IAS School of Social Science

ECONOMICS WORKING PAPERS

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Paper Number 105
October 2014

Decentralised bilateral trading, competition for bargaining partners and the “law of one price ” *

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16 October, 2014

Abstract

This paper analyses a model of price formation in a market with a finite number of non-identical agents engaging in decentralised bilateral interactions. We focus mainly on equal numbers of buyers and sellers, though we discuss other cases. All characteristics of agents are assumed to be common knowledge. Buyers simultaneously make targeted offers, which sellers can accept or reject. Acceptance leads to a pair exiting and rejection leads to the next period. Offers can be public, private or “ex ante public” . As the discount factor goes to 1, the price in all transactions converges to the same value.

keywords: Bilateral Bargaining Outside options Competition Uniform price.

1 Introduction

In this paper, we study price formation in a market with small numbers of buyers and sellers, where transactions are bilateral between a single buyer and a single seller. For a broad range of variants of a dynamic bargaining game with many sellers and buyers, in which only one side of the market makes offers, we find that, as the discount factor goes to 1, there is a stationary equilibrium where prices in different transactions converge to a single value.

*We thank the Human Capital Foundation (www.hcfoundation.ru), and especially Andrey P. Vavilov, for support to The Pennsylvania State University's Department of Economics. Dr Chatterjee would also like to thank the Institute for Advanced Study, Princeton, and the Richard B. Fisher endowment for financial support of his membership of the Institute during the year 2014-15.

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1.1 Motivation for the problem studied

Most modern markets consist of a small number of participants on each side. These participants buy from and sell to each other, write contracts with each other and sometimes merge with each other. The transactions in these markets are often bilateral in nature, consisting of an agreement between a buyer and a seller or a firm and a worker. These bilateral trades occur without any centralised pricing mechanism, in a series of bargains in which the "outside options" for a current bargaining pair are, in fact, endogenously given for each by the presence of alternative partners on the other side of the market. However, these potential alternative partners, by their presence, implicitly compete with each other and one question that arises naturally is whether the "competitive" pressure of the outside options leads to an approximately uniform price for non-differentiated goods. It is this basic question, about endogenous outside options and a uniform price, that this paper seeks to study. We focus on complete information. [1]

1.1.1 Examples

Whilst the models we study are going to be highly stylised representations of these examples, they at least have some features in common with them. A standard example used in these settings is the housing market, for a given location and a given type of home (to reduce the extent of differentiation). Sellers list their houses, buyers visit, inspect and then convey their offers to the sellers-one offer from each buyer. Sellers can accept or reject the offers they have; possibly they then make counter-offers or often wait for the buyer to come back again with new higher offers. Whether counter-offers are made or not distinguishes different extensive forms or bargaining protocols. The offers are privately made to sellers, who typically do not know what other sellers receive.

Another example is of a firm being acquired. Here the potential acquirer makes a public targeted offer for a particular firm, which the shareholders of the potential acquisition have to accept or reject (based on a recommendation by the management). A rejection could lead to the acquirer raising its offer. There could be competition on both sides, perhaps from another potential buyer called in by management of the target as a "white knight" and other possible targets with the same attractive characteristics as the one in play. In this particular context, it makes sense to think of offers as being one-sided, from the potential buyers, and publicly announced.

Private targeted offers occur in negotiations for joint ventures. For example, the book [1] describes the joint venture talks between industrial gas companies and chemical companies

[1] An incomplete information analysis has been done in a companion paper [4].

in the 1980s, in which the players were Air Products, Air Liquide and British Oxygen on one side and DuPont, Dow Chemical and Monsanto on the other. After some bargaining, two joint ventures and an acquisition resulted.

1.2 Main features of our model

Our model begins from a setting of two buyers with common valuation v, two sellers with valuations M, H and complete information about these values. We assume that $v > H > M > 0$. We then extend the model by adding both buyers, sellers to the basic model. There is a one-time entry of players, at the beginning of the game, and a buyer-seller pair who trade leave the market.

Players discount with a common discount factor $\delta \in (0, 1)$. We consider equilibria for high values of δ and consider the limit of equilibria as $\delta \to 1$. The extensive forms we consider have two main features; offers are one-sided and offers are simultaneous. Simultaneous offers seems to us to be the right way to capture the essence of competition. Targeting an offer to one individual on the other side of the market enables us to endogenise matching between buyers and sellers as a strategic decision. Once the offers have been made, one per proposer, recipients simultaneously accept or reject. A rejection ensures that the game continues to the following period, where payoffs are discounted by δ.

Our main results, starting with the basic model, can be simply described. There is a unique stationary equilibrium outcome under complete information, involving non-degenerate mixed strategies for all players. As $\delta \to 1$, the mixed strategies collapse to a single price and the price in all matches goes to H. In equilibrium, there could be one-period delay with positive probability, but the cost of delay, of course, goes to 0 as $\delta \to 1$. The price H might be thought of as a competitive equilibrium price in the complete information setting. (Given our assumptions, any price between v and H will equate supply and demand.)

For the general n buyer-n seller case, we show the **uniqueness** of the limiting payoff for buyers and the convergence of prices in all transactions to a single value as $\delta \to 1$, for any stationary subgame perfect equilibrium *(Theorem 1)*. **The main equilibrium characterisation** result for the general case is given in *Theorem 2*, which builds on the analysis preceding it in the paper. .

In the next sub-section, we discuss the relevant literature and compare our results to some of the existing work.

1.3 Related literature.

We now qualitatively describe the existing literature and compare our model with it. The first attempts to obtain micro foundations for markets using bilateral bargaining were the papers by Rubinstein and Wolinsky [15], [3] Gale [8] and [9], . These papers were all concerned with large anonymous markets, in which players who did not agree in a given period are randomly and exogenously rematched in succeeding periods with someone they had never met before. Rubinstein and Wolinsky [15] and Gale [9] consider bargaining frictions given by discounting and characterise the limiting price as the discount factor goes to 1. The limiting price depends on exogenously given probabilities of being matched in the following period.

Rubinstein and Wolinsky [16] (see also [13], Chapter 9.2, 9.3 for an exposition of their models) is the first paper to consider the issue of price determination in small markets. They consider buyers and sellers, with the number of buyers being more than that of sellers. Each buyer-seller pair is capable of generating a surplus of 1 unit. In their basic model with random matching and no discounting, they construct a class of non-stationary equilibria and show that these do not give the entire surplus to the short side of the market; however, the stationary equilibrium does. With discounting, some of their equilibrium constructions become difficult to sustain for random matching, so they introduce a seller choosing a buyer who is "privileged", as in their equilibrium construction without discounting. They do not consider equal numbers of buyers and sellers, heterogeneity in surpluses and direct competition for the same seller by two different buyers (with discounting). Both the current paper and the paper by Chatterjee and Dutta [5] may be viewed as extensions of the Rubinstein-Wolinsky model to richer strategic settings.

Chatterjee and Dutta [5] attempt a project similar to this one, also with public and private targeted offers and ex ante offers, but both sides of the market are allowed to make offers. It turns out that this difference with the current paper is crucial. The paper [5] does not, in general, obtain an asymptotically single price as $\delta \to 1$; under public targeted offers, there is a pure strategy equilibrium and all pure strategy equilibria involve two different prices. In general, the mixed strategy equilibrium with private offers remains non-degenerate even as $\delta \to 1$, unlike this paper, even though the expected player payoffs converge. The intuition behind these results in Chatterjee-Dutta is that there is a tension in every period between two opposing forces acting on the price. Since the two sides of the market alternate in making offers, a single rejection in a period (in the game with two buyers and two sellers) will generate a "Rubinstein bargaining subgame". In the presence of heterogeneous agents, this leads to pressure on the prices to diverge towards the two different bargaining solutions. However, there is also competition in each period to try to match with the player who offers a higher surplus because of simultaneity of offers and, therefore, undercutting or overbidding.

This conflict is impossible to resolve with two agreements taking place in the same period but there is a pure strategy equilibrium with agreement taking place in different periods at different prices. The current paper keeps the aspect of competition but eliminates the complication caused by the two sides of the market making alternating offers. This explains why one sided offers leads to unique stationary equilibrium with competitive price in the limit as players get patient.

Gale and Sabourian [11] and Sabourian [17] use notions of strategic complexity to select the competitive equilibrium in games of the kind studied by Rubinstein and Wolinsky.

Hendon and Tranaes [12], also following Rubinstein and Wolinsky [16] study a market with two heterogeneous buyers and one seller, and random matching after initial disagreement.

To summarise, this current paper differs from the existing literature by considering one or more of the following: (i) Small numbers and strategic matching. (ii) Extensive forms with different assumptions about whether offers are public or targeted and private. (iii) Simultaneous offers. Despite this variety and the number of differences with the papers mentioned above, the results we get are surprisingly consistent with an asymptotic single price. It is clear that the fact that we consider one-sided rather than alternating offers has much to do with this, and this might be considered one of the takeaways from this paper, namely that the intuition for the single price result holds broadly provided alternating offers don't push prices apart when buyer-seller valuations are heterogeneous.

In the next section, we discuss the basic model with two buyers and two sellers under complete information. In Section 3, we analyse the general case where there are n buyers and n sellers, for general finite n and obtain similar results on the asymptotic buyer payoffs being the same. In section 4 we consider possible extensions.

2 The Basic Framework

2.1 The model

2.1.1 Players and payoffs

In the basic model we address, there are two buyers and two sellers. As mentioned in Section 1.2, there are two buyers B_1 and B_2 with a common valuation of v for the good (the maximum this buyer is willing to pay for a unit of the indivisible good). There are two sellers. Each of the sellers owns one unit of the indivisible good. Sellers differ in their valuations (we can also interpret these as their costs of producing to order). One of the sellers, (S_M) has a value of M for one or more units of the good. The other seller, (S_H) similarly has a value of H

where

$$v > H > M > 0$$

This inequality implies that either buyer has a positive benefit from trade with either seller. Alternative assumptions can be easily accommodated but are not discussed in this paper. In the basic complete information framework all these valuations are commonly known. Finally, all players are risk neutral. Players (buyers or sellers) have a common discount factor δ where $\delta \in (0,1)$. Suppose a buyer agrees on a price p with seller S_j in period t. Then the buyer has an expected discounted payoff of $\delta^{t-1}(v-p)$ and S_j has the payoff of $\delta^{t-1}(p-j)$, where $j = M, H$.

We shall discuss the informational assumptions along with the extensive forms in the next subsection.

2.1.2 The extensive form

We consider an infinite horizon multi-player bargaining game with one-sided offers. The extensive form of the game is described as follows.

At each time point $t = 1, 2, ...$ offers are made simultaneously by the buyers. The offers are targeted. This means an offer by a buyer consists of a seller's name (that is S_H or S_M) and a price at which the buyer is willing to buy the object from the seller he has chosen. Each buyer can make only one offer per period. Two settings could be considered; one in which each seller observes all offers made (public targeted offers) and one in which each seller observes only the offers she gets (private offers). (Similarly for buyers after the offers have been made.) In the present section we shall focus on the first and consider the latter in a subsequent section. A seller can accept at most one of the offers she receives. Acceptances or rejections are simultaneous. Once an offer is accepted, the trade is concluded and the trading pair leave the game. Leaving the game is publicly observable. The remaining players proceed to the next period in which buyers again make price offers to the sellers. As is standard in these games, time elapses between rejections and new offers.

2.1.3 Strategies and equilibrium

We will not formally write out strategies, since this is a standard "multi-stage game with observable actions"[7] . Since we have public targeted offers, a seller's response (and subsequent actions by all players) can condition on the history of offers made to the other seller, in addition to those she receives herself. Our equilibrium notion here will be the standard *subgame perfect equilibrium.*

2.2 Equilibrium in the basic model

2.2.1 Stationary equilibria

We consider "stationary" equilibria, that is, equilibria in which buyers when making offers condition only on the set of players remaining in the game and the sellers, when responding, condition on the set of players remaining and the offers made by the buyers (We emphasise that this is not a restriction on strategies, only on the equilibria considered.). Clearly these are particular sub-game perfect equilibria in our public targeted offers extensive form. We shall demonstrate that the equilibrium outcome we find in this way is the unique stationary equilibrium outcome. We shall proceed in this subsection by showing that a candidate strategy profile, in fact, does constitute an equilibrium. In the next subsection, we shall show that the stationary equilibrium payoff vector is unique upto choice of the buyer who makes an offer to both sellers.

The conjectured equilibrium is as follows:

1. Consider a game in which only two players, buyer B_i and seller S_j remain in the market and j denotes the valuation/cost of S_j. Then it is clear that (i) B_i offers j and (ii) S_j accepts any offer at least as high as j and rejects otherwise.

2. Now consider the four-player game[2]. We consider the following strategies:

(a) One of the buyers, B_1 say, makes offers to each seller with positive probability and the other buyer B_2 makes an offer only to S_M. Let q be the probability with which B_1 offers to S_H. B_1 offers H to S_H. B_1 randomises an offer to S_M, using a distribution $F_1(\cdot)$ with support $[p_l, H]$, where p_l is to be defined later. The distribution $F_1(\cdot)$ consists of an absolutely continuous part from p_l to H and a mass point at p_l. B_2 randomises by offering M to S_M(with probability $q^{'}$) and randomising his offers in the range $[p_l, H]$ using an absolutely continuous distribution function F_2. The distributions $F_i(\cdot)$ are explicitly calculated later.

