (here it is p) and the probability of B given A. The latter probability is calculated as follows: Since i is not an implant (with the product being of G type), it means that the good-type implant has not been placed there, which happens with probability $1-\alpha_i$, where α_i could be 0 but is, of course, not 1. Also none of $i's$ neighbors except $i-1$ can be an innovator, with resulting probability $(1-\rho)^{d_i-1}$. Likewise i is not an innovator, which has probability (1-ρ). Conditional on i not being the $G-$ implant, $i-1$ could be the G implant, which has probability $\frac{\alpha_{i-1}}{1-\alpha_i}$ or $i-1$ could be an innovator and none of $i's$ other neighbors could be the $G-$implant, this having probability $\rho(1-\frac{\Sigma_{j\in N_i}\alpha_j}{1-\alpha_i})$.

The numerator is then $p\ (1-\rho)(1-\alpha_i)(1-\rho)^{d_i-1}[\frac{\alpha_{i-1}}{1-\alpha_i}+\rho(1-\frac{\Sigma_{j\in N_i}\alpha_j}{1-\alpha_i})]$.

$= p(1-\rho)(1-\rho)^{d_i-1}[\alpha_{i-1}+\rho(1-\Sigma_{j\in N_i\cup i}\alpha_j)]$ for $\alpha_i \neq 1$. (We can take limits if $\alpha_i = 1$)

The denominator is this quantity plus another term $P(B \mid A^C)P(A^C)]$. This is the probability of all this happening if the product is a bad one. The second term in the denominator is therefore $(1-p)(1-\rho)(1-\beta_i)\frac{\beta_{i-1}}{1-\beta_i}$ again assuming $\beta_i < 1$.Note that if player $i-1$ is an innovator and not a $B-$implant, she would not have passed on a recommendation to i, so we need only account for $i-1$ being a $B-$implant, unlike the case when the product was good, when $i-1$ could be either an innovator or an implant and would have passed on a recommendation in either event. Also for a similar reason there is no $(1-\rho)^{d_i-1}$ term necessary when the product is bad.

Hence, cancelling $(1-\rho)$ from both numerator and denominator, we get

$$\eta^1_{i,i-1} = \frac{p(1-\rho)^{d_i-1}\left[\alpha_{i-1}+(1-\sum_{j\in N_i}\alpha_j-\alpha_i)\rho\right]}{p(1-\rho)^{d_i-1}\left[\alpha_{i-1}+(1-\sum_{j\in N_i}\alpha_j-\alpha_i)\rho\right]+(1-p)\beta_{i-1}} \tag{1}$$

Some "special" cases illustrate the nature of the updating process. Suppose that the type G firm uses a pure strategy so that for some site m, $\alpha_m = 1$. Suppose i is a neighbor of m, but receives only one recommendation from some $j \neq m$. Then, i must conclude that j is a bad implant - if the product had been good, then there would have been a good implant at m who would then have passed on a recommendation to her. This argument generalizes even when the type G firm uses a strategy whose support is some set M containing more than one node. Suppose now that i is a *common* neighbor of all nodes in M. Again, if i does not receive a recommendation from some member of M, she will conclude that any other recommendation comes from a bad implant. Next, suppose again that $\alpha_m = 1$ and that m receives a recommendation from some neighbor. Of course, such recommendations are not credible to m - she would have been used as an implant by the type G firm if the product was good. These inferences are confirmed by equation (1) - in all cases, the numerator is 0.

Of course, if i receives a recommendation from *two* or more neighbors, then i concludes that the product is G with probability one- if there is a bad implant at j, then there cannot be a bad implant at $j' \neq j$.

Suppose next that i receives a recommendation from $i-1$ in some period $t > 1$, but no recommendation from any other neighbor. If i has not received any recommendations before period t and receives one from $i-1$ in period t, this can happen because the product is Bad, there is an implant at $i-1$ and the implant chooses to speak at period t. Alternatively, the product is Good, $i-1$ heard a recommendation from one of her neighbors in the previous period, but none of i's other neighbors received a recommendation from any of their neighbors in period $t-1$. Explicit computations of these probabilities are hard to describe since these depend on the structure of the network. But, notice that some recommendations are easy to dismiss. For instance, suppose Γ is a *line*, and let i be an extreme point of Γ, with degree one and either $\alpha_i = 0$ or the G implant at i would have spoken in period 1.[10] Then, any recommendation from i coming in period $t > 1$ is not credible to i's neighbor since i could not have received a recommendation from someone else in period $t-1$.

Diffusion Equilibria

We shall mainly, though not exclusively, limit ourselves to the consideration of "optimal diffusion equilibria" (ODE), adopting the viewpoint of, say, a development agency (or social planner) that wants a *good* idea or product to be spread through the entire population as quickly as possible, given the constraints of the network structure and the technology of diffusion. From the collective viewpoint of the (non-

[10] Also if the node were occupied by an innovator, he or she would speak immediately.

innovator) buyers, the best equilibrium is one in which the good product diffuses in the shortest time to the entire population of buyers and the bad product does not sell at all. Such an equilibrium is, of course, impossible without full information about the product's type. Suppose W is the collective benefit buyers obtain from the good product (including the discounting because of delayed purchase) and w is the disutility obtained from the bad product. Then the ex ante expected consumer utility will be $pW-(1-p)w$, if all recommendations are accepted with probability 1. If recommendations are rejected with positive probability, W is multiplied by some q and w by some Q, where typically $q \leq Q$ (the G product traverses a longer diffusion path and there are more opportunities for the diffusion to be stopped). Thus an equilibrium in which recommendations are rejected will reduce the expected payoff from the G type product more than it increases the payoff from not buying the B type product; hence existence of a sequence of recommendations with acceptance probability of 1 is beneficial. Note also that the B type in our model can expect to sell only to the initial adopters, since there will not be any further recommendations from the initial adopters once they find that the product is bad. For networks in which the maximum degree is small compared to the total number of nodes, $B's$ sales will only have a small effect on the total ex ante consumers utility. The study of diffusion through the network is most relevant for the G type, and hence the focus on ODE.

Consider an environment in which Firm G is the only type of firm, so that the issue of credibility of recommendations does not arise. For instance, consumers may be initially unaware about the existence of the product, but are willing to buy the product after receiving a recommendation. Then, Firm G will want to "seed" the network by using an implant. In the absence of any issue of credibility of recommendations, the product will diffuse throughout the network with probability one if an implant is used. Since the firm discounts the future, it will want to place its implant so as to maximize the *speed* of diffusion. The optimal site(s) for a Good implant is derived below.

Let $\psi_{ij}(t) =$ Prob (recommendation from i reaches j at time t.)

Note that $\psi_{ij}(t)$ refers to the probability that j gets a message at time t, either from a recommendation emanating from node i (we have in mind the implant located at node i) or from some innovator. So, if τ is the maximum distance between any two nodes in the (connected) network, $\sum_{t=1}^{\tau} \psi_{ij}(t) = 1$, i.e. the recommendation reaches every node by time τ.

Define the expected payoff of the G firm who locates an implant at node i, such that there is a sequence of recommendations accepted with probability 1,[11] as

[11] A question has arisen about what happens if a recommendation is rejected. In this case the

$$v(i,\Gamma) = \sum_j \sum_{t=1}^{\tau} \psi_{ij}(t)\delta^t.$$

A G type firm will want to choose as its implant a node which maximizes this expected payoff.

This provides the motivation for the following definition.

Definition 1 *A node i maximizes diffusion centrality in a graph Γ if it maximizes $v(i,\Gamma)$.*

The location of the node maximizing diffusion centrality will, of course, depend on ρ and on the structure of the network. Thus, if j already receives a message with high probability in period 2, increasing that probability by placing an implant next to j is dominated by placing an implant so that node k, who would receive the message in say period 10 with high probability, gets it in an earlier period. The implant will be placed to skew this probability distribution towards low values of t. When $\rho = 0$, diffusion must be from i and this corresponds to the definition present in the literature and known as *decay centrality. ([17]* p 64, for example). See the Appendix for an example of how these could differ for sufficiently high values of ρ.