(b) The sellers' strategies in the four-player game are as follows. S_H accepts the highest offer greater than or equal to H and rejects if all offers are less than H. S_M accepts the highest offer with a payoff from accepting at least as large as the expected continuation payoff from rejecting it (which is actually determined endogenously). Throughout our analysis it is assumed that a seller who is indifferent between accepting or rejecting an offer, always accepts.

3. The expected payoff of a buyer B_i in equilibrium is $v - H$. The expected payoff of S_H is 0 and that of S_M is positive.

[2]Note that, since we start with the same number of players on both sides of the market and since players can leave only in pairs, any possible subgames will also have the same number of buyers and sellers.

Proposition 1 *There exists a stationary, subgame perfect equilibrium with the characteristics described above.*

Proof.

We break up the proof into a sequence of two lemmas, which are stated below. The details are in the appendix. ■

The first lemma explicitly calculates the equilibrium $F_i(.)$, q and q', given a definition of p_l. In the second lemma, we demonstrate the existence of the p_l as defined.

Lemma 1 *Suppose there exists a p_l such that*

$$p_l - M = \delta(E(y) - M)$$

,where y (a random variable) represents the maximum price offer to S_M under the proposed strategies. Then the strategies in 1,2 and 3 above constitute an equilibrium with

(i)

$$F_1(s) = \frac{(v-H)(1-\delta(1-q)) - q(v-s)}{(1-q)[(v-s) - \delta(v-H)]} \tag{1}$$

(ii)

$$F_2(s) = \frac{(v-H)(1-\delta(1-q^{'})) - q^{'}(v-s)}{(1-q^{'})[(v-s) - \delta(v-H)]} \tag{2}$$

(iii)

$$q = \frac{[v-H](1-\delta)}{(v-M) - \delta(v-H)} \tag{3}$$

(iv)

$$q^{'} = \frac{[v-H](1-\delta)}{(v-p_l) - \delta(v-H)} \tag{4}$$

Proof.

The above expressions are derived with the help of the indifference conditions of the buyers. We relegate the formal proof to appendix (A). ■

We now show that there indeed exists a p_l as described above. The following lemma does this.

Lemma 2 *There exists a unique $p_l \in (M, H)$, such that,*

$$p_l - M = \delta(E(y) - M)$$

where $E(y)$ is same as defined before.

Proof.

Consider any $x \in (M, H)$. Let $F_1^x(.)$, $F_2^x(.)$, q^x, $q^{'x}$, and $E^x(y)$ be the expressions obtained from $F_1(.)$, $F_2(.)$, q, $q^{'}$ and $E(y)$ respectively by replacing p_l by x. That is we compute the distributions and the probabilities according to the above described strategy profile by assuming $p_l = x$.

All we now need to show is that there exists a unique $x^* \in (M, H)$ such that,

$$x^* - M = \delta(E^{x^*}(y) - M)$$

From our description given above, we can posit that $E^x(y)$ ca be written as follows

$$\begin{aligned} E^x(y) &= q^x[q^{'x}M + (1 - q^{'x})E_2^x(p)] + (1 - q^x)[q^{'x}E_1^x(p) \\ &\quad + (1 - q^{'x})E(\text{highest offer})] \end{aligned}$$

where, $E_i^x(p)$ is derived from $F_i^x(.)$, $(i = 1, 2)$ and is the expected price offer by the buyer B_i,when his offers are in the range $[x, H]$.

We claim that as x increases by 1 unit, increase in $E^x(y)$ is by less than 1 unit. See appendix B for the proof of this claim.

Now we define the function $G(.)$ as,

$$G(x) = x - [\delta E^x(y) + (1 - \delta)M]$$

Differentiating $G(.)$ w.r.t x we get,

$$G^{'}(x) = 1 - (\delta)\frac{\partial E^x(y)}{\partial x}$$

From our above claim we can infer that

$$G^{'}(x) > 0$$

From the equilibrium strategies we know that $M < E^x(y) < H$ for any $x \in (M, H)$. Since $\delta \in (0, 1)$ we have,

$$\lim_{x \to M} G(x) < 0 \text{ and } \lim_{x \to H} G(x) > 0$$

Since $G(.)$ is a continuous and monotonically increasing function, using the *Intermediate Value Theorem* we can say that there exists a unique $x^* \in (M, H)$ such that,

$$G(x^*) = 0$$
$$\Rightarrow x^* = \delta E^{x^*}(y) + (1-\delta)M$$

This x^* is our required p_l.

Thus we have,

$$G(p_l) = 0$$
$$\Rightarrow p_l = (1-\delta)M + \delta E(y)$$

■

2.2.2 Uniqueness of the stationary equilibrium outcome

In this section we will show that the outcome derived above is the unique stationary equilibrium outcome in this game, so that the expected payoff to each of the buyers is $v - H$[3]. By *outcome* we mean the vector of payoffs obtained by the buyers and sellers. We will adopt the methodology of *Shaked* and *Sutton* [18].

Let M^* and m^* be the maximum and the minimum payoffs[4] obtained by a buyer in any stationary equilibrium of the complete information game. Also let Λ_H and Λ_M be the maximal stationary equilibrium payoffs for sellers S_H and S_M respectively.

The following lemma rules out the possibility of having each buyer offering to both the sellers with positive probability.

Lemma 3 *In any stationary equilibrium, when all four players are present, both buyers cannot make offers to both sellers with positive probability.*

Proof.

In a stationary equilibrium when both the buyers are offering to both the sellers, each buyer should randomise its offer while offering to any of the sellers. Given the buyers' behaviour, each seller accepts an offer(or the maximum of the received offers) if and only if the payoff from acceptance is at least as large as the discounted continuation payoff from rejection. This implies that in a stationary equilibrium we need not worry about the deviations by the sellers.

[3]In fact there is another stationary equilibrium where B_2 offers to both the sellers with positive probability and B_1 to S_M only. The qualitative nature will be the same and the buyer with valuation v obtains a payoff of $v - H$.

This does not necessarily mean that the *price* is H. However, we shall show this is true asymptotically, as $\delta \to 1$.

[4]We assume (without needing to) that the supremal and infimal payoffs are actually achieved.

Let $\overline{s}_i^M$ be the upper bound of the support of offers to S_M from the buyer B_i, $i = 1, 2$.

Let $\overline{s}_i^H$ be the upper bound of the support of offers to S_H from the buyer B_i, $i = 1, 2$.

If $\overline{s}_1^H \neq \overline{s}_2^H$ then the buyer having a higher upper bound (say B_1) can profitably deviate by offering $(\bar{s}_1^H - \epsilon)$ to S_H, where $\epsilon > 0$ and $\bar{s}_1^H - \epsilon > \bar{s}_2^H$.

Thus ,

$$\overline{s}_1^H = \overline{s}_2^H = \overline{s}^H$$

By similar reasoning we can say that,

$$\overline{s}_1^M = \overline{s}_2^M = \overline{s}^M$$

Next we would argue that we must have $\overline{s}^H = \overline{s}^M$. Suppose not . W.L.O.G let $\overline{s}^H > \overline{s}^M$. In this case one of the buyers can profitably deviate by offering p to S_M such that $\overline{s}^H > p > \overline{s}^M$. Thus we have,

$$\overline{s}^H = \overline{s}^M = \overline{s}$$

Let q_2 be the probability with which B_2 offers to S_H. Let $F_2^M(.)$ and $F_2^H(.)$ be the conditional distributions of offers by B_2 given that he makes offers to S_M and S_H respectively. Take $s \in [\underline{s}_1^M, \overline{s}] \cap [\underline{s}_1^H, \overline{s}]$. B_1's indifference (from making offer to S_M or S_H) relation tells us that:

$$\begin{aligned} & (v-s)[q_2 + (1-q_2)F_2^M(s)] + (1-q_2)(1-F_2^M(s))\delta(v-H) \\ = \; & (v-s)[(1-q_2) + q_2 F_2^H(s)] + q_2(1-F_2^H(s))\delta(v-M) \end{aligned}$$

Since $\delta(v-M) \neq \delta(v-H)$, $(1-q_2)(1-F_2^M(s)) \neq q_2(1-F_2^H(s))$. W.L.O.G we take,

$$(1-q_2)(1-F_2^M(s)) > q_2(1-F_2^H(s))$$

$$\Rightarrow (1-q_2)(1-F_2^M(\overline{s})) > q_2(1-F_2^H(\overline{s}))$$

The above inequality suggests that B_2 puts a mass point at the upper bound of one of the supports. If not then both $(1-q_2)(1-F_2^M(\overline{s}))$ and $q_2(1-F_2^H(\overline{s}))$ are 0 and the above inequality is not satisfied. This implies that B_1 can profitably deviate. ■

Next we show that it is never the case that in a stationary equilibrium both buyers offer to the seller with higher cost. This is described in the following lemma

Lemma 4 *In any stationary equilibrium, when all four players are present, both buyers cannot offer to S_H with positive probability.*

Proof.

Clearly both offering to S_H only is not possible in equilibrium. Similarly one of the buyers offering to S_H only and the other one making offers to both the sellers with positive probability is not possible. In that case the buyer who is offering to both can profitably deviate by offering M to S_M . Thus if both are offering to S_H it must be the case that both are making offers to both the sellers with positive probability. From lemma 3 we know that this is not possible in a stationary equilibrium. This concludes the proof. ■

Hence in stationary equilibrium it must be the case that only one buyer makes offer to the seller with higher cost (S_H). With the help of the previous two lemmas, we will now show in the following lemma that the seller with cost H can never obtain a strictly positive payoff in a stationary equilibrium.

Lemma 5 *In any stationary equilibrium, the seller with a higher valuation (i.e S_H) never gets an offer which is strictly greater than H. Thus $\Lambda_H = 0$*

Proof.

Suppose not. That is let it be the case that in a particular stationary equilibrium S_H obtains a strictly positive payoff ($\Lambda_H > 0$). From Lemma 3 and Lemma 4 we know that a single buyer is making this offer to S_H. Since $\Lambda_H > 0$, this buyer is offering x_H (where $x_H \geq H + \Lambda_H$) with positive probability and his payoff is less than or equal to $v - x_H$.

Suppose this buyer deviates and makes an offer of $x_H^{'}$ such that,

$$x_H^{'} = H + \epsilon\Lambda_H$$

where $0 < \delta < \epsilon < 1$.

This offer will always be accepted by S_H, irrespective of what the other seller's strategy is. This is because if she rejects this offer then next period she can at most obtain a payoff of Λ_H which is worth $\delta\Lambda_H$ now. However by accepting this offer she gets $\epsilon\Lambda_H > \delta\Lambda_H$.

Since,

$$x_H - x_H^{'} \geq H + \Lambda_H - H - \epsilon\Lambda_H$$

$$= \Lambda_H(1 - \epsilon) > 0,$$

this deviation is profitable for the buyer. Thus we must have $\Lambda_H = 0$. Fom this we infer that in a stationary equilibrium S_H never gets an offer greater than H. ■

Thus in any stationary equilibrium, S_H always gets a payoff of zero. The following lemma describes that the price offer to S_M is bounded above by H.

Lemma 6 *In a stationary equilibrium, S_M cannot get an offer greater than H with positive probability.*

Proof.

Suppose S_M gets an offer $H + \triangle, \triangle > 0$ with positive probability. From lemma (5) we know that H never gets an offer greater than H in equilibrium. Thus the buyer making the above offer to M can profitably deviate by offering $H + \lambda\triangle, (0 < \lambda < 1)$ to S_H. Thus in equilibrium S_M cannot get an offer greater than H with positive probability. ■

We now show that $m^* = M^*$. This is described by the following two lemmas. ([18]).

Lemma 7 *The minimum payoff obtained by a buyer in a stationary equilibrium is bounded below by $v - H$. Thus*

$$m^* \geq v - H \textit{ for } i = 1, 2$$

Proof.

From Lemma 5 and Lemma 6 we can posit that none of the sellers gets any offer greater than H with positive probability. Thus in a stationary equilibrium buyers' offers are always in the interval $[M, H]$. Hence m^* is bounded below by $v - H$. This proves that

$$m^* \geq v - H$$

■

Lemma 8 *The maximum payoff obtained by a buyer in a stationary equilibrium is bounded above by $v - H$. Thus*

$$M^* \leq v - H \textit{ for } i = 1, 2$$

Proof.

Suppose there exists a stationary equilibrium such that B_i obtains a payoff of M^* such that $M^* > v - H$.

(i) Consider the situation when the buyers play pure strategies. It must be true that the offer made by B_i is accepted. Let p^* be the equilibrium price offer by B_i. Since,

$$M^* = v - p^* > v - H$$

we have,

$$p^* < H$$

This implies that this offer is accepted by seller S_M.

Thus either B_j $(j \neq i)$ is offering to S_H or it is offering a price lower than p^* to S_M. In both cases B_j can profitably deviate by offering a price p to S_M such that $p^* < p < H$.

Hence it is not possible for B_i to obtain a payoff of $M^* > v - H$ in a stationary equilibrium when both buyers play pure strategies.

(ii) Suppose at least one of the buyers plays a non-degenerate mixed strategy. It is easy to note that B_i cannot obtain a payoff of $M^* > v - H$, if he offers to S_H with positive probability. Thus we only need to consider the situations when B_i is offering to S_M only.

Suppose both B_1 and B_2 are offering to S_M only. There does not exist a stationary equilibrium where one of the buyers plays a pure strategy. Thus both B_1 and B_2 play mixed strategies. It is trivial to check that in equilibrium the supports of their offers have to be the same. Let $[\underline{s}, \overline{s}]$ be the common support of their offers, where $\underline{s} \geq M$. Since B_i obtains a payoff higher than $v - H$ we must have $\overline{s} < H$. Let $F_j(.)$ be the distribution [5] of offers by B_j, where $j = 1, 2$ and $j \neq i$. Thus for any $s \in [\underline{s}, \overline{s}]$ buyer B_i's indifference relation gives us

$$(v - s)F_j(s) + (1 - F_j(s))\delta(v - H) = M^*$$
$$\Rightarrow F_j(s) = \frac{M^* - \delta(v - H)}{(v - s) - \delta(v - H)}$$

Since $F_j(s)$ is always positive, B_j puts a mass point at $\underline{s}$. From lemma 7, we know that $m^* \geq v - H$. Thus by applying similar reasoning we can show that B_i also puts a mass point at $\underline{s}$.

We will now show that B_i can profitably deviate. Suppose B_i shifts the mass from $\underline{s}$ to $\underline{s} + \epsilon$ where $\epsilon > 0$ and ϵ is arbitrarily small. The change in payoff of B_i is given by,

$$\triangle_\epsilon = F_j(\underline{s} + \epsilon)(v - (\underline{s} + \epsilon)) - \frac{F_j(\underline{s})}{2}(v - \underline{s}) \tag{5}$$

We will show that for small values of ϵ the above change in payoff is positive. For $\epsilon > 0$, from (5) we have,

$$\triangle_\epsilon = [F_j(\underline{s}) + \epsilon F_j^{'}(x)](v - (\underline{s} + \epsilon)) - \frac{F_j(\underline{s})}{2}(v - \underline{s})$$

where $x \in (\underline{s}, \epsilon)$.

This implies

$$\triangle_\epsilon = F_j(\underline{s})(v - \underline{s}) + \epsilon F_j^{'}(x)(v - \underline{s}) - \epsilon F_j(\underline{s}) - \epsilon^2 F_j^{'}(x) - \frac{F_j(\underline{s})}{2}(v - \underline{s})$$

$$= F_j(\underline{s})(\frac{v - \underline{s}}{2} - \epsilon) + \epsilon F_j^{'}(x)(v - \underline{s}) - \epsilon^2 F_j^{'}(x)$$

[5] We assume that $F_j(.)$ is differentiable

Since ϵ is arbitraily small, we have $\epsilon^2 F_j^{'}(x) \approx 0$.

$$\text{Thus } \triangle_\epsilon = F_j(\underline{s})(\frac{v-\underline{s}}{2} - \epsilon) + \epsilon F_j^{'}(x)(v - \underline{s}) > 0$$

This shows that B_i has a profitable deviation.

Next, consider the case when B_i offers to S_M and B_j offers to S_H. If B_i is playing a pure strategy then his offer must be less than H. If B_i is playing a mixed strategy then the upper bound of the support must be less than H. In both cases B_j can profitably deviate.

Lastly, consider the case when B_i is offering to S_M and B_j is offering to both the sellers. If B_i obtains a payoff of $M^* > v - H$ then the upper bound of the support of his offers must be less than H. Since the other buyer is offering to S_H, his payoff is bounded above by $v - H$. This implies that B_j can profitably deviate.

Hence from the above arguments we can infer that,

$$M^* \leq v - H \tag{6}$$

■

Proposition 2 *The outcome implied by the asymmetric equilibrium of Proposition 1 is the unique stationary equilibrium outcome of the basic game.*

Proof. From Lemma 7 and Lemma 8 we have,

$$M^* \leq v - H \leq m^* \tag{7}$$

By construction we have,

$$m^* \leq M^*$$

This implies that,

$$M^* = v - H = m^*$$

This concludes the proof. ■

2.2.3 Asymptotic characterisation

We now determine the limiting equilibrium outcome when the discount factor δ goes to 1.

From (3) we know that the probability with which the buyer B_1 offers to S_H is given by,

$$q = \frac{(v-H)(1-\delta)}{(v-M) - \delta(v-H)} \tag{8}$$

From (8) it is clear that as $\delta \to 1$, $q \to 0$.

From section **2.2.1** recall the equation,

$$G(x) = x - [\delta E^x(y) + (1-\delta)M]$$

Since the fixed point x^* is a function of δ, we denote it by $x^*(\delta)$.

The following lemma shows that the lower boun of price offer

Lemma 9 *There exists a $\delta^* \in (0,1)$, such that for any $\delta \in (\delta^*, 1)$, the fixed point $x^*(\delta)$ is always less than δH.*

Proof.

We know that for any $\delta \in (0,1)$, $\lim_{x\to H} G(x) > 0$.

Since the function $G(x)$ is continuous and monotonically increasing in x, there exists a $\delta^* \in (0,1)$ such that, $G(\delta H) > 0$ for all $\delta \in (\delta^*, 1)$. Thus for any $\delta \in (\delta^*, 1)$, the fixed point $x^*(\delta)$ is always less than δH. ■

Next, we show that as agents become patient enough, the probability with which the buyer B_2 offers M to S_M, goes to zero. This is described in the following lemma.

Lemma 10 *As $\delta \to 1$, $q^{'} \to 0$.*

Proof. We have,

$$\begin{aligned} q^{'} &= \frac{(v-H)(1-\delta)}{(v-p_l) - \delta(v-H)} \\ &= \frac{1}{\frac{v}{v-H} + \frac{\delta H - p_l}{(1-\delta)(v-H)}} \end{aligned}$$

where $p_l = x^*(\delta)$.

From Lemma 9 we have $\delta H - p_l > 0$. Thus we have

$$q^{'} \to 0 \text{ as } \delta \to 1$$

■

We will now show that as the discount factor goes to 1, the distributions of offers to S_M collapse. The following proposition shows this.

Proposition 3 *As $\delta \to 1$, $p_l \to H$.*

Proof.

The offers from B_2 to S_M in the range $[p_l, H]$, follows the distribution function

$$F_2(s) = \frac{(v-H)[1-\delta(1-q^{'})] - q^{'}(v-s)}{(1-q^{'})[v-s-\delta(v-H)]}$$

$$\Rightarrow 1 - F_2(s) = \frac{H - s}{(1 - q')[v - s - \delta(v - H)]}$$

.

Note that,

$$1 - F_2(H) = 0$$

From Lemma 10 we know that as $\delta \to 1$, $q^{'} \to 0$. Thus as $\delta \to 1$, for s arbitrarily close to H we have,

$$1 - F_2(s) \approx \frac{H - s}{H - s} = 1$$

Thus the support of the distribution F_2 collapses. This implies that as $\delta \to 1$, $p_l \to H$. ■

This shows that as agents become patient enough, the unique stationary equilibrium outcome of the basic complete information game implies that in presence of all players both the buyers almost surely offer H to seller S_M. Hence although trading takes place through decentralised bilateral interactions, asymptotically we get a uniform price for a non-differentiated good.

2.3 Possibility of other equilibria in the public offers case[6]

In the public offers model there is a possibility of other subgame perfect equilibria for high values of δ. These equilibria can be constructed on the basis of the stationary equilibrium described above. This is as follows.

1. Suppose in the beginning both the buyers offer $p = M$ to S_M.

2. S_M accepts one offer by choosing each seller with probability $\frac{1}{2}$.

3. If any buyer offers slightly higher than p (but less than some $p^{'}$ as described below), then S_M rejects all offers and next period players revert to the stationary equilibrium.

4. If any of the buyer offer a price grater than or equal to $p^{'}$, then the seller S_M accepts that price.

5. If both buyers offer p and S_M rejects them, then next period buyers offer p to S_M again.

The equilibrium payoff to S_M from accepting any of the equilibrium offer is 0. If buyers stick to their equilibrium strategies and S_M rejects an equilibrium offer then next period his payoff is 0. Thus S_M has no incentive to deviate.

On the other hand buyers' equilibrium payoff is $\frac{1}{2}(v-M)+\frac{1}{2}\delta(v-H)$. From the proposed equilibrium strategies we know that if a buyer deviates then the continuation payoff to S_M by rejecting all offers is close to $\delta(H - M)$, since from the previous section we know that

[6] Asher Wolinsky directed our attention to this type of equilibrium, similar to constructions elsewhere in the literature.

the payoff to S_M from the stationary equilibrium approaches $H - M$ as δ goes to 1. Hence if a deviating buyer wants his offer to be accepted by S_M then he must offer $p^{'}$ or higher to her where $p^{'} = \delta H + (1-\delta)M$. In that case his deviating payoff would be $v - p^{'}$, which is strictly lower than $\frac{1}{2}(v-M) + \frac{1}{2}\delta(v-H)$ for high values of δ. If a buyer deviates by offering slightly higher than M (i.e less than $p^{'}$) then all offers are rejected by S_M and next period he obtains a payoff of $v - H$ (which is worth $\delta(v-H)$ now). Thus he has no incentive to offer something higher than M. Lastly observe that it is optimal for S_M to reject all offers if some buyer offers something higher than M(and less than $p^{'}$). This is because his continuation payoff from rejection will be higher than his payoff from acceptance.

We can get equilibria of this kind for all $p \in [M, p_l)$ when δ is close to 1.

As is usual in equilibria of this kind, a small change in a buyer's equilibrium offer leads to a large change in the expected continuation payoff for all players.

However, if an ϵ (for arbitrary positive ϵ) deviation by a player from the proposed equilibrium path is considered as a mistake, then there is no change in the expected continuation payoff. In that case a buyer can profitably deviate by offering a price little above p to S_M. If this does not change the expected equilibrium path, S_M accepts this offer with probability 1, the buyer deviation is profitable and this candidate equilibrium is destroyed. We feel this argument has some validity, though a full formal development is outside the scope of this paper (and similar arguments have been suggested earlier in different contexts by other authors).

In the next section we do the analysis of the general case, i.e when there are n buyers and n sellers, for some general finite $n \geq 3$.

3 n buyers and n sellers, $n \geq 3$

3.1 Players and payoffs

There are n buyers ($n > 2$ and n finite) and n sellers. Each buyer's maximum willingness to pay for a unit of an indivisible good is v. Each of the sellers owns one unit. Sellers differ in their valuations. We denote seller S_j's valuation ($j = 1, ..., n$) by u_j where,

$$v > u_n > u_{n-1} > ... > u_2 > u_1$$

The above inequality implies that any buyer has a positive benefit from trade with any of the sellers. All players are risk neutral. Hence the expected payoffs obtained by the players in any outcome of the game are identical to that in the basic model. For our notational convenience, we re-label $u_1 = L$ and $u_n = H$.

3.2 The extensive form

This is identical to the one in the basic complete information game. We consider the infinite horizon, public and targeted offers game where the buyers simultaneously make offers and each seller either accepts or rejects an offer directed towards her. Matched pairs leave the game and the remaining players continue the bargaining game with the same protocol.

3.3 Equilibrium

We seek, as usual, to find stationary equilibria. Thus buyers' offers at a particular time point depend only on the set of players remaining and the sellers' responses depend on the set of players remaining and the offers made by the buyers. We first show that in any arbitrary stationary equilibrium, as agents become patient, buyers' payoff is unique and all price offers converge to same value. Thereafter we construct one such equilibrium for the described extensive form.

3.3.1 Uniqueness of the asymptotic stationary equilibrium payoff and prices.

For the basic two-buyer, two-seller game, we have demonstrated the uniqueness of the stationary equilibrium. Following the same method of proof would be difficult in the general case because of the large number of special cases one would have to consider. We therefore adopt a different route here and use the uniqueness of the stationary equilibrium in the basic two-by-two model to demonstrate that *in any stationary subgame perfect equilibrium, the accepted price offer by any seller* converges to H as $\delta \to 1$. Thus in the general case as well, the buyers' payoffs converge to the same value $(v - H)$ as $\delta \to 1$. This payoff is "as if" the price in all transactions were the same but, of course, this is not literally true since $\delta \in (0, 1)$. We thus show that the asymptotic outcome implied by the particular stationary equilibrium demonstrated is the unique asymptotic outcome obtained in any stationary equilibrium. The following theorem demonstrates this result.

Theorem 1 *In any stationary equilibrium of a game with n buyers and n sellers ($n \geq 3$, n finite), prices in all transactions converge to H as players become patient enough($\delta \to 1$).*

The above theorem directly follows from the uniqueness result of the 2-buyer, 2 seller game of the basic model (proposition (2)) and iterating on the following proposition (proposition 4).

Proposition 4 *Let $n \geq 1$. If for any $m = 1, 2, ..., n$; (n finite) it holds that in a game with m buyers and m sellers accepted price offers converge to H for high values of δ, then this is also the case with any stationary equilibrium of a game with $n + 1$ buyers and $n + 1$ sellers*

Proof.