Let $D(\Gamma)$ denote the set of nodes maximizing diffusion centrality. While it is not easy to compute this set in general graphs, the set is easily identified in special cases. For instance, if Γ is a line, then the median(s) must be maximizing diffusion centrality. Or if Γ is a star, then the hub (that is a node with degree $n-1$) is obviously the node maximizing diffusion centrality.

We will often assume for simplicity that the following assumption is true.

Assumption S: There is a unique node maximizing diffusion centrality.

We are particularly interested in a PBE (the *optimal diffusion equilibrium* (ODE)) with two properties -(i) the good product will diffuse throughout the network with probability one, and (ii) the good implant is placed at some node maximizing diffusion centrality.

So, if (α, β) denote the probability distributions with which the G-type and B-type implants are placed at different nodes, then the support of α must be contained in the set $D(\Gamma)$. Moreover, there must be at least one sequence of recommendations

recommendation stops and the outcome is not complete diffusion. See also footnote 9. We do not discuss equilibria in this section, since this is devoted to describing the model. However, the section on the line network discusses examples of diffusion equilibria.

originating from all nodes with $\alpha_i > 0$ which are accepted with probability one- otherwise the good product will not diffuse through the entire network with probability one. Hence, conditional on the product being Good, an ODE maximizes consumer surplus.[12]

Since ODE may not exist for some networks and parameter values, we also focus on PBE in which the good product spreads throughout the network with probability one, even if this dos not happen in the fastest possible time. We call such equilibria *complete diffusion equilibrium* (CDE).

Of course, we also need to identify when G and B will decide to use an implant. This must depend on a comparison of the increase in expected profit resulting from an implant and the cost c incurred by employing an implant. We assume that $c > 0$, but is "small" so that the expected net gain from using an implant will be non-negative.

3 The Role of the Various Assumptions.

We make several assumptions in formulating the model. These include:

1. At most one implant, no direct negative recommendations (though silence is a noisy negative recommendation) and the B type cannot produce a G good, even by good luck. The last section on Extensions discusses the possible nature of the results when these assumptions are relaxed. However, these are maintained assumptions for our analysis.

2. Complete information about the structure of the network. Whilst we recognise that this is strong, it is essential in our analysis to obtain the full flavour of Bayesian rational players. These buyers take into account, given the equilibrium strategies, what events outside a buyer's own direct neighborhood affect her private history in each period. In order to account for these events, a buyer has to understand the various paths by which a recommendation can reach her and this involves knowledge of the network.

3. Assumption S simplifies the exposition but is not needed for most of our results. Its implication is that in any ODE, the G-type firm will play a *pure* strategy - the implant will be located at the unique node maximizing diffusion centrality. This simplifies the computation of ODE considerably.

[12] As mentioned earlier, if the product is Bad, the upper bound on the extent of diffusion is given by the maximal degree of any node in the network.

4. The same is true of myopic players. Suppose node i is at a distance of two from node k where the G-type implant is supposed to be placed in equilibrium. Suppose also that i receives a recommendation from j in period 1 where j is not on the path between i and k. If node i waits one more period, then she will receive one more recommendation if the product is good and none if the product is not good. In either case, she will know with probability one whether the good is good or bad. The "cost" of acquiring this confirmation is that one period elapses and time is costly because of discounting. We assume away the computational complications that arise because of non-myopic behaviour because there are no qualitative differences in the results.[13]

5. We assume that there is a single, fixed price of the good. There cannot be any separating equilibrium in prices. For suppose G and B type firms announce different prices with that of the former being higher. Then, the B firm can always deviate and announce the G-price. Since quality can be inferred only after consumers have bought the good and since the B-type firm can only get one round of customers, this must be a profitable deviation. So, there can be only one price which we normalize to give a profit of one.

4 The Role of Innovators

Recall that ρ is the probability of an innovator. In order to understand the role of innovators in sustaining CDE or ODE, we consider what happens in the stark case when there are no innovators; that is, when $\rho = 0$.

The notion of *effective degree* plays an important role in our analysis. Consider a PBE (α, β), where the support of α is say a set M of nodes. Suppose node i is connected to *every* node in M; that is, $i \in \cap_{m \in M} N_m$. Suppose also that i receives a recommendation only from $j \notin M$. Then, i *must* conclude that j is an implant of the B-type firm - if the product was good, then i must have received a recommendation from some $m \in M$. So, credibility considerations imply that a B type implant may not be able to get her recommendations accepted by all her neighbours. This consideration gives rise to the notion of effective degree.[14]

Definition 2 *Fix a PBE (α, β), and let M be the support of α. For any node, $i \notin M$, let $\bar{N}_i = N_i \setminus (\cap_{m \in M} N_m)$. Then, we define $\bar{d}_i = |\bar{N}_i|$ to be the effective degree of i in Γ.*

[13] We allowed for non-myopic behaviour in an earlier version of the paper.

[14] From the description above, it should be clear that these terms are defined with respect to a specific PBE.

The effective degree of $m \in M$ will be d_m itself.

Consider an obvious implication of $\rho = 0$. In this case, for any PBE (α, β), only a node in the support of α can make a recommendation in period 1. A bad implant at other sites will have to wait to speak. For instance, a neighbor of node m in the support of α can speak in period 2 by pretending to have received a recommendation from m in period 1, a site at a distance of 2 from m can speak in period 3 and so on. This constrains the possibility of the good product diffusing throughout the network. The following theorem describes the possibilities. Before stating it, we introduce a couple of definitions, which will be used again in Section 5.

Definition 3 *A link $ij \in \Gamma$ is critical if $\Gamma - ij$ has more components than Γ.*

That is, if a critical link is removed from a connected network Γ, then the network Γ no longer remains connected.

Definition 4 *A node i is critical in Γ if all links of i are critical.*

Of course, if the network is a tree, then all links are critical, and so all nodes are critical.

Theorem 1 *Suppose $\rho = 0$. Then,*

(i) If Γ is complete, then there is no CDE.

(ii) If the nodes maximizing diffusion centrality are both critical and have highest effective degree, then there is no ODE.

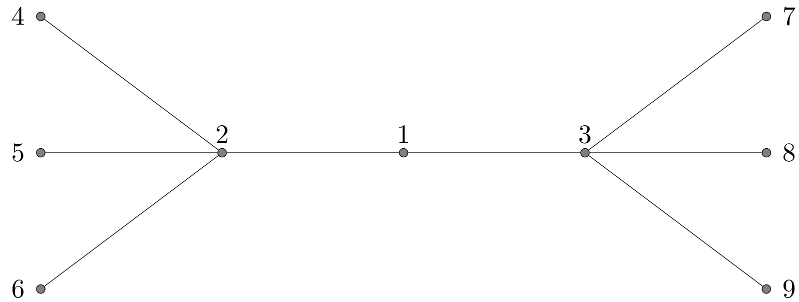
Proof. The proof of this and all other results is in Appendix A. ■

This theorem of course immediately implies that when $\rho = 0$, the line or, more generally, a tree where the node maximizing diffusion centrality also has maximal effective degree cannot support an ODE. It is easy to construct examples of trees which support an ODE if m does not have maximal effective degree. Consider the following example.

Example 1 *Let $n \geq 9$ and n be odd.*[15] *Let Γ be as follows. Individual 1 has just 2 links to 2 and 3. Divide the set $\{4, \ldots, n\}$ into two equal subsets, let 2 be connected to all individuals in the first subset, each of whom have no other link. Similarly, let,individual 3 be connected to all agents in the second subset, each of whom have no other link.*

[15] See Figure 1 for the case $n = 9$.