Please refer to appendix (J)

■

Whilst the formal proof of the proposition is in the appendix, we give some intuition here for the result. We start from the two buyers and two sellers case, where there is a unique stationary equilibrium with the price going to H as $\delta \to 1$. Thus, if there are more sellers and buyers, a seller knows that if she manages to stay in the game until there is only one other seller left, namely the one with value H, she can expect to get a price close to H. If discounting is small, she will only be willing earlier to accept prices close to H, because of the "outside option" of waiting. This holds for all the other sellers. The buyers, on the other hand, compete up the prices of the low-valued sellers and for them, the seller with value H is an outside option. The prices for the low valued sellers therefore get bid up in earlier periods to something close to H; as discounting goes to zero ($\delta \to 1$), this goes to H. Note this argument requires stationarity because the expectations of future play are not affected by the history of offers and counter-offers.

3.3.2 Characterisation of stationary equilibrium

We have shown that the asymptotic outcome of an arbitrary stationary equilibrium of the described extensive form is unique. We now provide a characterisation of a stationary equilibrium for the extensive form described. Since we start out with equal numbers of buyers and sellers, any possible subgame will also have that. Depending on the parametric values we can have three types of equilibria. However, as δ becomes greater than a threshold value, there is only one type of equilibrium.

From the basic complete information game, for each $i = 1, ..., n-1$, we calculate p_i such that,

$$p_i = (1-\delta)u_i + \delta E(y_i) \tag{9}$$

where $E(y_i)$ is defined as the equilibrium expected maximum price offer which S_i gets in the four-player game with S_i and S_n as the sellers and two buyers with valuation v.[7]

For each $i = 1, ..., n-1$ we define $\bar{q}_i$ as,

$$\bar{q}_i = \frac{H - p_i}{(v - p_i) - \delta(v - H)} \tag{10}$$

[7]Note that p_i is given by the equilibrium of the appropriate four-player game, which has already been described earlier. It can essentially be treated as an exogenously given function of the parameters of the problem for the purposes of the $n-player$ analysis.

and q^H as ,

$$q^H = \frac{(v-H)(1-\delta)}{(v-L)-\delta(v-H)} \tag{11}$$

Let $\mathcal{P} = \sum_{i=1,..,n-1} \bar{q}_i$. The following three propositions fully characterise the equilibrium behavior in the present game[8]. In all of them, sellers' strategies are as follows: (i) S_n accepts any offer greater than or equal to H. (ii) Seller S_i $(i = 1,..,n-1)$ accepts the highest offer with a payoff from accepting at least as large as the expected continuation payoff from rejecting it. The following theorem summarizes the equilibrium characterisations of the extensive form defined.

Theorem 2 *The equilibrium in the general case is given by the propositions (5), (6) and (7). For δ close to 1 and $n > 2$, proposition (7) gives the relevant characterisation.*

Proposition 5 *If for $\delta \in (0,1)$, $\mathcal{P} < 1$ and $1 - \mathcal{P} > q^H$, then a stationary equilibrium is as follows:*

(i) Buyer B_1 makes offers to S_1 only. B_1 puts a mass of $q_1^{'}$ at L and has a continuous distribution of offers $F_1(.)$ with $[p_1, H]$ as the support. B_n makes offers to S_1 with probability q_1. He randomises his offers to S_1 with a probability distribution $F_n^1(.)$ with $[p_1, H]$ as the support. $F_n^1(.)$ puts a mass point at p_1 and has an absolutely continuous part from p_1 to H.

(ii) For $i = 2, ..., n-1$, B_i makes offers to S_i only. B_i's offers to S_i are randomised with a distribution $F_i(s)$. $F_i(.)$ puts a mass point at p_i and has an absolutely continuous part from p_i to H. B_n makes offers to S_i $(i = 2,..,n-1)$ with probability $q_i = \bar{q}_i$. B_n's offers to S_i are randomised by an absolutely continuous probability distribution F_n^i with $[p_i, H]$ as the support.

(iii) B_n offers to S_n with probability q^H. He offers H to S_n.

(iv) In equilibrium, all buyers obtain an expected payoff of $v - H$.

The analytical details are in appendix D

Proof.

Please refer to appendix (D). ■

We now consider the case when $\mathcal{P} < 1$ and $1 - \mathcal{P} < q^H$.

Proposition 6 *If for a $\delta \in (0,1)$ $\mathcal{P} < 1$ and $1 - \mathcal{P} < q^H$, then a stationary equilibrium is as follows:*

(i) For $i = 1, 2, ..., n-1$, buyer B_i makes offers to S_i only. B_i's offers to S_i are random with a distribution $F_i(s)$. $F_i(.)$ puts a mass point at p_i and has an absolutely continuous part

[8] Note that all quantities used in these propositions are defined with respect to the exogenously given parameters of the game.

from p_i to H. B_n makes offers to S_i ($i = 1, .., n-1$) with probability $q_i = \bar{q}_i$. B_n's offers to S_i are random with an absolutely continuous probability distribution F_n^i with $[p_i, H]$ as the support.

(ii) B_n offers to S_n with probability $q_n = 1 - \mathcal{P}$. He offers H to S_n.

(iii) In equilibrium, all buyers obtain an expected payoff of $v - H$.

The analytical details are in appendix E

Proof.

Please refer to appendix (E) ■

Finally we consider the case when $\mathcal{P} > 1$.

Proposition 7 *If $\mathcal{P} \geq 1$, then a stationary equilibrium is as follows:*

For $i = 1, .., n-1$, buyer B_i makes offers to seller S_i only. B_i's offers to S_i are randomised using a distribution function $F_i(.)$, with $[p_i, \bar{p}]$ as the support. The distribution $F_i(.)$ puts a mass point at p_i and has an absolutely continuous part from p_i to $\bar{p}$. Buyer B_n offers to all sellers except S_n. B_n's offers to S_i ($i = 1, ..n-1$) are randomised with a continuous probability distribution F_n^i. The support of offers is $[p_i, \bar{p}]$. The probability with which B_n offers to S_i ($i = 1, .., n-1$) is q_i. If $\mathcal{P} = 1$ then $\bar{p} = H$. If $\mathcal{P} > 1$ then $\bar{p} < H$ and as $\delta \to 1$, $\bar{p} \to H$. In equilibrium, all buyers obtain an expected payoff of $v - \bar{p}$. The following relations formally define the equilibrium:

Further if for $\delta = \delta^*$, $\mathcal{P} > 1$ then for all $\delta > \delta^*$, $\mathcal{P} > 1$ and $\bar{p} \to H$ as $\delta \to 1$.

The analytical details are in appendix F

Proof.

Please refer to appendix (F)

■

Proposition (7) tells us that as agents become patient enough, prices in all transactions tend towards H.[9]. *Note, from equation (10) and the definition of $\mathcal{P}$, that for δ close to* 1 *and $n > 2$, $\mathcal{P} \geq 1$. Therefore, proposition (7) is the appropriate case to consider for high enough δ.* The following observation can be made about the asymptotic result. *For δ high enough, the prices tend towards the valuation of the highest seller, independently of the distributions of the valuations of the other sellers.* Hence even if the distribution of the valuations of the sellers S_i ($i = 1, ..n-1$) is heavily skewed towards L, the uniform asymptotic price will still be H.

[9] We have seen earlier (in the 2×2 game analysis) that p_i goes to H as $\delta \to 1$. In this propsition, we show that $\bar{p} \to H$ as $\delta \to 1$. Thus the supports of the randomised strategies also collapse as $\delta \to 1$

We conclude this section by providing a verbal description of the nature of the stationary equilibrium described above.

It can be observed that in all of the above stationary equilibria(propositions 5, 6 and 7)) each buyer, other than B_n,is assigned to a seller to make offers to-buyer B_i to seller S_i. The remaining buyer (B_n) offers to all (or all but one) the sellers. This creates some competition among the buyers, since each seller(except S_n) gets two offers with positive probability. The probability q^H is the probability with which B_n should offer to S_n in equilibrium if B_1 puts a mass point at $u_1(= L)$. The quantity $\bar{q}_i$ is the probability with which B_n should offer to S_i in equilibrium, if B_i puts a mass point at p_i and B_n offers to all the sellers. Further, in any stationary equilibrium, a buyer who is assigned to a seller S_j has to put a mass point either at u_j or at p_j. Hence, for a given δ, if B_n has to make offers to all the sellers then it is necessary to have $\mathcal{P} < 1$. Further if $1 - \mathcal{P} > q^H$, then it is possible to have the buyer B_1 put a mass point at L; the equilibrium is then described by proposition (5). Otherwise the equilibrium is described by proposition (6). On the other hand if $\mathcal{P} \geq 1$ it is not possible to have B_n offering to all the sellers in equilibrium. In that case he offers to all but the highest valued seller. The equilibrium is then described by proposition (7). In the 2×2 case, the conditions $\mathcal{P} < 1$ and $1 - \mathcal{P} > q^H$ are satisfied for all values of $\delta \in (0, 1)$. This is because in the 2×2 case $\mathcal{P} = \frac{H-p_l}{(v-p_l)-\delta(v-H)}$, which is less than 1 for all values of $\delta \in (0, 1)$. Further $1 - \mathcal{P} = \frac{(v-H)(1-\delta)}{(v-p_l)-\delta(v-H)} > q^H = \frac{(v-H)(1-\delta)}{(v-M)-\delta(v-H)}$ as $p_l > M$. Hence the qualitative nature of the equilibrium described in proposition (5) is identical to the one described in the basic model. However for $n > 2$, the conditions satisfied by the 2×2 configuration need not hold for all values of δ.

4 Extensions

In this section we consider possible extensions by having offers to be private, buyers being heterogeneous and number of buyers being more than the number of sellers. An extension of the basic model to ex-ante public offers is available in the Appendix.

4.1 Private Offers

In this subsection, we consider a variant of the extensive form of both the basic model (2 buyers-2 sellers) and the general model (n buyers-n sellers; $n \geq 3$) by having offers to be private. This means in each period a seller observes only the offer(s) she gets and a buyer does not know what and to whom offers are made by the other buyer(s).

Our equilibrium notion here will be *public perfect equilibrium*. The only public history in each period is the set of players remaining in the game. Clearly in the private targeted offers model, the response of a seller can condition only on her own offer. Hence, in the basic model, the equilibrium of proposition 1 is a public perfect equilibrium of the game with private targeted offers. Further, in proving this stationary equilibrium outcome to be unique (proposition 2) we have never used the fact that each seller while responding observes the other seller's offer. Thus the same analysis will hold good in the *private offers model*. Hence the outcome implied by the stationary equilibrium of proposition (1) is the unique *public perfect equilibrium* outcome of the basic complete information game with private targeted offers.

Next, consider the general model with n buyers and n sellers. In proposition (7), the highest valued seller does not get any offer when all the players are present. Hence the continuation game faced by a seller from rejection is always the same irrespective of whether she gets one offer or two offers. A seller knows that by rejecting all the offer(s) she will face a four-player game with S_n as the other seller and two buyers with valuation v. Thus the seller S_i,$(i = 1, .., n-1)$ knows the continuation game for sure and this does not require her to observe the offers received by other sellers or the seller to whom buyer B_n is making his offer. Since for high values of δ, $\mathcal{P} \geq 1$, we have the following corollary:

Corollary 1 *With private offers, Proposition (7) describes a public perfect equilibrium of the game for high values of* δ*.*

Theorem (1) extends to the game with private offers with some minor modification of the details[10].

4.2 Heterogeneous buyers

Suppose, in the basic model, buyers too are heterogeneous. That is, buyer B_i has a valuation of v_i where,

$$v_1 > v_2 > H > M$$

Analysis of the basic model holds good. Next, consider a model with n heterogeneous buyers and n heterogeneous sellers such that

[10]This is when in a stationary equilibrium all sellers get offers with positive probability. A seller S_k getting only one offer does not know for sure the continuation game on rejection. Hence her minimum acceptable price when she gets only one offer is always greater than her valuation u_k.This is the only change in detail of the proof of theorem 1. The rest of the proof holds good for private offers as well.

$$v_N > v_{N-1} > ... > v_2 > v_1 > H > u_{N-1} > .... > L$$

u_i $(i = 1, 2, .., n)$is the valuation of seller S_i with $u_1 = L$ and $u_N = H$. v_i $(i = 1, 2, ..., n)$ is the valuation of buyer B_i. An analogue of Theorem 2 holds in this case. The discussion is in Appendix (H).

We conclude this subsection by providing an example to show that even if there is potential of trade for both the sellers, such trades need not take place in the equilibrium of our model. Suppose there are two buyers with valuation v_1 and v_2 and two sellers with valuations H and M such that

$$M < v_2 < H < v_1$$

In equilibrium, both the buyers offer v_2 to the seller with valuation M and the trade takes place between the M-seller and the v_1-buyer. (If, in equilibrium, the v_2 buyer were concluding the trade with positive probability, the v_1 buyer would offer $\varepsilon > 0$ more and have a profitable deviation.) Note that, in this case, any price between v_2 and H would be a competitive equilibrium in which the demand and supply would equate.

4.3 n buyers, $n - 1$ sellers

We now consider the case when there are more buyers than sellers. That is, there are n buyers and $n - 1$ sellers such that $n \geq 2$. Buyers are homogeneous and their common valuation exceeds the valuation of the highest valued seller. The extensive form of the game is same as before.

For $n = 2$, the solution is quite trivial. Both buyers would compete for the only available seller and hence they would pull up the equilibrium price to v, the common valuation of the buyers.

Appendix (L) shows that for $n > 2$, we can construct a stationary equilibrium such that when agents become patient, prices in all transactions converge to a single value v, the common valuation of the buyers. Thus in the limit only the short side of the market gets positive surplus. This is equivalent to the Walrasian outcome of the present setup.

5 Conclusion

This paper has considered a dynamic strategic matching and bargaining game, with the feature that only one side of the market makes offers. Unlike other papers in the field, the offers are made *simultaneously* to capture competition. We find that stationary equilibria give a single price asymptotically in all the transactions.

Previous work has shown that this conclusion is not true when buyers and sellers take it in turns to make offers (a game of which the Rubinstein bargaining game is a special case). Alternating offers with heterogeneity in valuations tends to drive valuations apart.

Other authors [6] have mentioned the difficulty of solving dynamic bargaining and matching games with many players if there is heterogeneity of valuations on both sides, though she was specifically concerned with alternating offers. This turns out mostly not to be an issue for us.

One interesting heterogeneity would be to consider settings in which the value of buyer i for seller $j's$ good is v_{ij}, as in the housing market. In this setting it seems appropriate to assume that sellers' valuations do not depend upon the identity of the potential buyers. This is kept for future research, though it seems feasible that techniques similar to the ones used in this paper would enable us to characterise equilibrium prices in such markets as well.

acknowledgements: We thank Bhaskar Dutta, Drew Fudenberg, Faruk Gul, Ed Green, Vijay Krishna, Selçuk Özyurt, Larry Samuelson, Adam Slawski and Asher Wolinsky for their insightful comments and suggestions. We also thank an aasociate editor and two anonymous referees for their comments due to which the exoposition of the paper has become much better.

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Appendix

A Proof of Lemma 1

The proof proceeds as follows: We first show that $F_1(\cdot)$, $F_2(.)$ as given are probability distributions and have the desired properties. Next we show that q, q' are in (0,1). Assuming

p_l is between M and H, we then show that the strategies are an equilibrium. In the lemma (2), we show that there is a unique p_l implied by all these conditions and it is between M and H.

Since both buyers offer to S_M, it is clear that in equilibrium the offers to S_M from both the buyers have to be randomised.

To begin with, we figure out the continuation payoff for S_M from rejecting her offer(s). Consider the case when rejecting an offer leads her to face a 2-player game next period. This gives her a continuation payoff of zero.

When rejection leads S_M to face a 4-player game next period, the continuation payoff needs to be endogenously determined from the equilibrium strategies of the buyers. (Recall y is the maximum price S_M gets in equilibrium in the next period (a random variable this period)). Thus if p_l is the minimum acceptable price for S_M in this situation, we must have,

$$p_l - M = \delta(E(y) - M)$$
$$\Rightarrow p_l = (1-\delta)M + \delta(E(y))$$

Given the buyers' strategies, $E(y)$ is given by,

$$E(y) = q[q^{'}M + (1-q^{'})E_2(p)] + (1-q)[q^{'}E_1(p) + (1-q^{'})E(\text{highest offer})]$$

where $E_1(p)$ is the conditional expectation of B_1's offers given that he is offering to S_M and $E_2(p)$ is the conditional expectation of B_2's offers given that he is not offering M to S_M.

Since, as per our proposed strategies, competition takes place for S_M only, it is easy to note that $E(y) > M$. The fact $\delta \in (0,1)$ implies that we must have $p_l > M$.

Consider the region $[p_l, H]$ first, where both B_2 and B_1(if he does make one to S_M) make an offer. In equilibrium both buyers must be indifferent for all price offers in this region.

According to the proposed strategies the support of B_1's offer to S_M is $[p_l, H]$. Also we know that B_1 in equilibrium can obtain a payoff of $v - H$ by offering H to S_H. Hence for any $s \in (p_l, H]$ we should have the following indifference relation:

$$(v-s)[q^{'} + (1-q^{'})F_2(s)] + (1-q^{'})(1-F_2(s))[\delta(v-H)] = v - H$$

, which gives us,

$$F_2(s) = \frac{(v-H)(1-\delta(1-q^{'})) - q^{'}(v-s)}{(1-q^{'})[(v-s) - \delta(v-H)]}$$

as per (2).

As stated earlier, p_l is the minimum acceptable price for S_M, when on rejection she faces a 4-player game next period. This implies that on the equilibrium path p_l is the minimum acceptable price for S_M when she gets two offers. Thus B_1's offer of p_l to S_M is accepted only when B_2 offers M to S_M. Hence for $s = p_l$, B_1's indifference relation is,

$$(v - p_l)[q^{'}] + (1 - q^{'})[\delta(v - H)] = v - H$$

which implies,

$$q^{'} = \frac{[v - H](1 - \delta)}{(v - p_l) - \delta(v - H)}$$

as per (4).

Since $H > p_l$, from (4) we have,

$$q^{'} = \frac{[v - H](1 - \delta)}{(v - p_l) - \delta(v - H)} < \frac{[v - H](1 - \delta)}{[v - H](1 - \delta)} = 1$$

This implies that $q^{'} \in (0, 1)$.

For $F_2(.)$ to be a distribution function as conjectured, we must have $F_2(H) = 1$ and $F_2(P_l) = 0$. From 2 we have,

$$1 - F_2(s) = \frac{H - s}{(1 - q')[(v - s) - \delta(v - H)]}$$

From (4) we can infer that,

$$1 - F_2(p_l) = \frac{H - p_l}{(1 - q')[(v - p_l) - \delta(v - H)]} = \frac{(1 - q^{'})}{(1 - q')} = 1$$

and $1 - F_2(H) = 0$. Thus we have $F_2(H) = 1$ and $F_2(p_l) = 0$. Hence F_2 has the conjectured properties .

Now consider the behavior of B_2 in the selected region. Since B_2 can obtain a payoff of $v - H$ by offering H to S_M, for any $s \in [p_l, H]$ we should have,

$$(v - s)[q + (1 - q)F_1(s)] + (1 - q)(1 - F_1(s))[\delta(v - H)] = v - H$$

which gives us (1).

Next, consider other regions. According to the conjectured equilibrium strategies, B_2 offers M to S_M with probability $q^{'}$(i.e, he puts a mass point at M). Also B_1 offers H to S_h with probability q. At equilibrium B_2 should be indifferent for all price offers he makes.

Therefore we should have,

$$(v - M)q + (1 - q)\delta(v - H) = v - H$$

which gives us (3). Since $H > M$, from (3) we have,

$$q = \frac{[v - H](1 - \delta)}{(v - M) - \delta(v - H)} < \frac{[v - H](1 - \delta)}{[v - H](1 - \delta)} = 1$$

This implies that $q \in (0, 1)$.

For F_1 to satisfy the conjectured properties we should have $F_1(p_l) > 0$(since B_1 puts a mass point at p_l while offering to S_M) and $F_1(H) = 1$. From (1) and (3) we have,

$$1 - F_1(p_l) = \frac{H - p_l}{(1 - q)[(v - p_l) - \delta(v - H)]}$$

$$= \frac{(1 - q^{'})}{(1 - q)}$$

Since $p_l > M$, $q > q^{'}$. Thus

$$\frac{(1 - q^{'})}{(1 - q)} < 1 \tag{12}$$

From (12) we can infer that,

$$1 - F_1(p_l) < 1 \Rightarrow F_1(p_l) > 0$$

Also it is easy to note that $1 - F_1(H) = 0$. Hence $F_1(H) = 1$.Thus $F_1(.)$ satisfies the conjectured properties.

Lastly, to conclude the proof it needs to be verified that above specified strategies constitute a *subgame perfect equilibrium*. We use the one deviation property to do this.

Consider the sellers first. Since we are considering public offers, a seller's history constitutes the set of players, the offer she receives and the other seller's received offer(s). On the equilibrium path there are only two possible histories. One has all the players present with both sellers getting equilibrium offers. The other one is when only two players are present and an equilibrium offer is made. It is easy to observe that in the two-player game no seller has a profitable one-shot deviation. This is because offers are one sided. Thus we need to verify equilibrium for the 4-player game only. In the 4-player game irrespective of S_M's offer, it is always optimal for S_H to accept any offer greater than or equal to H. If she rejects then next period period either she will face a 4-player game or a 2-player game. In either case, given that other players adhere to their equilibrium strategies the maximum payoff which

S_H can obtain is 0. Also S_H has no incentive to accept any offer less than H, (which gives her a negative payoff), as she can always guarantee a zero payoff by rejecting the offer.

Next let us look at the possible one-shot deviations for S_M on the equilibrium path. Suppose in the event when she gets two offers, she rejects an offer greater than or equal to p_l. Her continuation payoff would then be $p_l - M$. This is less than or equal to the payoff obtained by accepting the offer. Thus on the candidate equilibrium path there is no profitable one-shot deviation by S_M. Finally, the way we have specified S_M's strategy, there exists no profitable one shot deviations for S_M for any off-path history.

Now consider the buyers. After any history there can be only two possible situations. Either all the players are present or only one pair remains. Given other players' strategies and the one-deviation property it is easy to note that buyers cannot profitably deviate.

This concludes the proof of the lemma.

B Proof of the claim that $\frac{\partial E^x(y)}{\partial x} < 1$

We prove this in the following steps:

(i) From the expression obtained for $q^{'}$ we can say that $q^{'x}$ is increasing in x.

(ii) Next we show that as we raise x by 1 unit, there is an increase in $E_2^x(p)$ by less than 1 unit.

Increasing x by 1 unit means raising the lower bound of support of $F_2^x(.)$ by 1 unit. Thus we need to show that

$$E_2^{x+1}(p) < E_2^x(p) + 1$$

Consider the distribution $\tilde{F}_2^x(.)$ with $[x+1, H+1]$ as the support such that,

$$\tilde{F}_2^x(s) = F_2^x(s-1)$$

Let $\widetilde{E_2^x(p)}$ be the expectation obtained under $\tilde{F}_2^x(s)$. Thus,

$$\widetilde{E_2^x(p)} = \int_{x+1}^{H+1} s \, d\tilde{F}_2^x(s)$$

$$\Rightarrow \widetilde{E_2^x(p)} = [\int_{x+1}^{H+1} (s-1) \, d\tilde{F}_2^x(s)] + 1$$

$$= [\int_{x+1}^{H+1} (s-1) \, dF_2^x(s-1)] + 1$$

$$= [\int_{x}^{H} (s)\, dF_2^x(s)] + 1$$

$$= E_2^x(p) + 1$$

$F_2^{x+1}(p)$ is obtained from $\tilde{F}_2^x(s)$ by transferring the mass from the interval $(H, H+1]$ to $[x+1, H]$, i.e transferring mass from higher values to lower values. Thus it is clear that,

$$E_2^{x+1}(p) < \widetilde{E_2^x(p)} = E_2^x(p) + 1$$

By similar reasoning we can say that ,

$$E_1^{x+1}(p) < E_1^x(p) + 1$$

These imply that the increase in E(highest offer) following a unit increase in x is less than 1.

Hence from the above arguments it follows that,

$$\frac{\partial E^x(y)}{\partial x} < 1$$

C Proof of lemma 12

As before, define the function $G(.)$ as,

$$G(x) = x - [\delta E^x(y) + (1-\delta)M]$$

where $E^x(y)$ is obtained from $E(y)$ as before(i.e by replacing p_l by x). Using lemma **??** we can argue that $G^{'}(x)$ is monotonically increasing in x for $x \in (p_l^{'}, H)$. Next, from the above prescribed strategies it is easy to see that for any $x \in (p_l^{'}, H)$,we have $E^x(y) > p_l^{'}$. Thus we can infer that there exists a $\delta^* \in (0,1)$ such that,

$$\lim_{x \to p_l^{'}} G(x) = x - [\delta^* E^x(y) + (1-\delta^*)M] = 0$$

Thus for any $\delta > \delta^*$, we have $\lim_{x \to p_l^{'}} G(x) < 0$. Also since for all $x \in (p_l^{'}, H)$, $E^x(y) < H$, we have $\lim_{x \to H} G(x) > 0$. Hence by applying the *Intermediate Value Theorem* we can infer that there exists a unique $x^* \in (p_l^{'}, H)$ such that $G(x^*) = 0$. This x^* is our required p_l. Thus there is a unique $p_l \in (p_l^{'}, H)$ such that for all $\delta > \delta^*$,

$$G(p_l) = 0 \Rightarrow p_l = \delta E(y) + (1-\delta)M$$

D Analytical details and proof of Proposition 5

Analytical Details:

The distributions $F_1(.)$, $F_n^1(.)$, q_1 and $q_1^{'}$ are given by:

$$F_1(s) = \frac{(v-H)[1-\delta(1-q_1^{'})] - q_1^{'}(v-s)}{(1-q_1^{'})[(v-s)-\delta(v-H)]} \tag{13}$$

$$F_n^1 = \frac{(v-H)[1-\delta q_1] - (1-q_1)(v-s)}{q_1[(v-s)-\delta(v-H)]} \tag{14}$$

$$q_1^{'} = \frac{(v-H)(1-\delta)}{(v-p_1)-\delta(v-H)} \tag{15}$$

$$q_1 = \bar{q}_1 + (1-\mathcal{P}-q^H) \tag{16}$$

For $i = 2, .., n-1$, $F_i(.)$ and $F_n^i(.)$ are given by,

$$F_i = \frac{(v-H)(1-\delta)}{(v-s)-\delta(v-H)} \tag{17}$$

$$F_n^i = \frac{(v-H)[1-\delta q_i] - (1-q_i)(v-s)}{q_i[(v-s)-\delta(v-H)]} \tag{18}$$