Figure 1: Example 1



Assume that

$$\delta \geq \max\left(\frac{4}{n-3}, \frac{n-5}{n-3}\right)$$

First, notice that if type G places his implant at 1 *and all subsequent recommendations are accepted with probability one, then his payoff is* $2\delta + (n-3)\delta^2 - c$. *On the other hand, if he places his implant at* 2 *or* 3, *then his payoff is* $(((n-3)/2)+1)\delta + \delta^2 + ((n-3)/2)\delta^3 - c$. *The inequality* $\delta \geq \frac{n-5}{n-3}$ *ensures that the first sum is at least as large.*

Second, assume also that $\underline{p}$ *and* $\bar{p}$ *are such that the type B can put probability* $1/2$ *on each of the nodes* 2 *and* 3, *and still make a credible recommendation in period 2. That is, the neighbors of* 2 *and* 3 *have to infer whether sites 2 and 3 have received a recommendation from the implant of type G placed at 1 in the previous period or whether it is the type B implant who is speaking in period 2, having strategically kept silent in period* 1. *A probability weight of* $\beta_2 = \beta_3 = 1/2$ *brings their updated belief that the recommendation is being passed on from 1 to the threshold* $\overline{p}$, *given the initial parameter values.*

Then, the following is an ODE. The type G puts his implant at 1 *with probability 1, the type B randomizes between 2 and 3 with equal probability. The type G implant speaks immediately while the type B implant speaks in period 2. All recommendations are accepted with probability one.*

These constitute an equilibrium because if $\delta \geq \frac{4}{n-3}$, *then the type B implant has no incentive to deviate and place his implant at 1 - he gets* $\delta^2(n-3)/2 - c$ *in equilibrium, whereas he would get* $2\delta - c$ *by placing his implant at* 1. *The response decisions are optimal because updated beliefs are not below the threshold.*

In general, $\rho = 0$ gives "less cover" for the B type to hide itself. It also creates more zero-probability events, giving rise, for example, to a no-trade equilibrium, in which neither type of firm chooses to enter with an implant. If anyone does enter and recommends the product, it is believed to be a B type and no buyer buys.

How does a positive ρ impact on the possibility of optimal diffusion? Fix all other parameters and the network structure Γ. Now, ρ influences the nature of equilibrium in two ways. First, the higher the value of ρ, the lower is the net gain from having an implant for both firm types. So, there will be some value of $\bar{\rho}$ such that if $\rho \geq \bar{\rho}$,then one or both types of firm F will refrain from employing an implant.

Second, the value of ρ influences the updating process according to equation 1. Suppose ρ changes. How does this affect $\eta^1_{i,i-1}$ for a fixed value of β_i and distribution α? An increase in ρ makes it more likely that $i-1$ is an innovator, but also makes it less likely that none of the *other* neighbors of i are innovators. These effects move in opposite directions and an unambiguous answer is difficult to provide.

5 The Line Network

In this section, we provide a discussion of a specific network structure - the *line* , in order to illustrate the conditions required for complete and optimal diffusion equilibrium. Much of the intuition underlying the model can be obtained by analysing what happens in the case of the line. In particular, the importance of *largeness* for optimal diffusion is perfectly illustrated by this network.

Assume, for simplicity, that n is odd, so that Assumption S is satisfied. So, let Γ be a line, with the sites ordered so that 1 and n are the end-points of the line having degree one, whilst all other sites have degree 2. Since n is odd, the unique median m maximizes diffusion centrality.

We discuss informally the ODE type of equilibrium. A theorem and formal proof of our results on the line follow this informal discussion. We will conclude this section with examples illustrating other types of CDE.

If (α, β) is to be an ODE, then the recommendation coming from m has to be accepted with probability one. That is, $m-1$ and $m+1$ must accept any recommendation coming from m with probability one. Since the bad type can always mimic the Good type, this implies that wherever the bad implant is placed, her recommendation must be accepted by two neighbors. Out-of-equilibrium, if an offer is rejected, the recommendation does not proceed farther along the path where there is a rejection, though it might elsewhere. No player who has a subsequent move can observe this deviation; all they can observe is that no recommendation reaches them, which occurs with positive probability along any path.

But, of course, $\beta_m \neq 1$. For, if $\beta_m = 1$, then from equation (1),

$$\eta^1_{m-1,m} - p = \eta^1_{m+1,m} - p = p[\frac{((1-\rho)-1)(1-p)}{p(1-\rho)+(1-p)}] < 0,$$

and neither of m's neighbors would buy the product after receiving a recommendation from m, which is a contradiction. So, while the support of β can *include* m, it cannot coincide with $\{m\}$. Over what set of nodes can the bad type "distribute" β? The answer of course is that the support of β must be contained in those nodes who can make "credible" recommendations to both their neighbors, since m's recommendations should be credible. That is, it must be the set of nodes with effective degree 2. Set

$$E_2 \equiv N \setminus \{1, m-2, m-1, m+1, m+2, n\}$$

It is clear that 1 and n cannot be in E_2 since they have only one neighbor. Although $m-2$ and $m+2$ have degree 2, notice that $m-1$ (respectively $m+1$) will not believe a single recommendation from $m-2$ (respectively from $m+2$).[16] Of course, $m-1$ and $m+1$ cannot sell to m. Also, notice that if $n < 9$, then m will be the sole member of E_2, and there will not be any ODE.

It is also clear when the bad type will want to use an implant when there is an ODE. Let the implant be placed at i. Since i can be an innovator with probability ρ (in which case he would buy the product anyway), the effective cost of an implant at i is

$$\rho + c$$

The benefit is the additional probability that $i-1$ and $i+1$ buy the product. With probability $(1-\rho)^2$, neither is an innovator. With probability $2(1-\rho)\rho$, one of the two is an innovator. Hence, the benefit is

$$2(1-\rho)^2\delta + 2(1-\rho)\rho\delta = 2\delta(1-\rho)$$

So,the net gain of an implant for B is given by

$$2\delta(1-\rho) - \rho - c$$

where c is the cost of an implant. So, B will use an implant if

$$c \leq c^B(\delta, \rho) \equiv 2\delta(1-\rho) - \rho$$

[16] Both $m-1$ and $m+1$ know that if the product is good, then m would have made a recommendation.

Not surprisingly, the higher the value of ρ, the lower is the expected gain from employing an implant.

Consider G. Let $\underline{\pi}^G(\delta, \rho)$ denote the profit of G in the absence of an implant, and $\bar{\pi}^G(\delta, \rho)$ denote the profit of G if an implant is used.

Then, G will use an implant if

$$c \leq c^G(\delta, \rho) \equiv \bar{\pi}^G(\delta, \rho) - \underline{\pi}^G(\delta, \rho)$$

The fact that the B-type implant's recommendation has to be credible imposes an upper bound on the value of β_i. The bound can be determined from equation 1.

It is now easy to describe what an ODE looks like when $n \geq 9$ is odd, and the cost of an implant does not exceed c.

(i) The type G firm puts its implant at the median of the line with probability one.

(ii) The support of β is contained in E_2, and the recommendation from each node in the support is credible.

(iii) The implant (irrespective of type) "speaks" in period 1.

Response decisions must support full diffusion - we spell this out in the proof of the theorem.

In the theorem below, we make a very mild assumption on out-of-equilibrium beliefs to show that for "large" networks, the only PBE is an ODE.

Assumption D1. Let the sender of type $t \in T$ choose to send a message $m \in M$. Let the receiver respond with an action a from A. Consider the set of responses to the message m that induces type t to weakly prefer to send the message m rather than the candidate equilibrium message m^*. Denote this set by $A_t \subseteq A$.If there is a type t' who strictly prefers sending message m (to the equilibrium message) if the responses are in A_t, then the belief of the receiver on receiving message m must accord probability 0 to type t.(See [3]).

The D1 refinement of sequential equilibrium essentially requires that if an out-of-equilibrium message is observed, the responder puts the entire weight of her updated beliefs on the set of types that gain maximally from the deviation. The attractive properties of this refinement in generic signaling games are examined in detail in [4]

Fix $(\rho, p, \bar{p})$. The integer bounds specified in the theorem below are with respect to this vector of parameter values. We will also assume that Assumption S is satisfied. However, the proof makes it clear that this just simplifies the computation of the bounds - there will be no qualitative change in the result if Assumption S is not satisfied.

Theorem 2 *Suppose Assumption S holds. Then, there are integers n^1, n^2 such that $n^2 \geq n^1 > 7$ such that*

(i) There is an ODE iff $n \geq n^1$.