Proof:

Consider Buyer B_1 first. He puts a mass point at L and his equilibrium payoff is $v-H$. Since we are considering public offers, S_1 will accept an offer of L only when B_n is offering to S_n. This is because only in that contingency would the continuation payoff to S_1 from rejection be zero. Thus we must have,

$$(v-L)q^H + (1-q^H)\delta(v-H) = v-H \tag{19}$$

Solving for q^H we get (11). Consider the region $[p_1, H]$, where both B_1 and B_n make offers. In equilibrium each buyer should be indifferent among all the points in the support. Thus for $s \in [p_1, H]$, B_1's indifference relation is given by:

$$(v-s)[(1-q_1) + q_1 F_n^1(s)] + q_1(1-F_n^1(s))\delta(v-H) = v-H$$

Solving for $F_n^1(.)$ from the above relation we get (14). Similarly for $s \in [p_1, H]$, B_n's indifference relation from offering to S_1 is,

$$(v-s)[q_1^{'} + (1-q_1^{'})F_1(s)] + (1-q_1^{'})(1-F_1(s))\delta(v-H) = v-H$$

Solving for $F_1(.)$ we get (13). Putting $s = p_1$ in B_n's indifference relation we get,

$$(v-s)q_1^{'} + (1-q_1^{'})\delta(v-H) = v-H$$

which gives us (15). Note that from (14) and (10) we have,

$$1 - F_n^1(p_1) = \frac{H-p_1}{q_1[(v-p_1)-\delta(v-H)]} = \frac{\bar{q}_1}{q_1}$$

From (16) we know that $q_1 > \bar{q}_1$. Hence we have $1-F_n^1(p_1) < 1$ which implies that $F_n^1(p_1) > 0$. This confirms our conjecture that B_n, while offering to S_1 puts a mass point at p_1. It is easy to check that $F_n^1(H) = 1$. Similarly from (13) and (15) we have,

$$1 - F_1(p_1) = \frac{H-p_1}{(1-q_1^{'})[(v-p_1)-\delta(v-H)]} = \frac{(1-q_1^{'})}{(1-q_1^{'})} = 1$$

which implies $F_1(p_1) = 0$. Again it is easy to observe that $F_1(H) = 1$.

Next, consider buyer B_i, $i = 2, ..., n-1$. Consider the region $[p_i, H]$, where both B_i and B_n make offers. In equilibrium both buyers should be indifferent between any offers in the region. For $s \in [p_i, H]$, B_n's indifference relation is given by,

$$(v-s)[F_i(s)] + [1-F_i(s)]\delta(v-H) = v-H$$

Solving for $F_i(.)$ from above, we get (17). We can easily infer that $F_i(p_i) > 0$ and $F_i(H) = 1$. This confirms the conjecture that B_i puts a mass point at p_i. Similarly, B_i's indifference relation is given by:

$$(v-s)[(1-q_i) + q_iF_n^i(s)] + q_i(1-F_n^i(s))\delta(v-H) = v-H$$

which gives us (18). Putting $s = p_i$ in B_i's indifference relation we get $q_i = \frac{v-p_i}{(v-p_i)-\delta(v-H)} = \bar{q}_i$. Hence we have,

$$1 - F_n^i(p_i) = \frac{H-p_i}{q_i[(v-p_i)-\delta(v-H)]} = \frac{q_i}{q_i} = 1$$

Thus $F_n^1(p_i) = 0$ and $F_n^i(H) = 1$. Also note that,

$$\sum_{i=1,,,n-1} q_i + q^H = \bar{q}_1 + (1 - \mathcal{P} - q^H) + \sum_{j=2,..,,n-1} \bar{q}_j + q^H = 1$$

Since $u_j > L$ for $j > 1$, from (19) we know that,

$$(v - u_j)q^H + (1 - q^H)\delta(v - H) < v - H \text{ for } j = 2, ...n - 1$$

Hence B_i $(i = 2, ..., N)$ does not have any incentive to offer u_j to seller S_j. Further, B_i cannot obtain a payoff higher than $v - H$ by deviating unilaterally and making offers to any other sellers. Lastly, the way we have specified sellers' strategies it is easy to check that none of the sellers has a unilateral profitable deviation on the equilibrium path. This concludes the proof.

E Analytical details and proof of Proposition 6

Analytical details: For $i = 1.., n - 1$, $F_i(.)$ and $F_n^i(.)$ are given by,

$$F_i = \frac{(v - H)(1 - \delta)}{(v - s) - \delta(v - H)} \tag{20}$$

$$F_n^i = \frac{(v - H)[1 - \delta q_i] - (1 - q_i)(v - s)}{q_i[(v - s) - \delta(v - H)]} \tag{21}$$

Proof: This proof is identical in many respects to the proof of proposition (5). Consider the region $[p_i, H]$,$(i = 1, .., n - 1)$. In this region both B_i and B_n make offers with positive probability. By considering the indifference relations of B_i and B_n in this region, we can get (20) and (21) in the same manner as we obtained (17) and (18) in the proof of the previous proposition. Similarly, we can infer that $F_i(p_i) > 0; F_i(H) = 1$ and $F_n^i(p_i) = 0; F_n^i(H) = 1$. Since $q_n = 1 - \mathcal{P} < q^H$, from (19) we know that,

$$(v - L)q^H + (1 - q^H)\delta(v - H) = v - H \text{ and}$$

$$(v - u_j)q^H + (1 - q^H)\delta(v - H) < v - H$$

for all $j = 2, .., n - 1$. Since $q_n < q^H$, for all $j = 1, ..n - 1$ we have,

$$(v - u_j)q_n + (1 - q_n)\delta(v - H) < v - H$$

Hence B_i $(i = 1, ..., n-1)$ has no incentive to offer u_i to seller S_i. Finally note that,

$$\sum_{i=1,..,n} q_i = \sum_{i=1,..,n-1} \bar{q}_i + (1 - \mathcal{P}) = 1$$

This concludes the proof.

F Analytical detail and proof of Proposition 7

Analytical Detail: For $i = 1, ..., n-1$ we have

$$F_i(s) = \frac{(v - \bar{p}) - \delta(v - H)}{(v - s) - \delta(v - H)} \tag{22}$$

$$F_n^i = \frac{(v - H)[1 - \delta q_i] - (1 - q_i)(v - s)}{q_i[(v - s) - \delta(v - H)]} \tag{23}$$

$$q_i = \frac{\bar{p} - p_i}{(v - p_i) - \delta(v - H)} \tag{24}$$

Proof:

Consider the region $[p_i, \bar{p}]$ $(i = 1, ..., n-1)$, where both the buyers B_i and B_n make offers. Hence the indifference relation of B_n is given by,

$$(v - s)F_i(s) + (1 - F_i(s))\delta(v - H) = v - \bar{p}$$

This gives us (22). One can easily figure out from (22) that $F_i(p_i) > 0$ and $F_i(\bar{p}) = 1$. This confirms our conjecture that $B_i(i = 1, .., n-1)$ puts a mass point at p_i. Buyer B_i's indifference relation is given by,

$$(v - s)[(1 - q_i) + q_i(F_n^i(s))] + q_i(1 - F_n^i(s))\delta(v - H) = v - \bar{p}$$

Solving for $F_n^i(.)$ we get (23). By substituting $s = p_i$ in B_i's indifference relation we get (24). From (23)and (24) it is easy to see that $F_n^i(p_i) = 0$ and $F_n^i(H) = 1$. For consistency in the expressions obtained we must have,

$$\sum_{i=1,..,n-1} q_i = 1 \Rightarrow \sum_{i=1,..,n-1} \frac{\bar{p} - p_i}{(v - p_i) - \delta(v - H)} = 1 \tag{25}$$

From the hypothesis of the proposition we know that $\mathcal{P} \geq 1$. If $\mathcal{P} = 1$, from (25) we have $\bar{p} = H$. If $\mathcal{P} > 1$, from (25) we can infer that $\bar{p} < H$.

From the analysis of the basic complete information game we know that for each $i = 1, ..., n-1$, $\frac{H-p_i}{(v-p_i)-\delta(v-H)} \to 1$ and $p_i \to H$ as $\delta \to 1$. Thus if $\mathcal{P} > 1$ for a particular $\delta^* \in (0,1)$,[11] it will be so for all $\delta > \delta^*$. Thus, the equilibrium behavior will remain the same for all higher values of δ. Hence we can characterise the equilibrium for values of δ close to one. Using (25) we have,

$$\sum_{i=1,..,n-1} (1 - \frac{\bar{p} - p_i}{(v-p_i) - \delta(v-H)}) = n-2 \Rightarrow \sum_{1,..,n-1} \frac{(v-\bar{p}) - \delta(v-H)}{(v-p_i) - \delta(v-H)} = n-2$$

$$\Rightarrow \bar{p} = v - (n-2)[\frac{\prod_{i=1,..,n-1}[(v-p_i) - \delta(v-H)]}{\sum_{j=1,..n-1}[\prod_{k=1,..,n-1;k\neq j}\{(v-p_k) - \delta(v-H)\}]}] - \delta(v-H) \tag{26}$$

From the basic model we know that for each $i = 1, .., n-1$, $p_i \to H$ as $\delta \to 1$. Hence $[(v-p_i) - \delta(v-H)] \to 0$ as $\delta \to 1$. From (26) we have,

$$\bar{p} = v - [\frac{n-2}{\sum_{j=1,..n-1}[\frac{\prod_{k=1,..,n-1;k\neq j}\{(v-p_k)-\delta(v-H)\}}{\prod_{i=1,..,n-1}[(v-p_i)-\delta(v-H)]}]}] - \delta(v-H)$$

As $\delta \to 1$, $[\frac{n-2}{\sum_{j=1,..n-1}[\frac{\prod_{k=1,..,n-1;k\neq j}\{(v-p_k)-\delta(v-H)\}}{\prod_{i=1,..,n-1}[(v-p_i)-\delta(v-H)]}]}] \to 0$. Hence as $\delta \to 1$, $\bar{p} \to H$. This concludes the proof.

G Ex-ante public offers model

G.1 Ex ante public offers model

G.1.1 Players and payoffs

These are identical to those described in the basic model.

G.1.2 The extensive form

We consider an infinite horizon[12] multi-person bargaining game. At each time point $t = 1, 2...$ offers are made by the buyers only. However the offers are not targeted. Instead each buyer posts a price at which he is willing to purchase the good. Sellers can accept either of the posted prices. If two sellers accept the price posted by the same buyer, the buyer selects between them with equal probability. Also if a seller accepts a price and fails to sell his good, all other offers expire. Matched pairs leave the game. The unmatched players move on to

[11] In fact, as δ increases we will eventually have $\mathcal{P} > 1$

[12] Hoon-Sik Yang's question in a seminar presentation prompted us to add this section; our thanks to him.

the next period and the bargaining is continued under the same protocol. In this setting the sellers can be compared to *workers* who sell their *services* and buyers can be compared to firms who seek to purchase *services* from the *workers.*

G.1.3 Stationary equilibrium for ex ante public offers

We intend to find a stationary equilibrium of this (modified) extensive form. The qualitative nature of the equilibrium, analogous to the one we have studied before, is as follows. One of the buyers B_1 randomises between posting a price of H and posting something less than H. He randomises his prices if his posting is less than H. The other buyer B_2's posted price is randomised along a support whose upper bound is H.

In order to describe the candidate equilibrium, we note that the two player game (one buyer-one seller) is identical to that in the targeted offers model. We consider only the four-player game. Consider the following strategies:

(a) One of the buyers, B_1 say, puts a mass of q at H and a continuous distribution of posts ($(1-q)F_1(.)$) from p_l to H, where p_l will be defined later. The conditional distribution $F_1(.)$ consists of an absolutely continuous part from p_l to H and a mass point at p_l. B_2, on the other hand randomises his posts by putting a mass point at $p_l^{'}$ and an absolutely continuous part $F_2(.)$ from p_l to H, with $p_l^{'} < p_l$. The price $p_l^{'}$ is defined as,

$$p_l^{'} = \frac{M+H}{2} \tag{27}$$

The distributions $F_i(.)$ will be explicitly calculated.

(b) The sellers' strategies in the four-player game are as follows:

Suppose p_1 and p_2 are the posted prices such that $M \leq p_1 \leq p_2$. If $p_2 \geq H$, then S_M accepts p_1 (p_2) if $p_1 \geq \frac{M+p_2}{2}$ ($p_1 < \frac{M+p_2}{2}$). If $p_2 < H$ then S_M accepts p_2 only if the payoff from accepting it is at least as large as the continuation payoff from rejecting it. S_H accepts p_2 provided $p_2 \geq H$.

2. The expected payoff of a buyer i in equilibrium is $v - H$. The expected payoff of S_H is zero and that of S_M is positive.

The following lemma explicitly calculates the equlibrium described above assuming its existence.

Lemma 11 *Suppose there exists $p_l \in (p_l^{'}, H)$ such that,*

$$p_l - M = \delta(E(y) - M)$$

,where p (a random variable) represents the highest price post $\leq H$ under the proposed

strategies. Then the proposed strategies constitute an equilibrium with,

(i)

$$F_1(s) = \frac{(v-H)(1-\delta(1-q)) - q(v-s)}{(1-q)[(v-s)-\delta(v-H)]}$$

(ii)

$$F_2(s) = \frac{(v-H)(1-\delta(1-q')) - q'(v-s)}{(1-q')[(v-s)-\delta(v-H)]}$$

(iii)

$$q = \frac{[v-H](1-\delta)}{(v-p_l')-\delta(v-H)}$$

(iv)

$$q' = \frac{[v-H](1-\delta)}{(v-p_l)-\delta(v-H)}$$

Proof.

The proof is identical to the proof of lemma 1, if we replace M by p_l'. ■

The next lemma states that for sufficiently high values of δ there exists a unique p_l in the open interval (p_l', H)

Lemma 12 *There exists a $\delta^* \in (0,1)$ such that for all $\delta > \delta^*$, there exists a unique $p_l \in (p_l', H)$ that satisfies,*

$$p_l = \delta E(y) + (1-\delta)M$$

Proof.

Refer to appendix C ■

G.1.4 Asymptotic characterisation for ex ante public offers case

In the public offers model, as $\delta \to 1$, $p_l \to H$. Thus as agents become patient enough we get a uniform price for the non-differentiated goods. Since the proof of this is almost identical to the proof of Propositio 3 we omit it.

Note that the different versions of the extensive form give similar equilibria and the same asymptotic result, provided offers are one-sided

H n buyers and n sellers, buyers are heterogeneous

Heterogeneous buyers: Suppose the buyers are heterogeneous such that,

$$v_N > v_{N-1} > ... > v_2 > v_1 > H > u_{N-1} > ... > L$$

For each $i = 1, ..n-1$, define

$$p_i^h = (1-\delta)u_i + \delta E(y_i^h)\text{and}$$

$$\bar{q}_i^h = \frac{H - p_i^h}{(v_i - p_i^h) - \delta(v_i^h - H)}$$

, where $E(y_i^h)$ is defined as the equilibrium expected maximum price offer that S_i gets in the four-player game with S_i and S_n as the sellers and two buyers with valuation v_i and v_n. As before, let $\mathcal{P}_h = \sum_{i=1,..,n-1} \bar{q}_i^h$. Define $q_h^H = \frac{(v_1-H)(1-\delta)}{(v_1-L)-\delta(v_1-H)} \equiv q^H$ as $v_1 = v$.

Proposition 8 *With heterogeneous buyer valuations, analogues to propositions 5, 6 and 7 hold good for $\mathcal{P}_h < 1$ and $1-\mathcal{P}_h > q^H$, $\mathcal{P}_h < 1$ and $1-\mathcal{P}_h < q^H$ and $\mathcal{P}_h \geq 1$ respectively. For $\mathcal{P}_h < 1$ and $1-\mathcal{P}_h > q^H$ the lowest-valued buyer with valuation v offers to S_1. The specifics, however, are slightly different(see appendix (I)). Also with private offers, proposition (7) describes the equilibrium for high values of δ.*

Remark 1 *We omit the formal proof of the results for heterogeneous buyers since this is very similar to those of the previous propositions. Here, we explain why in the case of $\mathcal{P}_h < 1$ and $1-\mathcal{P}_h > q^H$ the **lowest-valued** buyer with valuation v offers to S_1, rather than one of the others.*[13] *In equilibrium, the buyer who is making offers to S_1 puts a mass point at the reservation value of that seller (i.e. at L). Since the buyer is indifferent between offering L to S_1 and making randomised offers in the range $[p_1, H]$, the probability (q^H) with which the buyer B_n makes offers to S_n must just make B_1 indifferent among the offers in the support of his randomised strategy.*[14] *This gives q^H as below.*

$$(v-L)q^H + (1-q^H)\delta(v-H) = v - H$$

$$\Rightarrow q^H = \frac{(v-H)(1-\delta)}{(v-L)-\delta(v-H)}$$

Buyer B_j ($j \neq 1; j \neq n$) makes randomised offers to the seller S_j with $[p_j, H]$ as the support. First, it is easy to see that B_j cannot profitably deviate by making offers to S_k ($j \neq k \neq n$) in the range $[p_k, H]$. To ensure that the proposed strategies constitute an equilibrium we need to show that this buyer with valuation $v_j (\neq v)$, has no incentive to offer u_i(or in the range (u_i, p_i)) to S_i, $i = 1, .., n-1$;. First consider $i = 2, ..n-1$. Since offers are public[15]*, a seller with valuation u_i will only accept an offer of u_i (or something in the range*

[13] This is a sufficient condition for the strategies described to be an equilibrium.

[14] W.L.O.G we assume that $v_1 = v$

[15] Note that the equilibrium for private offers is described by a different proposition.

$(u_i{,}p_i)$) if the buyer B_n makes an offer to S_n. Hence, the payoff to the buyer with valuation v_j of making an offer of u_i to S_i is,

$$(v_j - u_i)q^H + (1 - q^H)\delta(v_j - H)$$

Define q_j^H such that, $(v_j - u_i)q^H + (1 - q^H)\delta(v_j - H) = v_j - H$. This implies $q_j^H = \frac{(v_j - H)(1-\delta)}{(v_j - u_i) - \delta(v_j - H)}$. Since $v_j > v$ for all $j \neq 1$ and $u_i > L$, for all $i \neq 1$ we have

$$q_j^H = \frac{(v_j - H)(1-\delta)}{(v_j - u_i) - \delta(v_j - H)} > \frac{(v - H)(1-\delta)}{(v - L) - \delta(v - H)} = q^H$$

Since $(v_j - u_i) > \delta(v_j - H)$, $(v_j - u_i)q^H + (1 - q^H)\delta(v_j - H) < (v_j - H)$. The equilibrium payoff to the buyer with valuation v_j is $(v_j - H)$. This implies that the buyer has no incentive to offer u_i to seller S_i. This also proves that for $i = 1$, the buyer B_j has no incentive to offer L to S_1. To see this note that $\frac{(v_j - H)(1-\delta)}{(v_j - L) - \delta(v_j - H)} > \frac{(v - H)(1-\delta)}{(v - L) - \delta(v - H)}$. Since B_1 is also offering L to S_1 with some positive probability the payoff to B_j by offering L to S_1 is strictly less than $(v_j - L)q^H + (1 - q^H)\delta(v_j - H) < v_j - H$. Hence B_j has no incentive to offer anything in the range $[u_i, p_i)$ to S_i ($i = 1, ..n - 1$).

I Details of the equilibria defined in proposition (8)

We give here a more detailed description of the equilibrium for heterogeneous buyers for the $n \times n$ model.

I.1 $\mathcal{P}_h < 1$ and $1 - \mathcal{P}_h > q^H$

Buyer B_i ($i = 1, .., n - 1$) offers to seller S_i only. B_1 while making offers to S_1 puts a mass of $q_1^{'h}$ at L. With probability $(1 - q_1^{'h})$ he randomises his offers to S_1 using a continuous probability(conditional) distribution function F_1^h with $[p_1^h, H]$ as the support. B_n offers to S_1 with probability q_1^h. His offers are randomised using a probability distribution function F_{nh}^1 with $[p_1^h, H]$ as the support. F_{nh}^1 puts a mass point at p_i^h. The distributions F_1^h, F_{nh}^1 and the probabilities q_1^h and $q_1^{'h}$ are given by:

$$F_1^h = \frac{(v_n - H)[1 - \delta(1 - q_1^{'h})] - q_1^{'h}(v_n - s)}{(1 - q_1^{'h})[(v_n - s) - \delta(v_n - H)]}$$

$$F_{nh}^1 = \frac{(v - H)[1 - \delta q_1^h] - (1 - q_1^h)(v - s)}{q_1^h[(v - s) - \delta(v - H)]}$$

$$q_1'^h = \frac{(v_n - H)(1-\delta)}{(v_n - p_1^h) - \delta(v_n - H)}$$

$$q_1^h = \bar{q}_1^h + (1 - \mathcal{P}_h - q^H)$$

For $i = 2,..,n-1$, B_i's offers to S_i are randomised with a distribution $F_i^h(s)$. $F_i^h(.)$ puts a mass point at p_i^h and has an absolutely continuous part from p_i^h to H. B_n makes offers to $S_i (i = 2,..,n-1)$ with probability $q_i^h = \bar{q}_i^h$. B_n's offers to S_i are randomised using an absolutely continuous probability distribution F_{nh}^i with $[p_i^h, H]$ as the support. For $i = 2,...,n-1$, $F_i^h(.)$, $F_{nh}^i(.)$ are given by,

$$F_i^h = \frac{(v_n - H)(1-\delta)}{(v_n - s) - \delta(v_n - H)}$$

$$F_{nh}^i = \frac{(v_i - H)[1 - \delta q_i^h] - (1 - q_i^h)(v_i - s)}{q_i^h[(v_i - s) - \delta(v_i - H)]}$$

B_n offers to S_n with probability q^H. He offers H to S_n.

I.2 $\mathcal{P}_h < 1$ and $1 - \mathcal{P}_h < q^H$

Buyer B_i $(i = 1,..,n-1)$ offers to seller S_i only. B_i's offers to S_i are random with a distribution $F_i^h(s)$. $F_i^h(.)$ puts a mass point at p_i^h and has an absolutely continuous part from p_i^h to H. B_n makes offers to S_i $(i = 1,..,n-1)$ with probability $q_i^h = \bar{q}_i^h$. B_n's offers to S_i are random with an absolutely continuous probability distribution F_{nh}^i with $[p_i^h, H]$ as the support. For $i = 1,..,n-1$, $F_i^h(.)$ and $F_{nh}^i(.)$ are given by

$$F_i^h = \frac{(v_n - H)(1-\delta)}{(v_n - s) - \delta(v_n - H)}$$

$$F_{nh}^i = \frac{(v_i - H)[1 - \delta q_i^h] - (1 - q_i^h)(v_i - s)}{q_i^h[(v_i - s) - \delta(v_i - H)]}$$

B_n offers to S_n with probability $q_n^h = 1 - \mathcal{P}_h$. He offers H to S_n.

I.3 $\mathcal{P}_h \geq 1$

Buyer B_i makes offers to seller S_i only. B_i's offers to S_i are randomised using a distribution function $F_i^h(.)$ with $[p_i^h, \bar{p}^h]$ as the support. The distribution $F_i^h(.)$ puts a mass point at p_i^h and has an absolutely continuous part from p_i^h to $\bar{p}^h$. Buyer B_n makes offers to all sellers except S_n. B_n's offers to S_i $(i = 1,..,n-1)$ are randomised with a continuous probability distribution F_{nh}^i. The support of offers is $[p_i^h, \bar{p}^h]$. The probability with which B_n makes offers to S_i is q_i^h. If $\mathcal{P}_h = 1$ then $\bar{p}^h = H$. If $\mathcal{P}_h > 1$ then $\bar{p}^h < H$ and as $\delta \to 1$, $\bar{p}^h \to H$. $F_i^h(.)$, F_{nh}^i and q_i^h are given by the following expressions:

$$F_i^h(s) = \frac{(v_n - \bar{p}^h) - \delta(v_n - H)}{(v_n - s) - \delta(v_n - H)}$$

$$F_{nh}^i = \frac{(v_i - H)[1 - \delta q_i^h] - (1 - q_i^h)(v_i - H)}{q_i[(v_i - s) - \delta(v_i - H)]}$$

$$q_i^h = \frac{\bar{p}^h - p_i^h}{(v_i - p_i^h) - \delta(v_i - H)}$$

J Proof of Theorem 1

Consider an arbitrary stationary equilibrium of the game with $n+1$ buyers and $n+1$ sellers.

First of all it is to be observed that in any stationary equilibrium the seller with valuation H can never get offers from more than one buyers with positive probability. Refer to appendix (K) for a formal proof of this claim.

There can be two situations. Either all sellers are getting an offer with positive probability or at least one seller is not getting an offer with probability 1.

Case 1:*Suppose all sellers are getting offers with positive probability:*

This implies that the upper bound of the support of price offers to the sellers is H. Further except for seller H, each seller in the considered equilibrium gets offers from at least two buyers with positive probability. This is because if one of the sellers other than H is getting offers from only one buyer then she must be getting an offer equal to her valuation. This is not possible in equilibrium.

Also it is to be noted that in a stationary equilibrium it is never possible that at a time point there are remaining buyers and sellers to be matched and the seller with valuation H has already left. Suppose it is the case. Let the highest value seller remaining be $\tilde{H} < H$. Then from our hypothesis, the highest equilibrium price is $\tilde{H}$. Now consider the period when the seller with valuation H had left. It must be the case that a buyer B_H had offered a price $y \geq H$ to this seller. Hence the payoff obtained by that buyer is less than or equal $v - y \leq v - H$. In this stationary equilibrium, the highest expected price offer to the seller with valuation $\tilde{H}$ is $\tilde{H}$. Hence in equilibrium no buyer will offer more than $\tilde{H}$ to this seller. Suppose the buyer B_H deviates by offering to the seller with valuation $\tilde{H}$. Two things are possible if this seller rejects this offer from the deviating buyer. Either next period the same set of sellers and buyers will be present or there will be fewer buyers and sellers with the seller with valuation H being present. In the first case we know that the seller with valuation $\tilde{H}$ will accept any offer greater than or equal to $\tilde{H}$ and in the latter case by our hypothesis she will accept an offer of $H - \gamma$ where $\gamma > 0$ and is arbitrarily small for high values of δ. Since only one buyer can offer to the seller with valuation H in equilibrium, in either case B_H can profitably deviate by offering to the seller with valuation $\tilde{H}$.