*(ii) Moreover, if **D1** is satisfied and $n \geq n^2$, then the only PBE is an ODE.*

We now give examples of other types of equilibria on the line. It follows that either a different PBE exists along with an ODE or n is smaller than n^1 .

Our first example is one where $n = 7$ so that an ODE does not exist, but a CDE does exist. Recall that from equation 1, $b^1 \equiv \frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right]$ is the probability weight of the B-implant on i which ensures $\eta^1_{j,i} = \bar{p}$ when $\alpha_i = 0$. That is, this is the probability weight of the B-implant on i which makes neighbor j of i indifferent between accepting or rejecting a recommendation from i. Assume that

$$7b^1 \geq 1$$

Example 2 *Let $n = 7$. then, the following is a CDE.*

- $\alpha_1 = 1$.
- $\beta_i = b^1$ *for* $i = 3, 4, 5, 6$.
- $\beta_i = \max(0, \frac{1-4b^1}{3})$ *for* $i = 1, 2, 7$.

Any implant speaks in period 1. The response decisions are the following.

(i) For each $i \neq 7$, only $i + 1$ accepts the recommendation of i and does so with probability one in period 1.

(ii) Node 6 accepts the recommendation from 7 in period 1.

(iii) Node $1 + t$ accepts her left neighbor's recommendation in period $t > 1$.

So, each period 1 recommendation is accepted by exactly one node. In subsequent periods, diffusion of the good product happens from left to right.

Since the period 1 updated probability is exactly $\bar{p}$ for recommendations coming from $i = 3, 4, 5, 6$, it is an equilibrium response for $i + 1$ to accept and $i - 1$ to reject the recommendation. The updated probability of recommendations coming from 1,2,7 is either greater or equal to $\bar{p}$. Noting that 1 will not accept any recommendation from 2 (since $\alpha_1 = 1$), the other response decisions are also optimal.

The G-implant achieves complete diffusion. Any deviation yields lower expected payoff since the product does not diffuse to the left. The B-implant gets one acceptance. No deviation does better. Hence, this constitutes a CDE.

Since the G-implant locates at an extreme node, the set of possible nodes which can be the support of β includes nodes with effective degree one. This increase in possibility allows a CDE to exist for $n = 7$. That is, "small" networks which cannot sustain ODE can nevertheless sustain CDE.

The next example shows that a network size which allows ODE to exist can also sustain CDE where the G-type does not locate at m. In fact, in the example below, m is not in the support of either α or β. The B-type implant gets two acceptances in equilibrium and hence has no reason to deviate to m. The G-type implant would like to deviate to m but recommendations are not accepted for reasons we explain below.

Example 3 *Let $n = 9$, and*

$$\frac{p(1-\bar{p})(1-\rho)}{\bar{p}(1-p)} + 2\frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right] \geq 1$$

Let

- $\alpha_4 = 1, \alpha_i = 0$ *for all* $i \neq 4$.
- $\beta_4 = \frac{p(1-\bar{p})(1-\rho)}{\bar{p}(1-p)}$
- $\beta_7 = \beta_8 = \frac{1-\beta_4}{2} \leq \frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right]$
- $\beta_i = 0$ *for all other* i.

Response decisions are the following.
(i) 4 does not accept any recommendation.
(ii) Every other node accepts all other recommendations in period 1.[17]

Since $\alpha_4 = 1$, $E_2 = \{7, 8\}$. Readers can check that β_4 corresponds to β_m from the previous theorem, while $\beta_7 = \beta_8$ does not exceed the probability weight that makes recommendations just credible from nodes which are not in the support of α. So, these ensure that the response decisions of all nodes other than 4 are optimal.

What should 4 (if no implant is placed there) do if it receives a recommendation from 5, which is the median? Note that this is an out-of-equilibrium move if it comes from an implant (though the type of player making the recommendation is not observable). In equilibrium, it can only come from an innovator since 5 is not in the

[17] For $t > 1$, response decisions support complete diffusion. We do not specify these since they play no role.

*support of either α or β. But, 4 "knows" that if the product is good, then it should have a G-implant. So, the recommendation cannot be from an innovator - it must be from a deviant of either the G- or B-type. If **D1** is not imposed, then 4 can assign the belief that the recommendation is from the B-implant and hence not accept the recommendation. Given this, the G-type implant has no incentive to deviate to 5, which is the median.*

Hence, the prescribed strategies constitute a PBE which is a CDE. Of course, there is also an ODE where $\alpha_5 = 1$ with the support of β being $\{2, 5, 8\}$.

6 Constraining both types of implant to speak in the first period.

As a further simplification of the model, we could require that both the G and B implants must speak in the first period, with the innovators. This would imply that any recommendation received in the second period would be considered to be passed on from a recommendation received in the previous period and therefore a signal of a good type. For the equilibria discussed in the last section and, for general networks in which the probability of acceptance is either 1 or 0 by any node, this restriction does not affect the equilibria. In an ODE, even if the B type had the option to wait, he would not do so because a payoff of 2 is greater than one of 2δ.The G type, of course, will not wait.

The restriction will have some bite if $\sum_i \overline{\beta}_i < 1$, (see the previous section for the line) for the set of nodes in the support of the B type's random behavioral strategy. This means that B does not have enough nodes in the support of his mixed strategy to make the neighbors of these nodes indifferent between accepting and rejecting a recommendation. If the entire probability mass is distributed on these nodes, some neighbors will have $p < \overline{p}$ and will therefore reject the recommendation, even if they are myopic. In this case, the equilibrium (for $\rho > 0$) has to have the B type randomize between entering with an implant or not. The residual probability has to be on not entering. Any recommendation will be accepted with a probability that makes the B type's expected payoff equal to c, the cost of the implant. The G type will strictly prefer to enter with an implant because she will obtain payoffs after the first period as well, whilst the B type can only get payoffs from his immediate neighbors and not from their neighbors.

We can also see the effect of restricting both types to speak in the first period in the example in Figure 1. Here the B types locate at the two nodes that are connected to three leaves each, whilst the G type locates at the node that is connected to these two nodes only. This will only work if the ρ provides enough "cover" for the B type to put all his probability mass at these two nodes. If he were allowed to speak also in the second period, he could also exploit the feature of equilibrium that the G type's recommendation can be passed on to the leaves in the second period, thus reducing the amount of probability mass associated with a first period recommendation. If B is required to speak in the first period, a deviation by B to speaking in the second period is infeasible. B can then locate at one of three nodes so as to make neighbors indifferent between accepting and rejecting and put the remaining probability mass, if any, on not entering. Acceptance and rejection probabilities can be adjusted to make B indifferent between entering and not entering.

7 A Partial Characterization Result

In this section, we describe a sufficient condition for an ODE to exist, and then show that this condition is necessary for certain types of network structures.

Throughout this section, we restrict our attention to networks satisfying Assumption S.[18]

Given m, partition nodes into sets $S_1, \ldots, S_K$ such that S_1 is the set of nodes maximizing $\bar{d}_i$, S_2 is the set of nodes with the next highest value of $\bar{d}_i$, and so on.

Suppose that node i is in the support of β, and $j \in \bar{N}_i$. Whether j finds the recommendation credible or not depends on the *degree* of j since she does not receive a recommendation from any neighbor other than i. Clearly, the larger the number of neighbors of j, the lower is the updated probability after receiving i's recommendation.

For every node i, let k_i be the site in $\bar{N}_i$ that maximizes degree. Then,

$$\eta^1_{k_i,i} \leq \eta^1_{j,i} \text{ for all } j \in \bar{N}_i$$

For each $j \in \bar{N}_i$, let β_i^j be the value of β_i which sets $\eta^1_{j,i} = \bar{p}_j$.

$$\text{For each } i, \bar{\beta}_i \equiv \min_{j \in \bar{N}_i} \beta_i^j \tag{2}$$

So, if $\beta_i \leq \bar{\beta}_i$, all neighbors of i in $\bar{N}_i$ find i's recommendation credible.