Consider a seller S_k with valuation u_k. At least two buyers are offering to this seller with positive probability. Since all sellers are getting offers with positive probability, in any stationary equilibrium there can be the following three possibilities:

(i) S_k **is getting more than one offer but fewer than** $(n+1)$ **offers:** In such a case rejection of all offers would lead to a continuation game with n_1-buyers and n_1 sellers with $2 \leq n_1 < n+1$. Then by hypothesis, all price offers converge to H in the continuation game when $\delta \to 1$. For high values of δ we can thus approximate the expected highest price offer to S_k in the continuation game by a number $H - \epsilon$, where $\epsilon \to 0$ as $\delta \to 1$. Hence the minimum acceptable price offer this period should be at least as large as $p_k = [\delta(H - \epsilon) + (1 - \delta)u_k]$.

(ii) S_k **gets one offer but some other seller gets more than one offers:** In such a situation again rejection of all offers would lead to a continuation game with n_1-buyers and n_1 sellers with $2 \leq n_1 < n+1$. Thus the minimum acceptable price offer this period should be at least as large as $p_k = [\delta(H - \epsilon) + (1 - \delta)u_k]$.

(iii) S_k **gets one offer and all other sellers also get one offer:** In this situation rejecting the offer would lead to a game with seller S_k and one buyer next period. Hence the continuation payoff from rejection is 0 and thus the minimum acceptable price is u_k.

(iv) S_k **gets** $(n+1)$ **offers:** Rejection of all offers leads to a game with $n+1$-buyers and $n+1$ sellers. Let $\bar{p}$ be the minimum acceptable price to S_k in such a situation.

It is easy to note that

$$u_k < \bar{p}; u_k < p_k$$

Consider an arbitrary stationary equilibrium strategy profile.

First, suppose $\bar{p} < p_k$.

We would first argue that it is never possible that in equilibrium all buyers will make offers to S_k with probability 1. This actually follows from our proof of uniqueness of the basic model with two buyers and two sellers. If such is the case then we can show that at least two buyers would put a mass point at the lower bound of the support. This is not possible in equilibrium.

Claim 1 *:*

It is never a possibility that in equilibrium a buyer makes an offer $s \in (u_k, \bar{p})$ as a part of his behavioural randomised strategy.

Proof of the claim : An offer $s \in (u_k, \bar{p})$ is accepted when S_k gets only one offer. However if a buyer is offering something in this range and getting it accepted with positive probability, then he could have still made that offer accepted with the same probability by offering $s - \lambda \in (u_k, \bar{p})$. This is a profitable deviation and it is true for any $s \in (u_k, \bar{p})$. This proves our claim.

Claim 2 *:*

It is never a possibility in equilibrium that two buyers put a mass point at $\bar{p}$ as a part of their behavioural randomised strategies.

Proof of the claim: Suppose the claim is false. Consider a buyer B_k who puts a mass point at $\bar{p}$. This offer can get accepted only when S_k gets $(n+1)$ offers and all other offers are less than or equal to $\bar{p}$. Since there is another buyer B_j who puts a mass point at $\bar{p}$, B_k can profitable deviate by shifting the mass from $\bar{p}$ to $\bar{p} + \lambda$, λ arbitrarily small. This is because by shifting the mass to $\bar{p} + \lambda$, the probability of acceptance gets an upward jump but the payoff from acceptance remains the same. This proves our claim.

Since S_k is getting offers from two buyers with positive probability, there has to be a distribution of randomised offers to S_k from each buyer. As all sellers are getting an offer with positive probability, the upper bound of the support of the distribution is H. Hence with positive probability an offer greater than or equal to p_k is made to S_k.

Without loss of generality we can take p_k to be the lowest offer greater than or equal to p_k. Consider a buyer who is making this offer. Let q_k be the total probability that the offers by all other buyers are less than p_k. Since this buyer's payoff is $v - H$ (as offers are being made to all sellers with positive probability) we have

$$(v - p_k)q_k + (1 - q_k)\delta(v - H) = v - H$$

(By our hypothesis the continuation payoff of the buyer if his offer is rejected is $\delta(v - H)$.) This gives us:

$$q_k = \frac{(v-H)(1-\delta)}{(v-p_k)-\delta(v-H)} = \frac{1}{\frac{v}{v-H} + \frac{\delta H - p_k}{(1-\delta)(v-H)}}$$

Since p_k is bounded above by δH, as $\delta \to 1$, $q_k \to 0$..

If $\bar{p} \geq p_k$, then also we can show in the same way as above that the total mass put at u_k goes to zero as $\delta \to 1$.

Thus as δ goes to 1, the probability that all offers to S_k are less than p_k goes to zero. Hence the highest offer is always greater than or equal to p_k.

Case 2: *At least one seller is not getting an offer with probability* 1*:*

First of all it is easy to note that if exactly one seller is not getting an offer with probability 1 in a stationary equilibrium, then this seller must be the seller with valuation H. Otherwise the buyer who is making offer to the seller with valuation H can profitable deviate by making offer to the seller who in equilibrium is not getting an offer with probability 1.

Consider a seller S_k with valuation $u_k < H$ who is getting an offer. If she gets j offers $(j = 1, 2, ..., n)$ then rejecting all offers would lead to a game with n_1 buyers-n_1 sellers next period $(2 \leq n_1 < n+1)$. This is because at least one of the sellers is not getting an offer. Then by hypothesis, the minimum acceptable price in this case would be $p_k = \delta(H-\epsilon)+(1-\delta)u_k$. If S_k gets $n+1$ offers then rejection of all offers would lead to a game with $n+1$ buyers-$n+1$ sellers next period. Let $\bar{p}$ be the minimum acceptable price. Suppose $\bar{p} < p_k$. In equilibrium thus all offers are at least as large as $\bar{p}$. In equilibrium all buyers cannot put a mass at $\bar{p}$. Hence at least one buyer while making offers to S_k should not put a mass point at $\bar{p}$. In that case no buyer should ever put a mass point at $\bar{p}$. This is because an offer less than p_k is only accepted when the seller gets $n+1$ offers. However since at least one buyer is always offering more than $\bar{p}$, an offer of $\bar{p}$ is always rejected. This is not possible in equilibrium.

We would now argue that there can never be a distribution of offers to S_k with the lower bound of the support being less than p_k and greater than $\bar{p}$. Suppose it is the case and let $\underline{s}_k$ be the lower bound of the support with $\bar{p} < \underline{s}_k < p_k$. Consider the buyer who makes offers in the interval $(\underline{s}_k, \underline{s}_k + \tau)$ with positive probability where $\tau > 0$ and is sufficiently small. Offers in this range are almost surely (as $\tau \to 0$) rejected since all offers are almost surely greater than the offers in this range and an offer in this range would only be accepted if the seller gets $n+1$ offers. This is true for all $\underline{s}_k < p_k$. Hence offers to S_k are at least as large as p_k.

If $p_k \leq \bar{p}$, then all price offers are always at least as large as p_k.

Hence as $\delta \to 1$, price offers to S_k are almost surely greater than or equal to p_k.

$$p_k \to H \text{ as } \delta \to 1$$

Thus we have shown that as $\delta \to 1$, the accepted price offer by any seller in a stationary equilibrium with $(n+1)$ buyers and $(n+1)$ sellers converges to H.

This concludes the proof of this proposition.

K Seller with valuation H cannot get offer from more than one buyers with positive probability

Suppose this is true. Then there exists a stationary equilibrium where at least two buyers make offers with positive probability to the seller with valuation H. Let B_k and B_j be two such buyers. Clearly there has to be a distribution of offers to the H seller with an upper bound of the support $\bar{s}_H > H$. Thus equilibrium payoff to each of these buyers is $v - \bar{s}_H < v - H$. Consider the following deviation by one of these buyers(say B_k). Suppose B_k makes an unacceptable offer to H $(H - e, e > 0)$ with probability 1. Then next period according to our hypothesis his minimum payoff would be close to $(v - H)$ for high values of δ. This is because next period he will face a game which will have fewer than $(n+1)$ sellers and buyers. Either seller H would be present or she would be not. According to our hypothesis, for high values of δ the payoff to the buyer would be close to $v - H$ in the first case and strictly higher than $v - H$ in the later case. Hence for high values of δ, B_k's payoff from deviation is at least as large as $\delta(v - H)$. For high values of δ we would have $\delta(v - H) > v - \bar{s}_H$. Thus it is a profitable deviation by the buyer. This proves the claim

L A Stationary equilibrium of the game with n buyers and $n-1$ sellers

We will derive a stationary equilibrium of this extensive form. Thus buyers' offers at any time point depend only on the set of players remaining and the sellers' responses depend only on the set of players remaining, and the offers. Before we describe the equilibrium of this game formally we will verbally discuss its nature. In equilibrium, if all the players are present, buyer B_i $(i = 1, ..., n-1)$ makes offers to S_i only. His offers are randomised using a distribution function function $F_i(.)$, with $[p_i, \bar{p}]$ (p_i and $\bar{p}$ will be defined later) as the support. $F_i(.)$ puts a mass point at p_i and has an absolutely continuous part from p_i to $\bar{p}$. Buyer B_n makes offers to all the sellers with positive probability. B_n's offers to S_j $(j = 1, .., n-1)$ are randomised using a probability distribution $F_n^i(.)$. The support of offers is $[p_i, \bar{p}]$.

For each $i = 1, .., n-1$ we define p_i as ,

$$p_i = (1-\delta)u_i + \delta v \tag{28}$$

Let q_i be the probability with which B_n offers to seller S_i. The following proposition now formally defines the equilibrium of the game.

Proposition 9 *(i) The above conjectured strategies constitute a stationary equilibrium of the present game with,*

$$F_i(s) = \frac{v-\bar{p}}{v-s} \tag{29}$$

$$F_n^i(s) = \frac{(v-\bar{p}) - (1-q_i)(v-s)}{q_i(v-s)} \tag{30}$$

$$q_i = \frac{\bar{p}-p_i}{v-p_i} \tag{31}$$

$$\bar{p} = v - (n-2)\frac{\prod_{i=1,..,n-1}(v-p_i)}{\sum_{j=1,..,n-1}[\prod_{k=1,..,n-1;k\neq j}(v-p_k)]} \tag{32}$$

(ii) In equilibrium, each buyer obtains an expected payoff of $(v-\bar{p})$.

Proof. First consider the buyer B_i, $(i = 1, .., n-1)$. For $s \in [p_i, \bar{p}]$ his indifference relation is,

$$(v-s)[(1-q_i) + q_i F_n^i(s)] = v - \bar{p}$$

Solving the above relation for $F_n^i(.)$ we get (30). Putting $s = p_i$ in B_i's indifference relation we obtain (31). It is easy to note that $F_n^i(p_i) = 0$ and $F_n^i(\bar{p}) = 1$.

Next, consider the buyer B_n. The support of his offers to S_i $(i = 1, .., n-1)$ is $[p_i, \bar{p}]$. For $s \in [p_i, \bar{p}]$, B_n's indifference relation is given by

$$(v-s)[F_i(s)] = v - \bar{p}$$

which gives us (29). Note that $F_i(p_i) > 0$ and $F_i(\bar{p}) = 1$. This confirms our conjecture that B_i puts a mass point at p_i.

To have consistency in the expressions obtained we must have,

$$\sum_{i=1,..,n-1} q_i = 1 \Rightarrow \sum_{i=1,..,n-1} \frac{(\bar{p}-p_i)}{(v-p_i)} = 1$$

$$\Rightarrow \sum_{i=1,..,n-1} \frac{(v-\bar{p})}{(v-p_i})) = n-2$$

Rearranging the terms in the above relation we get (32).

Now we should check that the strategies constitute an equilibrium. First, observe that on the equilibrium path if a seller S_i rejects her offer(s) then next period she will face a game with two buyers and one seller. This will give her a discounted payoff of $\delta(v - u_i)$. Hence her minimum acceptable price should be p_i. From the analysis of the basic model one can infer that on the equilibrium path, there is no profitable deviation for the players. The way we have specified sellers' strategies these always constitute best responses in any off-path contingency. It is easy to check that buyers' strategies also constitute best responses in any off-path contingency. This concludes the proof. ∎

Remark 2 *Note that irrespective of whether a seller gets one offer or two offers, the continuation game faced by her from rejection is the same. Hence the stationary equilibrium constructed is a public perfect equilibrium for the case of private targeted offers.*

From (28)it is easy to observe that $p_i \to v$ as $\delta \to 1$. Thus as $\delta \to 1$, $(v - p_i) \to 0$ for $i = 1, .., n - 1$. This implies that the second term in (32) goes to zero as δ tends to one. Hence ,

$$\bar{p} \to v \text{ as } \delta \to 1$$

This implies that the distributions of the price offers by each buyer collapse to a single value in the limit.