The next theorem identifies a sufficient condition for a network structure to support an ODE. It also shows that this is "almost" necessary. These conditions are satisfied by the line, and so this theorem will include the line as a special case.

[18] This assumption results in somewhat more transparent results. In an earlier version, we did not use this assumption here. That version is available from the authors.

Theorem 3 *Suppose the cost of an implant is sufficiently low*[19] *for both types of the firm to use an implant. Then, an ODE exists if*

$$\sum_{i \in S_1} \bar{\beta}_i \geq 1. \tag{3}$$

Conversely, an ODE does not exist if m is critical, $m \in S_1$ and equation 3 does not hold.[20]

Equation 3 is easy to interpret. If there are a sufficient number of nodes maximizing effective degree, then an ODE is easy to support since the type B firm has enough "space" to distribute his probability. Conversely, if an ODE is to exist, then it must be possible for the type B firm to ensure that a recommendation from each node in the support of its mixed strategy is credible for $|S_1|$ neighbors.

Why is equation 3 not necessary without additional conditions? Suppose, that $\alpha_m = 1$, but $m \notin S_1$, but say in S_2. Also, assume that equation 3 does not hold. It is possible then to have another equilibrium in which (i) the type B uses a mixed strategy over nodes in $S_1 \cup S_2$, (ii) the probability of acceptance of recommendations coming from nodes in S_1 is adjusted below one so as to ensure that the expected payoff from an implant located in S_1 is the same as that from an implant in S_2. However, the value of δ cannot be too high. If δ is high, then the low discounting may induce the type B implant at some site in S_1 to strategically postpone her recommendation to a later period. The freedom to distribute some probability weight over nodes in S_2 may now help in ensuring existence of equilibrium. [21]

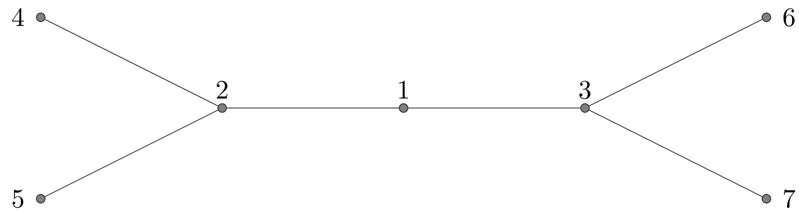
If $m \in S_1$ but is not critical, then there could be an equilibrium of the following kind. The good product may diffuse throughout the network with probability one even if some neighbors of m who do not constitute critical links with m refuse m's recommendations - the fact that some link mi is not critical obviously implies that there is some path from m to i not involving the link mi. An implication of this is that the type B firm needs fewer customers in equilibrium. Now, suppose each node $i \in S_1$ has one neighbor in $\bar{N}_i(\Gamma)$ with very high degree, say h_i, while the others have relatively low degree. Then, one option for the type B firm is to put

[19] For the B type, given the network, this is the condition that the effective cost $c + \rho$ is less than the maximal effective degree. For the G type, the upper bound on c is harder to express explicitly but it depends only on the total number of nodes, ρ and δ. Therefore, c is less than the minimum of these two upper bounds.

[20] Hence, equation 3 is necessary for the existence of an ODE in all trees satisfying Assumption S and where $m \in S_1$.

[21] An example of such an equilibrium is available in a previous version of the paper.

Figure 2: Example 4(*Tree 1*)



probability weights $\tilde{\beta}_i > \bar{\beta}_i$ on each i such that all nodes in $\bar{N}_i(\Gamma)$ except h_i accept i's recommendation.

The sufficiency condition in Theorem 3 is phrased in terms of effective degrees, and not in terms of "standard" network characteristics such as degree distribution or diameter of the network. This is almost inevitable because the type B firm will choose to place his implant at some node maximizing effective degree, and that latter does not easily translate into the familiar network characteristics. The following example shows that two networks with the same degree distribution can have quite different properties with respect to the existence of an ODE.

Example 4 *In this example both networks are trees with the same degree distribution, such that there is an ODE in one , but not in the other. Let* $n = 7$*.*

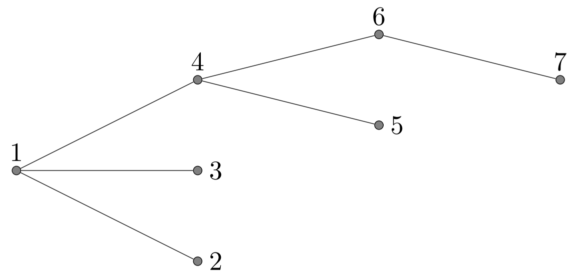
Tree 1: Node 1 is connected to nodes 2 and 3; node 2 is connected to 4 and 5, while node 3 is connected to 6 and 7.

Tree 2: Node 1 is connected to 2,3,4; 4 is connected to 5 and 6, while 6 is connected to 7.

Both trees have degree distribution $(3, 3, 2, 1, 1, 1, 1)$*. In tree 2, node 4 maximizes diffusion centrality for all* $\delta > 0$*. In tree 1, node 1 maximizes diffusion centrality for* $\delta > 1/2$*.*

Let $\delta > 1/2$*. Although both trees have the same degree distribution, the distribution of effective degrees is not identical. In tree 1, the distribution of effective degrees is* $(2, 2, 2, 0, 0, 0, 0)$*. To see this, check that nodes 1,2,3 all have effective degree 2, while the rest have effective degree 0. In tree 2, the distribution of effective degrees is* $(3, 2, 0, 0, 0, 0, 0)$*. For some parameter values, it is possible to sustain an ODE in tree 1, where* $\alpha_1 = 1$*, while the support of* β *is* $\{1, 2, 3\}$*. It is easy to see that no parameter values can sustain an ODE in tree 2.*

Figure 3: Example 4(*Tree 2*)



8 The Role of the Network Structure

In this section, we show that "dense" networks are not necessarily conducive for complete diffusion, a conclusion which is in stark contrast to the received wisdom in models where credibility is not an issue. We show that the complete network cannot support a CDE, while no network that contains a star encompassing all nodes can support an ODE.

In what follows, for any set $M \subset N$, and for any $i \notin M$, let $\bar{N}_i(M) = N_i \backslash \cap_{j \in M} N_j$, and $\bar{d}_i(M) = |\bar{N}_i|$. The interpretation of $\bar{N}_i(M)$ is that if M is the support of α, then $\bar{N}_i(M)$ is the set of potential customers of i. Since individuals in $\cap_{j \in M} N_j$ are connected to all nodes in M, they will not accept any recommendation from a node $i \notin M$.

Theorem 4 *(i) If* Γ *is the complete network, then it cannot support a CDE.*
(ii)Suppose Γ *contains a star as a subgraph. Then,* Γ *cannot support an ODE.*

Under Assumption S, we can also place an upper bound on the degree of m.

Theorem 5 *Let Assumption S hold, with* m *the unique node maximizing diffusion centrality in* Γ*. Then, if* Γ *is to support an ODE,* $d_m \leq \frac{n-1}{2}$*.*

Theorems 4 and 5 describe some network structures that cannot support an ODE. In particular, nodes maximizing diffusion centrality cannot be too well-connected since their connections tend to reduce the effective degree of those nodes which are "close" to them.

9 Extensions

We consider some possible extensions of the basic model.

9.1 Negative recommendations

We have assumed that "recommendations" can only be positive. However, one needs to consider negative recommendations as well if only to consider the robustness of the model. It is easy to check that Theorems 1, 4 and 5 continue to remain valid.

It also turns out that the possibility of negative recommendations will actually simplify calculations in one respect in that the probability calculations would not now depend on the potential recipient's degree. So, the analogue of equation 1 will now be

$$\eta^1_{i,i-1} = \frac{p\left[\alpha_{i-1} + (1 - \sum_{j \in N_i} \alpha_j - \alpha_i)\rho\right]}{p\left[\alpha_{i-1} + (1 - \sum_{j \in N_i} \alpha_j - \alpha_i)\rho\right] + (1-p)\beta_{i-1}} \tag{4}$$

However, it would complicate expected payoff calculations for B, where a high degree recipient of an implant's (positive) recommendation would be more likely to have countervailing negative information than one of low degree. Such a problem would not arise for regular graphs, but the general issue is illustrated below.

So, suppose B places implants at i and j with some positive probability. Then, his expected payoff from i is

$$E_i = \sum_{k \in \bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

where P_k is the probability with which the offer is accepted by k. The updated belief for all $k \in \bar{N}_i(\Gamma)$ will now be the same - it will just depend on β_i and not on the degrees of k.

So, since $E_i = E_j$, we need

$$\sum_{k \in \bar{N}_j} (1-\rho)^{d_k - 1} P_k = \sum_{k \in \bar{N}_i} (1-\rho)^{d_k - 1} P_k$$

The specification of a general sufficient condition is now more difficult because the derivation of the support of β is now more complicated. However, the qualitative result that a larger network is more conducive for optimal diffusion remains unchanged for the line and regular networks.

The possibility of negative recommendations may also help in sustaining optimal diffusion, particularly in dense networks where nodes have high degree. This is because the expected payoff of the type B firm will now be lower- and it will be lower the larger are the degrees of different sites that can be recipients of implant recommendations. So, the type B firm may simply not employ implants.

9.2 Multiple implants

If the firm can choose multiple implants, the qualitative features of the analysis will be similar. Clearly it does not make sense for the multiple implants to have overlapping supports (for the firm's randomized strategies). This suggests that for large networks, the firm will partition the networks in such a way as to have one implant randomly located (for B) in each element of the partition. If δ is close to 1 and an ODE exists as above, there is very little incentive for G to incur the cost of an additional implant, since this can only speed up the diffusion and the benefit from this might be low compared to the cost. Therefore, for low discounting, we would expect to have several B implants but only one G implant. This suggests that the B implants would either have to rely on a relatively high ρ for credibility or speak only at sufficiently late time periods to mimic a message transmitted along the network from a supposed good implant, which might be located some distance away.

9.3 The bad type's probability of producing a good product.

We have assumed so far that Firm F knows its type, where *type* is identified with the quality of the product that is produced. Let us redefine type as follows. The type G firm produces a good product with probability one, while type B produces a good product with small positive probability ϵ and the bad product with probability $1-\epsilon$. Suppose as before that firm F knows its type in the modified sense.[22]

In this case, the following cases could arise (this is not an exhaustive description):

(i) There is a unique node m maximizing diffusion centrality, which also maximizes degree centrality. In this case, an ODE will not exist. The reason is that both G and B will care about speed of diffusion, though B will care less, and therefore both will prefer to locate at m rather than at any other node. As pointed out earlier, both types locating with probability 1 at m cannot be an equilibrium.

(ii) There is a unique node m maximizing diffusion centrality but it does not maximize effective degree centrality. Now B will be better off not locating at m for ε small enough. If he locates at a site that has effective degree at least 1 more than

[22] Alternatively, suppose type is identified with quality of the product as before, but consumers who buy the bad quality product make a "mistake" with small probability - they make a positive recommendation with probability ϵ.

m, he can get some additional payoff. If he locates at m, he loses at least 1 for sure and obtains some additional payoff depending on ε and δ. For ϵ small enough, this is not a best response for B. In this case, the analysis from the firm's point of view will not change from that discussed earlier in this paper. Hence, an ODE will exist under the same conditions as before.

10 Conclusions

In this paper, we have explored the implications for diffusion of a product or a technology to a network of rational consumers and "innovators", where the seller of the product or technology has private information about its quality. Consumers are aware that firms may "seed" the network, and also know that both "good" and "bad" quality firms may do so. As a result, agents cannot take recommendations from their social neighbors at face value - the credibility of recommendations has to be evaluated with beliefs updated using Bayes' Rule. Within this framework, we show that a priori notions about what network structure is conducive to optimal diffusion may be misleading. In particular, "small" networks and highly-connected agents may actually impede the diffusion of the good product. The requirement of credibility of learning has bite; these results would not hold in a model without rationality. Also the entire structure of the network is important in determining whether optimal diffusion is possible or not, though the nodes with the highest degree and those that maximize "diffusion centrality", a notion of centrality that takes into account whether an agent is connected to other agents who are themselves central, play special roles in the equilibrium pattern of implants and diffusion.

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11 Appendix A : Proof of Theorems

Here, we provide proofs of all the results in the text.

Proof of Theorem 1

Proof.

(i) Suppose (α, β) is a CDE. Let M be the support of α.ny node i knows that if the product is good, then it should receive a recommendation in period 1 from some node in M. So, a recommendation from nodes outside M are not credible.

Hence, the support of β must be a subset of M. But, then equation 1 shows that the updated probability will be less than $\bar{p}$. This proves (i).

(ii) Let M be the set of nodes maximizing diffusion centrality. If an ODE exists, then the support of α is contained in M. For simplicity, assume it is M.

Let $m \in M$. If m is critical, then all of m's neighbors must accept m's recommendations. If some neighbor j does not accept m's recommendation with probability one, then the criticality of m implies that the good product will not diffuse to some segment of the network. Suppose $\beta_i > 0$, where $i \notin M$. Then, the implant at i cannot make its recommendation in period 1 since recommendations from nodes other than those in M are not credible in period 1. So, the payoff to the implant at i is at most $\delta d_i < d_m$, since m maximizes effective degree. So, if the bad implant could be located at m, it would get a higher payoff. This implies that the support of β should be contained in M. But, then the updated probability at some $m \in M$ would be less than $\bar{p}$. This shows that the type B firm does not have an equilibrium strategy that sustains optimal diffusion. ■

Proof of Theorem 2

Proof.

(i) From our informal discussion of an ODE, it is clear that an ODE exists iff $\alpha_m = 1$, and

$$\bar{\beta}_i \geq \beta_i \text{ for } i \in E_2 \tag{5}$$

$$\sum_{i \in E_2} \bar{\beta}_i \geq 1 \tag{6}$$

where E_2 is the set of nodes which have effective degree two, and $\bar{\beta}_i$ is the upper bound on β_i. It follows that $n \geq 9$ since otherwise $E_2 = \emptyset$ when $\alpha_m = 1$ at the median m.

Using equation 1, these are given by

$$\bar{\beta}_m = \frac{p(1-\bar{p})(1-\rho)}{\bar{p}(1-p)}, \text{and for } i \neq m, \bar{\beta}_i = \frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right]$$

Let n^1 be the *smallest* value of n such that

$$\frac{1-\bar{\beta}_m}{n-7} \leq \frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right] \tag{7}$$

So, there is no probability distribution β satisfying equations **??**, 5 iff $n < n^1$. This follows because the best the B-type implant can do to economize on the number of nodes in the support of β is to assign $\bar{\beta}_m$ to m and equal weight to all other nodes in E_2.

This shows why there is no ODE if $n < n^1$.

Now, suppose $n \geq n^1$. Then, the following is an ODE.

(i) $\alpha_m = 1$.

(ii $\beta_m = \bar{\beta}_m \equiv \frac{p(1-\bar{p})(1-\rho)}{\bar{p}(1-p)}$

(iii) $\beta_i = \frac{1-\beta_m}{n-7} \leq \frac{\rho(1-\rho)p}{1-p}\left[\frac{(1-\bar{p})}{\bar{p}}\right]$ for all $i \in E_2 \setminus \{m\}$.

(iii) $\beta_i = 0$ for all $i \notin E_2$.

Moreover, the implant (irrespective of type) "speaks" in period 1.

To complete the description of the equilibrium, we define the responses of all nodes.

First, all nodes accept recommendations received in period 1 with probability one.

In subsequent periods $t > 1$, a site $i < m$ accepts a recommendation from $i+1$ with probability one if the equilibrium response of $i+1$ was to accept the recommendation from $i+2$ in period $t-1$. Similarly, a site $i > m$ accepts a recommendation from $i-1$ if the equilibrium response of $i-1$ was to accept the recommendation from $i-2$ in period $t-1$.

Clearly, the Good product diffuses throughout the network, while the Bad product is bought by all innovators and the immediate neighbors of the B-type implant.

In order to see that these constitute a PBE, notice that after node i receives a recommendation from neighbor j in period t, $\eta^t_{i,j} \geq \bar{p}$. Hence, the response decision (of buying) of i is optimal.

The G-type implant does not want to deviate from $\alpha_m = 1$. This follows by noting that type G cannot be indifferent between m and any other site. At m, the implant will obtain an expected payoff of $\delta.2 + \delta^2.2 + ...$ for $m-1$ terms. At $m-k$, say, the payoff will be $\delta.2 + \delta^2.2 + ...\delta^t 2$ for $m-k-1$ terms and $\delta^{m-k}.1 + ..$ for an additional $2k$ terms, thus taking $m+k-1$ periods to diffuse completely rather than $m-1$ periods, if the good implant locates at m.[23] Thus the speed of diffusion is higher by locating at m, since there are two new buyers in each period for every period the diffusion continues, whilst at $m-k$, there is only one buyer for every period after $k-1$.

[23] This calculation does not take ρ into account, but it is obvious this wouldn't change the ranking because sites are independently innovators or non-innovators

The B-type implant has no incentive to deviate because at each node where $\beta_i > 0$, node i's recommendation is accepted by both neighbors of i. Since the B-type good will be recommended only by the implant, this is the best that the B-type can do.

Finally, we need to check whether the response decisions are optimal. Note that since $\beta_i \leq \bar{\beta}_i$ at all nodes, $\eta^1_{j,i} \geq \bar{p}$ for both neighbors j of i. Hence, it is optimal for each j to accept any recommendation received in period 1. Since any recommendation received in period $t > 1$ must be coming from a node who has tried the product in period $t - 1$ and found it to be good, acceptance of recommendations received in period $t > 1$ is also optimal.

In this equilibrium, a message from m in period 2 would be an event of probability 0. To check that neither type wants to deviate, we have to consider out-of-equilibrium beliefs, but all beliefs about the type sustain this equilibrium. The m implant's recommendation is accepted with probability 1 in the first period, so discounting makes it sub-optimal to wait, no matter what the belief.

This completes the proof of (i).

(ii) Suppose **D1** is satisfied and there is a PBE (α, β) which is not an ODE.

Case 1: Suppose first that the PBE is a CDE.

Of course, $\alpha_m = 0$. For if $\alpha_m > 0$, then the CDE would in fact be an ODE, contradicting our assumption. In fact, $\alpha_i = 0$ for *any* interior node i. For suppose $\alpha_i > 0$ where $i \neq 1, n$. Since this is a CDE by assumption, both of i's neighbours accept i's recommendation. So, the B-implant must also get two acceptances - otherwise the B-implant would deviate to i. If $\beta_m > 0$, then the G- implant would deviate to m. This follows because $m + 1$ and $m - 1$ accept the recommendation coming from m. Then, in period 2, $m - 1$'s and $m + 1$'s recommendations must be accepted by their neighbours. Continuing this chain of argument, the G-implant improves payoff by achieving complete diffusion in the fastest possible time- this justifies the deviation by the G implant.

Hence, $\beta_m = 0$. Now, suppose the G-implant deviates to m. When m makes a recommendation, either both $m - 1$ and $m + 1$ believe that m is an innovator, and accept the recommendation. Alternatively, one of them, say $m - 1$ concludes that this is an out-of equilibrium move by an implant. This happens if $\alpha_{m-1} = 1$ or $\alpha_{m-2} = 1$. But, since the B-implant already gets two acceptances, it has no strict incentive to deviate to m even if both neighbors accept $m's$ recommendation (and an incentive not to deviate if this is not the case). On the other hand, the G-implant by getting its recommendation accepted would achieve complete diffusion in the fastest possible time, and so would strictly prefer to deviate if $m's$ recommendation was accepted with probability 1. That is, the deviation from equilibrium is more likely to have come from the G-implant. So, applying **D1**, $m-1$ must conclude that either m is an innovator or the G-implant has deviated to m. In either case, $m-1$ accepts the recommendation and then recommends the good in the next period, if he finds it good. Continuing this argument, there must be complete diffusion from m once the G implant deviates. This again justifies the deviation.

So, the support of α is restricted to $1, n$.

Claim 1 If $i \in E_2$, then $\beta_i > 0$.

Suppose the claim is not true. Then, any recommendation from i must be coming from an innovator.[24] Take any $j \neq i$ such that $\beta_j > 0$. Then, both $j-1$ and $j+1$ accept j's recommendation with probability one - otherwise the B-type would shift his implant to i where he gets two acceptances. This also implies that the B-type implant makes a recommendation with probability one in period 1 itself since postponing a recommendation to period 2 does strictly worse given $\delta < 1$.

Hence, any recommendation in period 2 signals that the product is good. But, then the G-type will deviate to i since node i's recommendations will be accepted by $i-1, i+1$ in period 1, $i+1$ and $i-1$'s recommendations will be accepted in period 2, and so on. This will be a profitable deviation since complete diffusion from an interior node is faster than complete diffusion from an extreme node.

Hence, this establishes the claim.

Claim 2 : If $i \in E_2$ is an interior node, then recommendations from i are accepted with probability γ such that $0 < \gamma < 1$.

We know from Claim 1 that $\beta_i > 0$. So, $\gamma > 0$. Suppose $\gamma = 1$. Then, the B-implant does not make a recommendation in period 2. From what we have said in the proof of claim 1, this will imply that the G-type has a profitable deviation. Hence, $\gamma < 1$.

So, every node $i \in E_2$ is in the support of β and a recommendation from i is accepted with probability less than one. The latter means that the updated probability is exactly $\bar{p}$. From equation 1, this implies that $\beta_i = b > 0$ for some b. But, for some $i \in E_2$, $\beta_i \leq 1/k$ where $k = |E_2|$. Since $k \geq n-6$, there is n' such that $\beta_i < b$ for all $n \geq n'$. But, then $\eta^1_{j,i} > \bar{p}$ where j is a neighbor of i. So, both neighbors of i accept i's recommendation with probability one. This contradiction establishes the proof of (ii) in Case 1.

Case 2: Suppose the PBE is not a CDE.

Again, it must be the case that the recommendation of the B-implant is not accepted with probability one by the 2 neighbors. From arguments in Claims 1 and

[24] Note that out-of-equilibrium beliefs are not involved when $i \in E_2$.

2 in Case 1, it follows that for large enough n, this is not possible.

This concludes the proof. ■

Proof of Theorem 3

Proof. Suppose equation 3 is satisfied. Let $\alpha_m = 1$ be the strategy employed by the good type, and choose β such that

$$\beta_i \leq \bar{\beta}_i \text{ for all } i \in S_1$$

and

$$\beta_i = 0 \text{ if } i \notin S_1$$

The response strategies are straightforward. All sites accept all recommendations in all periods.

It is easy to check that these strategies constitute a PBE. Clearly, the type G firm has no incentive to deviate since her implant is at the site maximizing diffusion centrality and recommendations are accepted with probability one. Similarly, the type B firm has no incentive to deviate since she obtains a payoff of $\bar{d}_1 - c$, conditional on no innovators in $\bar{N}_1$. Clearly, no other site can yield a higher payoff. The response decisions are optimal because (i) any recommendation coming from a site not in the union of the supports of α and β must be coming from an innovator, and (ii) the updated belief of any i after receiving a recommendation from a potential implant is at least as large as the threshold value $\bar{p}$.

Consider now the necessity of this condition. First, the type G firm must be choosing an $\alpha_m = 1$ in any ODE. Since m is critical in Γ, all neighbors of m must accept a recommendation from m with probability one in an ODE. So, the type B firm by placing her implant at m can obtain a payoff of $|S_1| - c$. Hence, in equilibrium, the support of β must be contained in S_1. Moreover, if $\beta_i > 0$, every member of $\bar{N}_i(\Gamma)$ has to accept the recommendation of i. Hence, the maximum probability weight that the type B firm can put on i cannot exceed $\bar{\beta}_i$. This is not possible if equation 3 does not hold. ■

Proof of Theorem 4

Proof. (i) The proof is almost identical to that of part (i) of theorem 1. Let Γ be the complete network, and (α, β) be any PBE. If M is the support of α, then any i will only accept a recommendation from some $j \in M$. This means the support of β must also be in M. But, this will mean that the updated probability of any node in the support of β is less than $\bar{p}$. This shows that there cannot be complete diffusion.

(ii) Let $M = \{i \in N | d_i(\Gamma) = n - 1\}$. If Γ contains a star, this set is non-empty. Then, all members of M maximize diffusion centrality. If an ODE exists, the support of α is some $M' \subseteq M$. Take any site $i \notin M'$. Then, $\bar{d}_i(M') = 0$ since any site $j \neq i$

is connected to all nodes in M'. So, the support of β is also M'. Then, it follows from equation 1 that

$$\eta^1_{j,i} \leq p < \bar{p}$$

So, no neighbor of the bad implant at i buys the product after receiving a recommendation from i. Since the bad type is indifferent between all sites in the support of β, no site in the support of β can get her recommendation accepted. This implies that only the good type employs an implant. However, this cannot be an equilibrium since the bad type would then deviate and place an implant at some site in M'. ■

Proof of Theorem 5

Proof. Let Γ support an ODE, and let $d_m = k$. Then, for all $i \neq m$, $\bar{d}_i \leq n - k - 1$. Suppose $d_m > \bar{d}_i$ for all $i \neq m$. Then the bad type would prefer to put her implant at m since recommendations from m are accepted with probability one. But, if $\beta_m = 1$, then $\eta^1_{m,j} < p < \bar{p}$ and this is not consistent with m's recommendations being accepted. Hence, $d_m = k \leq \bar{d}_i \leq n - k - 1$ for some $i \neq m$. This implies $d_m \leq \frac{n-1}{2}$. ■

12 Appendix B: Example on diffusion and decay centrality

In this appendix, we illustrate the difference between diffusion and decay centrality for two networks with two hubs. The second network is obtained from the first by deleting a node. The maximization for different values of ρ is done numerically, by generating different configurations of innovators for each value of ρ and comparing the average performance (across configurations) of each node in the network in achieving complete diffusion.

Prior to presenting the numerical calculation for these two networks, we illustrate the nature of the concept of diffusion centrality by considering the following example. Suppose we have a line of four nodes, labelled $1, 2, 3$ and 4. We set $\rho = .2$ (recall this is the probability that any given node is an innovator) and consider the "diffusion centrality" measures for nodes 1 and 3. (That is we assume an implant is placed at 1 or 3 and calculate the appropriately discounted time for the innovation to diffuse in the whole network.) Thus, if there is one innovator placed at node 4, an event that has probability $(.2)(.8)^3$ of happening, if the implant is placed at node 1 then nodes 1 and 4 will know of the product immediately and 2 and 3 the following period, giving a total discounted time to diffuse of $2{+}2\delta$. Denoting by '0' the absence of an innovator and by '1', the presence of one, we can construct the following table:

Configuration	Prob	1	3
0000	$(.8)^4$	$1+\delta+\delta^2+\delta^3$	$1+2\delta+\delta^2$
0001	$(.2)(.8)^3$	$2+2\delta$	$2+\delta+\delta^2$
0010	$(.2)(.8)^3$	$2+2\delta$	$1+2\delta+\delta^2$
0100	$(.2)(.8)^3$	$2+\delta+\delta^2$	$2+2\delta$
1000	$(.2)(.8)^3$	$1+\delta+\delta^2+\delta^3$	$2+2\delta$
0011	$(.2)^2(.8)^2$	$3+\delta$	$2+\delta+\delta^2$
0101	$(.2)^2(.8)^2$	$3+\delta$	$3+\delta$
1001	$(.2)^2(.8)^2$	$2+2\delta$	$3+\delta$
0110	$(.2)^2(.8)^2$	$3+\delta$	$2+2\delta$
1100	$(.2)^2(.8)^2$	$2+\delta+\delta^2$	$3+\delta$
1010	$(.2)^2(.8)^2$	$2+2\delta$	$2+2\delta$
1110	$(.2)^3(.8)$	$3+\delta$	$3+\delta$
1101	$(.2)^3(.8)$	$3+\delta$	4
1011	$(.2)^3(.8)$	$3+\delta$	$3+\delta$
0111	$(.2)^3(.8)$	4	$3+\delta$
1111	$(.2)^3(.8)$	4	4

The diffusion centrality of node 1 can be found by multiplying the numbers in columns 2 and 3 and adding for all rows and likewise, columns 2 and 4 for the centrality measure of node 3.

The difference in the centrality measures for 3 and 1 can be calculated to be

$$(.8)^4(\delta-\delta^2)+(.2)(.8)^3(2\delta-\delta^2-\delta^3) \quad (8)$$

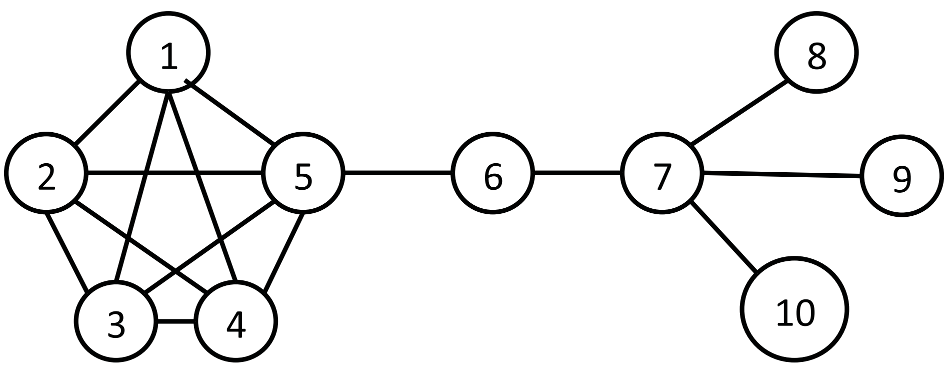
$$> \; 0. \quad (9)$$

Thus node 3 is more central than node 1, which is not a surprising result.

The numerical example is described in the following pages.

Setting: delta=0.8. Simulation for 3000 times

Source node	Decay Centrality	Diffusion Centrality					
		rho=0	rho=0.1	rho=0.3	rho=0.5	rho=0.7	rho=0.9
1	5.58	5.58	6.43	7.30	7.66	7.82	7.96
2	5.58	5.58	6.43	7.30	7.66	7.82	7.96
3	5.58	5.58	6.43	7.30	7.66	7.82	7.96
4	5.58	5.58	6.43	7.30	7.66	7.82	7.96
5	6.18	6.18	6.85	7.47	7.72	7.84	7.96
6	6.08	6.08	6.78	7.41	7.68	7.82	7.96
7	5.89	5.89	6.82	7.65	7.92	7.99	8.00
8	4.87	4.87	6.06	7.20	7.64	7.82	7.96
9	4.87	4.87	6.06	7.19	7.64	7.82	7.96
10	4.87	4.87	6.05	7.20	7.64	7.82	7.96



Setting: delta=0.8. Simulation for 3000 times

Source node	Decay Centrality	Diffusion Centrality					
		rho=0	rho=0.1	rho=0.3	rho=0.5	rho=0.7	rho=0.9
2	4.78	4.7808	5.6101	6.4510	6.8428	7.0267	7.1516
3	4.78	4.7808	5.6092	6.4560	6.8430	7.0260	7.1514
4	4.78	4.7808	5.6112	6.4571	6.8414	7.0243	7.1515
5	5.38	5.3760	6.0375	6.6428	6.9119	7.0441	7.1531
6	5.44	5.4400	6.0240	6.5843	6.8624	7.0251	7.1512
7	5.38	5.3760	6.1078	6.8260	7.0890	7.1811	7.1993
8	4.46	4.4608	5.3778	6.3499	6.8014	7.0171	7.1511
9	4.46	4.4608	5.3777	6.3477	6.8024	7.0174	7.1511
10	4.46	4.4608	5.3809	6.3536	6.8022	7.0174	7.1511